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GABOR FRAMES IN $l^2(\mathbb{Z})$ FROM GABOR FRAMES IN $L^2(\mathbb{R})$

Jineesh Thomas, Madhavan Namboothiri N M, and Eldo Varghese

Abstract. In this paper we discuss about the image of Gabor frame under a unitary operator and derive a sufficient condition under which a unitary operator from $L^2(\mathbb{R})$ to $l^2(\mathbb{Z})$ maps Gabor frame in $L^2(\mathbb{R})$ to a Gabor frame in $l^2(\mathbb{Z})$.

1. Introduction

To date, Hilbert space frame theory has gained applications in vast areas of pure mathematics, applied mathematics and engineering. In 1946, it was first initiated by D.Gabor [11] in his *Theory of Communication*, and formulated a fundamental approach to signal decomposition in terms of elementary signals. His approach has become the archetype for the spectral analysis associated with time-frequency methods and further uses are being found for the theory in areas such as, optics, filterbanks, signal detection and many more. The vital works of Janssen [14], along with the theoretical foundation of communication theory and signal processing, using time frequency analysis by Gabor made frame theory an independent topic of mathematical investigation in 1980's. Frames and their relatives are most often considered in the discrete case, for instance in signal processing [7]. Traditionally, frames were studied for the whole

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space or for the closed subspace. The importance of the beautiful theory of frames in modern signal processing and time frequency analysis is now ingrained (see [13], for example).

The concept of frames in Hilbert spaces were introduced in 1952 by Duffin and Schaeffer [7] for studying some profound problems in non harmonic Fourier series. With the intensity of research on frame theory, various generalizations of frames have been proposed; frame of subspaces [1], [2], pseudo-frames [16], oblique frames [5] and so on, in which Gabor frames or Weyl-Heisenberg frames bagged a prime position. These frames are generated by translations and modulations of a single element in the space. Gabor analysis took a new spin with the fundamental works of Daubechies, Gross- mann and Meyer in 1986 [6] and put forth the idea of combining Gabor analysis with frame theory. Systematic utilisation of time shifts (translations) and frequency shifts (modulations) lie at the heart of modern time-frequency analysis. Gabor analysis aims at representating functions(signals) $f \in L^2(\mathbb{R})$ as superpositions of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$. Gabor systems in $L^2(\mathbb{R})$ have the form $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ for some $g \in L^2(\mathbb{R})$ and $a, b > 0$. Using operator notation, we can write a Gabor system as $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. We will not go into a detailed explanations of Gabor analysis and its role in time-frequency analysis, but just refer to [13], [4], [8], [9], [10].

One can consider frames in the sequence space $l^2(\mathbb{Z})$ with a Gabor like structure without referring to frames in $L^2(\mathbb{R})$. The theory for these frames is very similar to the Gabor theory in $L^2(\mathbb{R})$. Janssen showed in [15] that there is a natural way to obtain discrete Gabor frames using Gabor frames for $L^2(\mathbb{R})$ through sampling. We are interested to get Gabor frame in $l^2(\mathbb{Z})$ as image of Gabor frame in $L^2(\mathbb{R})$ via a suitable unitary operator. In this paper, section 2 is just recalling of basics in general frame theory and basics of Gabor frames in the spaces $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$. In Section 3, we explain a method to construct Gabor frames in $l^2(\mathbb{Z})$ from Gabor frames in $L^2(\mathbb{R})$ via a unitary transformation. Theory about frames in finite dimensional spaces are available in [3], [12].

In this paper, $\mathcal{H}$ will denote a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Our main references for frame theory and the proofs of the statements in this article are [13], [4].

2. Preliminaries

A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $\alpha,\ \beta > 0$ such that

$$\alpha\|f\|^2 \ \leq \ \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \ \leq \ \beta\|f\|^2, \ \ \forall\ f \in \mathcal{H}.$$

If a sequence $\{f_k\}_{k=1}^{\infty}$ satisfies atleast the upper frame condition, then it is called a *Bessel sequence*. We say that $\{f_k\}_{k=1}^{\infty}$ is a *frame sequence*, if it is a frame for $\overline{span}\{f_k\}_{k=1}^{\infty}$. The numbers α, β are called *frame bounds*. If $\alpha = \beta$, then the corresponding frame is called a *tight* frame and in particular a tight frame with $\alpha = \beta = 1$ is called a *Parseval frame* or *normalized tight frame*.

Since a frame $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, the operator $T : l^2(\mathbb{N}) \to \mathcal{H}$ defined by $T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$ is bounded and T is called the *synthesis operator* or *pre-frame operator*. The adjoint operator of T is the operator $T^* : \mathcal{H} \to l^2(\mathbb{N})$ given by, $T^*f = \{\langle f, f_k\rangle\}_{k=1}^{\infty}$ and is called the *analysis operator*. By composing T and T^* we obtain the *frame operator*

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = TT^*f = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$$

Note that since $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, the series defining S converges unconditionally for all $f \in \mathcal{H}$. It can be seen that, frame operator of a tight frame is a scalar multiple of the identity operator and that of a normalized tight frame is the identity operator.
Following are some important properties of frame operator S associated with a frame in a Hilbert space.

Let $\{f_k\}_{k=1}^{\infty}$ be a frame in a Hilbert space $\mathcal{H}$ with frame operator S and frame bounds $\alpha,\ \beta$. Then the following hold.
(1) S is bounded, invertible, self-adjoint and positive. In fact $\alpha I \leq S \leq \beta I$.
(2) $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame bounds $\beta^{-1},\ \alpha^{-1}$; if α and β are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$, then the bounds β^{-1}, α^{-1} are the optimal frame bounds for $\{S^{-1}f_k\}_{k=1}^{\infty}$.
(3) The frame operator for $\{S^{-1}f_k\}_{k=1}^{\infty}$ is S^{-1}.
(4) $\{S^{-1/2}f_k\}$ is a normalized tight frame.

The frame $\{S^{-1}f_k\}_{k=1}^{\infty}$ is called the *canonical dual* frame of $\{f_k\}_{k=1}^{\infty}$ because it plays the same role in the frame theory as the dual of a basis in functional analysis.

Next we state a prime result in frame theory namely the frame decomposition. It says that if $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, then every element in $\mathcal{H}$ has a representation as a superposition of the frame elements. Furthermore, it says that all information about each $f \in \mathcal{H}$ is contained in the sequence $\{\langle f, S^{-1}f_k\rangle\}_{k=1}^{\infty}$. The numbers $\langle f, S^{-1}f_k\rangle$ are called the *frame coefficients* of f.

THEOREM 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame in a Hilbert space $\mathcal{H}$ with a frame operator S. Then for all $f \in \mathcal{H}$,* $f = \sum_{k=1}^{\infty} \langle f, S^{-1}f_k\rangle f_k$ *and* $f = \sum_{k=1}^{\infty} \langle f, f_k\rangle S^{-1}f_k$. *Both series converge unconditionally for all $f \in \mathcal{H}$.*

Among several classes of frames in frame theory, Gabor frames or Weyl-Heisenberg frames in $L^2(\mathbb{R})$ have received remarkable role as they are generated by a single element in the space. We now come out with some basics of Gabor frame analysis in $L^2(\mathbb{R})$. The theory for Gabor analysis in $L^2(\mathbb{R})$ is based on two classes of operators on $L^2(\mathbb{R})$, namely the translation and modulation operators. For $a, b \in \mathbb{R}$, the translation operator T_a on $L^2(\mathbb{R})$ is defined by $(T_af)(x) = f(x-a),\ x \in \mathbb{R}$, and the modulation operator E_b on $L^2(\mathbb{R})$ by $(E_bf)(x) = e^{2\pi ibx}f(x),\ x \in \mathbb{R}$.

DEFINITION 2.2. A frame in $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for some $g \in L^2(\mathbb{R})$ and $a, b > 0$ is called a Gabor frame or Weyl-Heisenberg frame.

Gabor analysis aims at representing functions $f \in L^2(\mathbb{R})$ as superpositions of translated and modulated versions of a fixed $g \in L^2(\mathbb{R})$.
It is well known that if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame, then there exists a function $h \in L^2(\mathbb{R})$ such that
$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g \quad \forall f \in L^2(\mathbb{R}).$$
The classical choice of h is $h = S^{-1}g$, where S is the frame operator. The function $h = S^{-1}g$ is called the canonical dual generator.

PROPOSITION 2.3. *For each pair of translation and modulation parameters a, b satisfying the condition $0 \leq ab \leq 1$, there exists a $g \in L^2(\mathbb{R})$*

such that (g, a, b) is a normalized tight Gabor frame which, therefore has the identity operator as its frame operator.

We now record some basics of Gabor frame analysis in the sequence space $l^2(\mathbb{Z})$. Most of the numerical calculation with elements in $L^2(\mathbb{R})$ will involve a discrete structure, where all calculations are done with sequences in $l^2(\mathbb{Z})$. Hence it is important to know that certain conditions on a Gabor frame $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ in fact imply that we can construct a frame for $l^2(\mathbb{Z})$ having a similar structure.

For each $b \in \mathbb{R}$, the modulation operator $\hat{E}_b : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is defined by, $\hat{E}_b g(j) = e^{2\pi i b j} g(j)$, for all $g = (...., g(-1), g(0), g(1), ...) \in l^2(\mathbb{Z})$, where the j^{th} coordinate of g is denoted by $g(j)$. Similarly for each $n \in \mathbb{Z}$ the translation operator $\hat{T}_n : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is defined by $\hat{T}_n g(j) = g(j - n)$, for all $g = (...., g(-1), g(0), g(1), ...) \in l^2(\mathbb{Z})$.

Even though the definition of $\hat{E}_b$ makes sense for all $b \in \mathbb{R}$ we will only use modulations of the form $\hat{E}_{m/M}$, where $M \in \mathbb{N}$ is fixed and $m \in \mathbb{Z}$. In the language used for Gabor systems in $L^2(\mathbb{R})$ this corresponds to having the modulation parameter equal to $1/M$. There is, however, one major difference between the two settings. In $L^2(\mathbb{R})$-setting modulation operator with different parameters are necessarily different, but this is not the case in discrete setting. In fact with the above definition,
$\hat{E}_k = \hat{E}_{\frac{m}{M}+k}$, for all $k \in \mathbb{Z}$. Therefore $\{\hat{E}_{m/M} g : m \in \mathbb{Z}\}$ can not be a Bessel sequence in $l^2(\mathbb{Z})$ unless $g = 0$. For this reason we will consider modulations $\hat{E}_{m/M}$ with $m = 0, 1, 2, ...M - 1$.

The discrete Gabor system generated by a sequence $g \in l^2(\mathbb{Z})$ with the modulation parameter $\frac{1}{M}$ and translation parameter $N, (M, N \in \mathbb{N})$ is now defined as the family of sequences $\{\hat{E}_{m/M}\hat{T}_{nN} g : m = 0, 1, ...M - 1, n \in \mathbb{Z}\}$. Specifically , $\hat{E}_{m/M}\hat{T}_{nN} g$ is the sequence in $l^2(\mathbb{Z})$ whose j- th coordinate is $\hat{E}_{m/M}\hat{T}_{nN} g(j) = e^{2\pi i (j - nN)\frac{m}{M}} g(j - nN)$.
For more detailed results about frame theory in the spaces $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$ one may refer [4].

3. Construction of Gabor frames in $l^2(\mathbb{Z})$ from Gabor frames in $L^2(\mathbb{R})$

From Proposition 2.3, there exists an element $g \in L^2(\mathbb{R})$ such that the collection $\{E_{\frac{m}{M}} T_{nN} g : m, n \in \mathbb{Z}\}$ is a Gabor frame in $L^2(\mathbb{R})$ for any two positive integers M, N with $\frac{N}{M} \leq 1$. In this section, we focused on the construction of a Gabor frame $\{\hat{E}_{\frac{m}{M}}\hat{T}_{nN}g : m = 0, 1, 2...M-1, n \in \mathbb{Z}\}$ in $l^2(\mathbb{Z})$ from a Gabor frame $\{E_{\frac{m}{M}}T_{nN}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ for any two positive integers M, N with $\frac{N}{M} \leq 1$.

The following lemma which is available in [4] is useful in our discussion and it ensures the existence of pseudo inverse for a bounded linear operator on a Hilbert space with closed range.

Lemma 3.1. *Let U be a bounded linear operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ with its range set R_U is closed. Then there exists a bounded operator $U^\dagger$ from $\mathcal{K}$ to $\mathcal{H}$ such that $UU^\dagger f = f$ for all $f \in R_U$. Moreover $UU^\dagger$ is the orthogonal projection of $\mathcal{H}$ onto R_U .*

Theorem 3.2. *Let $\{f_k\}_{k=1}^\infty$ be a frame in $\mathcal{K}$ with bounds A and B and let $U : \mathcal{K} \longrightarrow \mathcal{H}$ a bounded linear operator with non trivial closed range. Then $\{Uf_k\}_{k=1}^\infty$ is a frame sequence with bounds $A \parallel U^\dagger \parallel^{-2}$ and $B \parallel U \parallel^2$.*

Proof. First observe that, $\sum_{k=1}^\infty \mid \langle f, Uf_k\rangle \mid^2 = \sum_{k=1}^\infty \mid \langle U^*f,\ f_k\rangle \mid^2$

$$\leq B \parallel U^*f \parallel^2 \leq B \parallel U^* \parallel^2 \parallel f \parallel^2 = B \parallel U \parallel^2 \parallel f \parallel^2 .$$

Thus $\{Uf_k\}_{k=1}^\infty$ is a Bessel sequence in $\mathcal{H}$ with upper frame bound $B \parallel U \parallel^2$. For $h \in Span\{Uf_k\}_{k=1}^\infty$, there is $f \in Span\{f_k\}_{k=1}^\infty$ with $h = Uf$. Since $UU^\dagger$ is the orthogonal projection onto R_U, it is self adjoint and hence,

$$h = Uf = (UU^\dagger)(Uf) = (U^\dagger)^*U^*(Uf).$$

Thus, $\parallel h \parallel^2 \leq \parallel (U^\dagger)^* \parallel^2 \parallel U^*Uf \parallel^2 \leq \frac{\|U^\dagger\|^2}{A} \sum_{k=1}^\infty \mid \langle U^*Uf, f_k\rangle \mid^2$

$$= \frac{\|U^\dagger\|^2}{A} \sum_{k=1}^\infty \mid \langle h, Uf_k\rangle \mid^2.$$

Since $U : \mathcal{K} \longrightarrow \mathcal{H}$ is of non trivial closed range, the remaining assertions follow. □

Remark 3.3. Let $\{f_k\}_{k=1}^\infty$ be a frame in $\mathcal{K}$ with bounds A and B and $U : \mathcal{K} \longrightarrow \mathcal{H}$ a bounded linear surjective operator . Then $\{Uf_k\}_{k=1}^\infty$ is a frame in $\mathcal{H}$ with bounds $A \parallel U^\dagger \parallel^{-2}$ and $B \parallel U \parallel^2$.

Theorem 3.4. *Let M, N be natural numbers with $\frac{N}{M} \leq 1$. Suppose that the collection $\{E_{\frac{m}{M}}T_{nN}g : m,\ n \in \mathbb{Z}\}$ is a Gabor frame in $L^2(\mathbb{R})$ for some $g \in L^2(\mathbb{R})$. Then for any surjective bounded linear operator $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ with the property that $UE_{\frac{m}{M}}T_{nN} = \hat{E}_{\frac{m}{M}}\hat{T}_{nN}U$ for $m,\ n \in \mathbb{Z}$, the sequence $\{\hat{E}_{\frac{m}{M}}\hat{T}_{nN}Ug : m = 0, 1, 2....., M-1, n \in \mathbb{Z}\}$ is a Gabor frame in $l^2(\mathbb{Z})$.*

Proof. Let $g \in L^2(\mathbb{R})$ and M, N be two natural numbers such that $\frac{N}{M} \leq 1$ and assume that the collection $\{E_{\frac{m}{M}} T_{nN} g : m, n \in \mathbb{Z}\}$ is a Gabor frame in $L^2(\mathbb{R})$ and U be a surjective bounded linear operator from $L^2(\mathbb{R})$ to $l^2(\mathbb{Z})$ with the property that $UE_{\frac{m}{M}} T_{nN} = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} U$ for $m,\ n \in \mathbb{Z}$. Then by Remark 3.3 $\{U(E_{\frac{m}{M}} T_{nN} g) : m, n \in \mathbb{Z}\}$ is a frame in $l^2(\mathbb{Z})$. Since U satisfies $UE_{\frac{m}{M}} T_{nN} = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} U$ for $m, n \in \mathbb{Z}$, and $\hat{E}_{\frac{m}{M}+k} \hat{T}_{nN} g = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} g$ for any $k \in \mathbb{Z}$, we see that $\{UE_{\frac{m}{M}} T_{nN} g : m,\ n \in \mathbb{Z}\} = \{\hat{E}_{\frac{m}{M}} \hat{T}_{nN} Ug : m,\ n \in \mathbb{Z}\} = \{\hat{E}_{\frac{m}{M}} \hat{T}_{nN} Ug : m = 0,\ 1,\ 2,\ \ldots,\ M-1,\ n \in \mathbb{Z}\}$ and hence the frame $\{\hat{E}_{\frac{m}{M}} T_{nN} Ug : m = 0,\ 1,\ 2,\ \ldots,\ M-1, n \in \mathbb{Z}\}$ is a Gabor frame in $l^2(\mathbb{Z})$.

□

It is important to know that what kind of conditions on a Gabor frame $\{E_{\frac{m}{M}} T_{nN} g : m, n \in \mathbb{Z}\}$ really imply that we have a frame for $l^2(\mathbb{Z})$ having a Gabor like structure. The relevant conditions were discovered by Janssen [15]. He proved that there is a natural way to obtain discrete Gabor frames *via* Gabor frames for $L^2(\mathbb{R})$ through sampling. A detailed discussion of these theories are available in [4]. We consider a Gabor system for $L^2(\mathbb{R})$ of the form $\{E_{\frac{m}{M}} T_{nN} g : m, n \in \mathbb{Z}\}$, where $g \in L^2(\mathbb{R})$ is the *window function* or *generating function* and $M,\ N\ \in \mathbb{N}$. In searching a Gabor like system in $l^2(\mathbb{Z})$ the natural question arising is, "which type of linear transformations maps a Gabor frame in $L^2(\mathbb{R})$ to a Gabor like frame in $l^2(\mathbb{Z})$".

Let $h = \chi_{[0,1]}$, the characteristic function on $[0, 1]$. Then the collection $\{E_k T_j h : k, j \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ [4].
Let $g \in L^2(\mathbb{R})$ then, $g = \sum_{k,j\in\mathbb{Z}} \alpha_{kj} E_k T_j h,$ where $\alpha_{kj} = \langle g, E_k T_j h\rangle$.

Definition 3.5. Let $g \in L^2(\mathbb{R})$, then for each $m, n \in \mathbb{Z}$ and for each pair of positive integers M and N, there is a sequence of complex numbers $\{\zeta_{r,s}\}$ such that $E_{\frac{m}{M}} T_{nN} g = \sum_{r,s\in\mathbb{Z}} \zeta_{r,s} E_r T_s h$. This sequence is called window sequence of g with respect to the quadruple (m, n, M, N) and each terms of this sequence are called window constants.

Proposition 3.6. *Let $\{\zeta_{r,s}\}$ be the window sequence of $g \in L^2(\mathbb{R})$ with respect to the quadruple (m, n, M, N) where $m, n \in \mathbb{Z}$ and M, N*

are positive integers . Then for $\frac{m}{M} \notin \mathbb{Z}$

$$\zeta_{r,s} = \sum_{p\in\mathbb{Z}} \alpha_{r-p,\ s-nN} \frac{M}{2\pi i(m-pM)} e^{2\pi i \frac{m}{M} s}(e^{2\pi i \frac{m}{M}} - 1)$$

and for $\frac{m}{M} \in \mathbb{Z}$

$$\zeta_{r,s} = \alpha_{r-\frac{m}{M},\ s-nN}$$

where $\alpha_{kj} = \langle g, E_k T_j \chi_{[0,1]}\rangle$.

Proof. For each $m, n \in \mathbb{Z}$, the elements $E_{\frac{m}{M}} T_{nN} g$ in the Gabor frame $\{E_{\frac{m}{M}} T_{nN} g : m,\ n \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$ takes the form

$$\begin{aligned}
E_{\frac{m}{M}} T_{nN} g &= \sum_{k,j\in\mathbb{Z}} \alpha_{kj} E_{\frac{m}{M}} T_{nN} E_k T_j h \\
&= \sum_{k,j\in\mathbb{Z}} \alpha_{kj} E_{\frac{m}{M}} T_{nN} e^{2\pi i kj} T_j E_k h \\
&= \sum_{k,j\in\mathbb{Z}} \alpha_{kj} E_{\frac{m}{M}} T_{nN+j} E_k h \\
&= \sum_{k,j\in\mathbb{Z}} \alpha_{kj} e^{2\pi i \frac{m}{M}(nN+j)} T_{nN+j} E_{\frac{m}{M}} E_k h \\
&= \sum_{k,j\in\mathbb{Z}} \alpha_{kj} e^{2\pi i \frac{m}{M}(nN+j)} T_{nN+j} E_k (E_{\frac{m}{M}} h)
\end{aligned}$$

Note that, $E_{\frac{m}{M}} h = \sum_{p,q\in\mathbb{Z}} \beta_{pq} E_p T_q h$, where

$$\begin{aligned}
\beta_{pq} &= \langle E_{\frac{m}{M}} h, E_p T_q h\rangle \\
&= \int_{-\infty}^{\infty} E_{\frac{m}{M}} h(x) \overline{E_p T_q h(x)} dx \\
&= \int_{-\infty}^{\infty} e^{2\pi i \frac{m}{M} x} h(x) e^{-2\pi i p(x-q)} h(x-q) dx \\
&= \int_{-\infty}^{\infty} e^{2\pi i \frac{m}{M} x} e^{-2\pi i p(x-q)} \chi_{[0,1]\cap([0,1]+q)}(x) dx
\end{aligned}$$

when $q = 0$, we have $\beta_{p0} = \int_0^1 e^{-2\pi i(p-\frac{m}{M})x} dx$ and hence

$$\beta_{p0} = \begin{cases} \frac{M}{2\pi i(m-pM)}[e^{-2\pi i(p-\frac{m}{M})} - 1] & \text{if } p \neq \frac{m}{M} \\ 1 & \text{if } p = \frac{m}{M} \end{cases}$$

Also for $q \neq 0$, $\beta_{pq} = 0$

Hence,

$$\beta_{pq} = \begin{cases} 0 & \text{if } \; q \neq 0 \\ \frac{M}{2\pi i(m-pM)}[e^{-2\pi i(p-\frac{m}{M})} - 1] & \text{if } \; q = 0 \; \text{ and } \; p \neq \frac{m}{M} \\ 1 & \text{if } \; q = 0 \; \text{ and } \; p = \frac{m}{M} \end{cases}$$

Therefore for $\frac{m}{M} \notin \mathbb{Z}$, $\quad E_{\frac{m}{M}} h = \sum_{p \in \mathbb{Z}} \frac{M}{2\pi i(m-pM)} (e^{2\pi i \frac{m}{M}} - 1) E_p h$

Thus for any $m \in \mathbb{Z}$ with $\frac{m}{M} \notin \mathbb{Z}$, we have $E_{\frac{m}{M}} T_{nN} g$

$$= \sum_{k,j \in \mathbb{Z}} \alpha_{kj} e^{2\pi i \frac{m}{M}(nN+j)} T_{nN+j} E_k \sum_{p \in \mathbb{Z}} \frac{M}{2\pi i(m-pM)} (e^{2\pi i \frac{m}{M}} - 1) E_p h$$

$$= \sum_{k,j \in \mathbb{Z}} \alpha_{kj} e^{2\pi i \frac{m}{M}(nN+j)} (e^{2\pi i \frac{m}{M}} - 1) T_{nN+j} E_k \sum_{p \in \mathbb{Z}} \frac{M}{2\pi i(m-pM)} E_p h$$

$$= \sum_{p \in \mathbb{Z}} \sum_{k,j \in \mathbb{Z}} \alpha_{kj} e^{2\pi i \frac{m}{M}(nN+j)} (e^{2\pi i \frac{m}{M}} - 1) E_k T_{nN+j} \frac{M}{2\pi i(m-pM)} E_p h$$

$$= \sum_{p \in \mathbb{Z}} \sum_{k,j \in \mathbb{Z}} \alpha_{kj} \frac{M}{2\pi i(m-pM)} e^{2\pi i \frac{m}{M}(nN+j)} (e^{2\pi i \frac{m}{M}} - 1) E_{k+p} T_{nN+j} h$$

Take $r = k + p$ and $s = nN + j$. Then

$$E_{\frac{m}{M}} T_{nN} g = \sum_{r,s,p \in \mathbb{Z}} \alpha_{r-p,\ s-nN} \frac{M}{2\pi i(m-pM)} e^{2\pi i \frac{m}{M} s} (e^{2\pi i \frac{m}{M}} - 1) E_r T_s h$$

$$= \sum_{r,s \in \mathbb{Z}} \zeta_{r,s} E_r T_s h$$

where, $\zeta_{r,s} = \sum_{p \in \mathbb{Z}} \alpha_{r-p,\ s-nN} \frac{M}{2\pi i(m-pM)} e^{2\pi i \frac{m}{M} s} (e^{2\pi i \frac{m}{M}} - 1)$

Now for any $m \in \mathbb{Z}$ with $\frac{m}{M} \in \mathbb{Z}$, we have $\quad E_{\frac{m}{M}} T_{nN} g$

$$= \sum_{k,j \in \mathbb{Z}} \alpha_{kj} e^{2\pi i \frac{m}{M}(nN+j)} T_{nN+j} E_k E_{\frac{m}{M}} h$$

$$= \sum_{k,j \in \mathbb{Z}} \alpha_{kj} T_{nN+j} E_{k+\frac{m}{M}} h$$

Take $r = k + p$ and $s = nN + j$. Then,

$$E_{\frac{m}{M}} T_{nN} g = \sum_{r,s \in \mathbb{Z}} \zeta_{r,s} E_r T_s h$$

where, $\zeta_{r,s} = \alpha_{r-\frac{m}{M},\ s-nN}$. $\square$

The following theorem gives a sufficient condition for a unitary operator U from $L^2(\mathbb{R})$ to $l^2(\mathbb{Z})$ which maps a Gabor frame $(g, \frac{1}{M},\ N)$ in $L^2(\mathbb{R})$ to a Gabor frame $(Ug, \frac{1}{M},\ N)$ in $l^2(\mathbb{Z})$.

THEOREM 3.7. *Let $g \in L^2(\mathbb{R})$ and N, M are positive integers such that $\frac{N}{M} \leq 1$, $\{E_{\frac{m}{M}} T_{nN} g : m, n \in \mathbb{Z}\}$ is a Gabor frame in $L^2(\mathbb{R})$. Assume ϕ is a bijection from $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that, for each $r, s, m, n \in \mathbb{Z}$ if the window sequence of g with respect to the quadruple (m, n, M, N) given by $\{\zeta_{r,s}\}$ satisfies $\zeta_{r,s} = \alpha_{\phi^{-1}(\phi(r,s)+nN)} e^{2\pi i \frac{m}{M}\phi(r,s)}$, where $\alpha_{kj} = \langle g, E_k T_j h\rangle$, $h = \chi_{[0,1]}$, then there is a unitary operator $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ so that $\{\hat{E}_{\frac{m}{M}} \hat{T}_{nN} Ug : m = 0, 1, 2 ... M-1, n \in \mathbb{Z}\}$ is a Gabor frame in $l^2(\mathbb{Z})$.*

Proof. Define $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ by

$$U[E_{\frac{m}{M}} T_{nN} E_k T_j h] = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} e_{\phi(k,j)}$$

where $\{e_j\}$ is the standard orthonormal basis for $l^2(\mathbb{Z})$.
Note that $U(E_k T_j h) = e_{\phi(k,j)}$. Hence U is a unitary linear map since U maps the an orthonormal basis of $L^2(\mathbb{R})$ to an orthonormal basis for $l^2(\mathbb{Z})$.
Also we have $g = \sum_{k,j\in\mathbb{Z}} \alpha_{kj} E_k T_j h$, where $\alpha_{kj} = \langle g, E_k T_j h\rangle$.
Therefore, $U(g) = \sum\limits_{k,j\in\mathbb{Z}} \alpha_{kj} e_{\phi(k,j)}$ since $U(E_k T_j h) = e_{\phi(k,j)}$.

$$\begin{aligned}
\text{Hence,}\quad \hat{E}_{\frac{m}{M}} \hat{T}_{nN} U(g) &= \sum_{k,j\in\mathbb{Z}} \alpha_{kj} \hat{E}_{\frac{m}{M}} \hat{T}_{nN} e_{\phi(k,j)} \\
&= \sum_{k,j\in\mathbb{Z}} \alpha_{kj} e^{2\pi i \frac{m}{M}(\phi(k,j)-nN)} e_{\phi(k,j)-nN} \\
&= \sum_{r,s\in\mathbb{Z}} \alpha_{\phi^{-1}(\phi(r,s)+nN)} e^{2\pi i \frac{m}{M}\phi(r,s)} e_{\phi(r,s)} \text{..........(1)}
\end{aligned}$$

Now by definition of the window sequence $\{\zeta_{r,s}\}$,
$E_{\frac{m}{M}} T_{nN} g = \sum\limits_{r,s\in\mathbb{Z}} \zeta_{r,s} E_r T_s h$. Therefore,

$$U(E_{\frac{m}{M}} T_{nN} g) = \sum_{r,s\in\mathbb{Z}} \zeta_{r,s} e_{\phi(r,s)} \text{........(2)}$$

Since $\zeta_{r,s} = \alpha_{\phi^{-1}(\phi(r,s)+nN)} e^{2\pi i \frac{m}{M}\phi(r,s)}$, Eq(1)and (2) follows that

$$U E_{\frac{m}{M}} T_{nN} g = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} U g$$

Hence by Theorem 3.4 $\{\hat{E}_{\frac{m}{M}}\hat{T}_{nN}Ug : m = 0, 1, 2...M-1, n \in \mathbb{Z}\}$ is a Gabor frame in $l^2(\mathbb{Z})$. □

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Jineesh Thomas
Research department of Mathematics
St.Thomas College Palai
Kottayam, Kerala, India, 686574.
E-mail: jineeshthomas@gmail.com

Madhavan Namboothiri N M
Department of Mathematics
Government College Kottayam
Nattakom P O, Kerala, India, 686013.
E-mail: madhavangck@gmail.com

Eldo Varghese
Research department of Mathematics
St.Thomas College Palai
Kottayam, Kerala, India, 686574.
E-mail: eldovarghese1978@gmail.com