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APPROXIMATION OPERATORS AND FUZZY ROUGH SETS IN CO-RESIDUATED LATTICES

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Abstract. In this paper, we introduce the notions of a distance function, Alexandrov topology and $\ominus$-upper ($\oplus$-lower) approximation operator based on complete co-residuated lattices. Under various relations, we define $(\oplus, \ominus)$-fuzzy rough set on complete co-residuated lattices. Moreover, we study their properties and give their examples.

1. Introduction

Pawlak [15,16] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. For an extension of Pawlak's rough sets, many researchers [1-11,19,20,24] developed lower and upper approximation operators. Radzikowska et al.[17,18] investigated (I, T)-generalized fuzzy rough set where T is a t-norm and I is an implication. J.S.Mi et al.[14] investigated (S, T)-generalized fuzzy rough set where T is a t-norm and $S(a, b) = 1 - T(1 - a, 1 - b)$ is an implication.

Ward et al.[23] introduced a complete residuated lattice which is an algebraic structure for many valued logic [3-5]. It is an important mathematical tool as algebraic structures for many valued logics [1-11,19,20]. Using this concepts, fuzzy rough sets, information systems and decision rules were investigated in complete residuated lattices [1,2,7,20,25]. Moreover, Zheng et al.[25] introduced a complete co-residuated lattice as the generalization of t-conorm. Junsheng et al.[7] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \vee, \wedge, \odot, \&, 0, 1)$ where $(L, \vee, \wedge, \&, 0, 1)$ is a complete residuated lattice and $(L, \vee, \wedge, \odot, 0, 1)$ is complete co-residuated lattice in a sense [13].

As the study of rough set theory and topological structures, many researchers [1,6-9,12,14,15,17,21] investigated the Alexandrov topology and lattice structures of fuzzy rough sets determined by lower and upper sets. In particular, Kim [8-11] introduce the notion of Alexandrov topologies as a topological viewpoint of fuzzy rough sets and studied the relations among fuzzy preorders, lower and upper approximation operators and Alexandrov topologies in complete residuated lattices.

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In this paper, we introduce the notions of distance functions, Alexandrov topologies and $\ominus$-upper ($\oplus$-lower) approximation operators based on complete co-residuated lattices $(L,\vee,\wedge,\oplus,0,1)$. Under various relations, we define $(\oplus,\ominus)$-fuzzy rough set on complete co-residuated lattices $(L,\vee,\wedge,\oplus,0,1)$ where $\ominus$ is induced by $\oplus$. Moreover, we study their properties and give their examples.

2. Preliminaries

Definition 2.1. [7,25] An algebra $(L,\wedge,\vee,\oplus,0,1)$ is called a *complete co-residuated lattice* if it satisfies the following conditions:

(Q1) $L=(L,\leq,\vee,\wedge,0,1)$ is a complete lattice where 0 is the bottom element and 1 is the top element.

(Q2) $a=a\oplus 0$, $a\oplus b=b\oplus a$ and $a\oplus(b\oplus c)=(a\oplus b)\oplus c$ for all $a,b,c\in L$.

(Q3) $(\bigwedge_{i\in\Gamma} a_i)\oplus b=\bigwedge_{i\in\Gamma}(a_i\oplus b)$.

Let $(L,\leq,\oplus)$ be a complete co-residuated lattice. For each $x,y\in L$, we define

$$x\ominus y=\bigwedge\{z\in L\mid y\oplus z\geq x\}.$$

Then $(x\oplus y)\geq z$ iff $x\geq(z\ominus y)$.

In this paper, we assume $(L,\wedge,\vee,\oplus,\ominus,0,1)$ is a complete co-residuated lattice. For $\alpha\in L, A\in L^X$, we denote $(\alpha\ominus A),(\alpha\oplus A),\alpha_X\in L^X$ as $(\alpha\ominus A)(x)=\alpha\ominus A(x)$, $(\alpha\oplus A)(x)=\alpha\oplus A(x)$, $\alpha_X(x)=\alpha$.

Put $N(x)=1\ominus x$. The condition $N(N(x))=x$ for each $x\in L$ is called a *double negative law.*

Remark 2.2. (1) An infinitely distributive lattice $(L,\leq,\vee,\wedge,\oplus=\vee,0,1)$ is a complete co-residuated lattice. In particular, the unit interval $([0,1],\leq,\vee,\wedge,\oplus=\vee,0,1)$ is a complete co-residuated lattice [7,25].

$$\begin{aligned}x\ominus y&=\bigwedge\{z\in L\mid y\vee z\geq x\}\\&=\begin{cases}0, & \text{if } y\geq x,\\ x, & \text{if } y\not\geq x.\end{cases}\end{aligned}$$

Put $N(x)=1\ominus x=1$ for $x\neq 1$ and $N(1)=0$. Then $N(N(x))=0$ for $x\neq 1$ and $N(N(1))=1$. Hence N does not satisfy a double negative law.

(2) The unit interval with a right-continuous t-conorm $\oplus$, $([0,1],\leq,\oplus)$, is a complete co-residuated lattice [7.25].

(3) $([1,\infty],\leq,\vee,\oplus=\cdot,\wedge,1,\infty)$ is a complete co-residuated lattice where

$$\begin{aligned}x\ominus y&=\bigwedge\{z\in[1,\infty]\mid yz\geq x\}\\&=\begin{cases}1, & \text{if } y\geq x,\\ \frac{x}{y}, & \text{if } y\not\geq x.\end{cases}\end{aligned}$$

$$\infty\cdot a=a\cdot\infty=\infty,\forall a\in[1,\infty],\infty\ominus\infty=1.$$

Put $N(x)=\infty\ominus x=\infty$ for $x\neq\infty$ and $N(\infty)=1$. Then $N(N(x))=1$ for $x\neq\infty$ and $N(N(\infty))=\infty$. Hence N does not satisfy a double negative law.

(4) $([0,\infty],\leq,\vee,\oplus=+,\wedge,0,\infty)$ is a complete co-residuated lattice where

$$\begin{aligned}&y\ominus x=\bigwedge\{z\in[0,\infty]\mid x+z\geq y\}\\&=\bigwedge\{z\in[0,\infty]\mid z\geq -x+y\}=(y-x)\vee 0,\\&\infty+a=a+\infty=\infty,\forall a\in[0,\infty],\infty\ominus\infty=0.\end{aligned}$$

Put $N(x) = \infty \ominus x = \infty$ for $x \neq \infty$ and $N(\infty) = 0$. Then $N(N(x)) = 0$ for $x \neq \infty$ and $N(N(\infty)) = \infty$. Hence N does not satisfy a double negative law.

(5) $([0,1], \leq, \vee, \oplus, \wedge, 0, 1)$ is a complete co-residuated lattice where

$$\begin{aligned} & x \oplus y = (x^p + y^p)^{\frac{1}{p}} \; 1 \leq p < \infty, \\ & x \ominus y = \bigwedge\{z \in [0,1] \mid (z^p + y^p)^{\frac{1}{p}} \geq x\} \\ & = \bigwedge\{z \in [0,1] \mid z \geq (x^p - y^p)^{\frac{1}{p}}\} = (x^p - y^p)^{\frac{1}{p}} \vee 0, \end{aligned}$$

Put $N(x) = 1 \ominus x = (1 - x^p)^{\frac{1}{p}}$ for $1 \leq p < \infty$. Then $N(N(x)) = x$ for $x \in [0,1]$. Hence N satisfies a double negative law.

(6) Let $P(X)$ be the collection of all subsets of X. Then $(P(X), \subset, \cup, \cap, \oplus = \cup, \emptyset, X)$ is a complete co-residuated lattice where

$$\begin{aligned} & A \ominus B = \bigwedge\{C \in P(X) \mid B \cup C \supset A\} \\ & = A \cap B^c = A - B. \end{aligned}$$

Put $N(A) = X \ominus A = A^c$ for each $A \subset X$. Then $N(N(A)) = A$. Hence N satisfies a double negative law.

Lemma 2.3. [11] *Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $(x \oplus y) \leq (x \oplus z)$, $y \ominus x \leq z \ominus x$ and $x \ominus z \leq x \ominus y$.*

(2) *$(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$ and $x \ominus (\bigwedge_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$.*

(3) *$(\bigwedge_{i\in\Gamma} x_i) \ominus y \leq \bigwedge_{i\in\Gamma}(x_i \ominus y)$*

(4) *$x \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \ominus y_i)$.*

(5) *$x \ominus x = 0$, $x \ominus 0 = x$ and $0 \ominus x = 0$. Moreover, $x \ominus y = 0$ iff $x \leq y$.*

(6) *$y \oplus (x \ominus y) \geq x$, $y \geq x \ominus (x \ominus y)$ and $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.*

(7) *$x \ominus (y \oplus z) = (x \ominus y) \ominus z = (x \ominus z) \ominus y$.*

(8) *$x \ominus y \geq (x \oplus z) \ominus (y \oplus z)$, $y \ominus x \geq (z \ominus x) \ominus (z \ominus y)$ and $(x \oplus y) \ominus (z \oplus w) \leq (x \ominus z) \oplus (y \ominus w)$.*

(9) *$x \oplus y = 0$ iff $x = 0$ and $y = 0$.*

(10) *$(x \oplus y) \ominus z \leq x \oplus (y \ominus z)$ and $(x \ominus y) \oplus z \geq x \ominus (y \ominus z)$.*

(11) *If L satisfies a double negative law and $N(x) = 1 \ominus x$, then $N(x \oplus y) = N(x) \ominus y = N(y) \ominus x$ and $x \ominus y = N(y) \ominus N(x)$. Moreover, $N(\bigvee_{i\in\Gamma} x_i) = \bigwedge_{i\in\Gamma} N(x_i)$ and $N(\bigwedge_{i\in\Gamma} x_i) = \bigvee_{i\in\Gamma} N(x_i)$.*

Definition 2.4. [11] Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Let X be a set. A function $d_X : X \times X \to L$ is called a *distance function* if it satisfies the following conditions:

(M1) $d_X(x, x) = 0$ for all $x \in X$,

(M2) $d_X(x, y) \oplus d_X(y, z) \geq d_X(x, z)$, for all $x, y, z \in X$,

(M3) If $d_X(x, y) = d_X(y, x) = 0$, then $x = y$.

The pair (X, d_X) is called a *distance space*.

Remark 2.5. (1) We define a distance function $d_X : X \times X \to [0, \infty]$. Then (X, d_X) is called a pseudo-quasi-metric space.

(2) Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \to L$ as $d_L(x, y) = x \ominus y$. By Lemma 2.3 (5) and (8), (L, d_L) is a distance space. Moreover, we define a function $d_{L^X} : L^X \times L^X \to L$ as $d_{L^X}(A, B) = \bigvee_{x\in X}(A(x) \ominus B(x))$. Then (L^X, d_{L^X}) is a distance space.

(3) We define a function $d_{[0,\infty]^X} : [0,\infty]^X \times [0,\infty]^X \to [0,\infty]$ as $d_{[0,\infty]^X}(A,B) = \bigvee_{x\in X}(A(x) \ominus B(x)) = \bigvee_{x\in X}((B(x) - A(x)) \vee 0)$. Then $([0,\infty]^X, d_{[0,\infty]^X})$ is a pseudo-quasi-space.

(4) If (X, d_X) is a distance space and we define a function $d_X^{-1}(x,y) = d_X(y,x)$, then (X, d_X^{-1}) is a distance space.

(5) Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Let (X, d_X) be a distance space and define $(d_X \uplus d_X)(x,z) = \bigwedge_{y\in X}(d_X(x,y) \oplus d_X(y,z))$ for each $x, z \in X$. By (M2), $(d_X \uplus d_X)(x,z) \geq d_X(x,z)$ and $(d_X \uplus d_X)(x,z) \leq d_X(x,x) \oplus d_X(x,z) = d(x,z)$. Hence $(d_X \uplus d_X) = d_X$.

3. Approximation operators and fuzzy rough sets

Definition 3.1. A subset $\tau \subset L^X$ is called an *Alexandrov topology* on X iff it satisfies the following conditions:

(O1) $\alpha_X \in \tau$.

(O2) If $A_i \in \tau$ for all $i \in I$, then $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau$.

(O3) If $A \in \tau$ and $\alpha \in L$, then $A \ominus \alpha, \alpha \oplus A \in \tau$.

Definition 3.2. A map $\mathcal{J} : L^X \to L^X$ is called an *$\ominus$-upper approximation operator* if it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(J1) $\mathcal{J}(A \ominus \alpha) = \mathcal{J}(A) \ominus \alpha$,

(J2) $\mathcal{J}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{J}(A_i)$,

(J3) $\mathcal{J}(A) \geq A$ and $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$.

Definition 3.3. A map $\mathcal{H} : L^X \to L^X$ is called an *$\oplus$-lower approximation operator* if it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(H1) $\mathcal{H}(\alpha \oplus A) = \alpha \oplus \mathcal{H}(A)$,

(H2) $\mathcal{H}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{H}(A_i)$,

(H3) $\mathcal{H}(A) \leq A$ and $\mathcal{H}(\mathcal{H}(A)) = \mathcal{H}(A)$.

Let $\mathcal{H}$ (resp. $\mathcal{J}$) be $\oplus$-lower (resp. $\ominus$-upper) approximation operator on X. As a generalization of fuzzy rough set, the pair $(\mathcal{H}(A), \mathcal{J}(A))$ is called an *$(\oplus, \ominus)$-fuzzy rough set* for $A \in L^X$.

The map $\alpha : L^X \to L$ is an *fuzzy accuracy measure* defined, for $A \in L^X$

$$\alpha(A) = \bigvee_{x\in X} (\mathcal{J}(A)(x) \ominus \mathcal{H}(A)(x)).$$

Theorem 3.4. *Let $d_X \in L^{X\times X}$ be a distance function. Define $\mathcal{J}_{d_X}, \mathcal{H}_{d_X} : L^X \to L^X$ as follows*

$$\mathcal{J}_{d_X}(B)(x) = \bigvee_{y\in X}(B(y) \ominus d_X(x,y)),$$
$$\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_X(x,y)).$$

Then the followings hold.

(1) *$\mathcal{J}_{d_X}$ is an $\ominus$-upper approximation operator.*

(2) *$\mathcal{H}_{d_X}$ is an $\oplus$-lower approximation operator. Moreover, $(\mathcal{H}_{d_X}(A), \mathcal{J}_{d_X}(A))$ is an $(\oplus, \ominus)$-fuzzy rough set for $A \in L^X$.*

(3) *$\mathcal{J}_{d_X}(\alpha_X) = \alpha_X$, $\mathcal{J}_{d_X}(d_X(x,-)) = d_X(x,-)$ and $\alpha \oplus \mathcal{J}_{d_X}(A) \geq \mathcal{J}_{d_X}(\alpha \oplus A)$ for each $\alpha \in L, A \in L^X$ and $\mathcal{J}_{d_X}(A) \leq \mathcal{J}_{d_X}(B)$ for $A \leq B$.*

(4) $\mathcal{H}_{d_X}(\alpha_X) = \alpha_X$, $\mathcal{H}_{d_X}(d_X(x,-)) = d_X(x,-)$ *and* $\mathcal{H}_{d_X}(A) \ominus \alpha \le \mathcal{H}_{d_X}(A \ominus \alpha)$ *for each* $\alpha \in L, A \in L^X$ *and* $\mathcal{H}_{d_X}(A) \le \mathcal{H}_{d_X}(B)$ *for* $A \le B$.

(5) $\mathcal{H}_{d_X}(A) = \bigvee\{B \mid \mathcal{J}_{d_X}(B) \le A\}$ *and* $\mathcal{J}_{d_X}(\alpha \ominus A) = \alpha \ominus \mathcal{H}_{d_X^{-1}}(A)$*, for all* $A \in L^X$.

(6) $\mathcal{J}_{d_X}(B) = \bigwedge\{A \mid \mathcal{H}_{d_X}(A) \ge B\}$.

(7) *For each* $A, B \in L^X$, $\mathcal{H}_{d_X}(\mathcal{J}_{d_X}(B)) = \mathcal{J}_{d_X}(B)$ *and* $\mathcal{J}_{d_X}(\mathcal{H}_{d_X}(A)) = \mathcal{H}_{d_X}(A)$

(8) $\tau_{d_X} = \{A \in L^X \mid A(x) \oplus d_X(x,y) \ge A(y)\}$ *is an Alexandrov topology on* X *with* $d_X(x,-), (\alpha \ominus d_X(-,x)) \in \tau_{d_X}$. *Moreover,*

$$\begin{aligned}\tau_{d_X} &= \{\mathcal{H}_{d_X}(A) \mid A \in L^X\} = \{\textstyle\bigwedge_{y\in X}(A(y) \oplus d_X(y,-)) \mid A \in L^X\}\\ &= \{\mathcal{J}_{d_X}(A) \mid A \in L^X\} = \{\textstyle\bigvee_{y\in X}(A(y) \ominus d_X(-,y)) \mid A \in L^X\}.\end{aligned}$$

Proof. (1) (J1) For each $A \in L^X$ and $\alpha \in L$, by Lemma 2.3 (7),

$$\begin{aligned}&\mathcal{J}_{d_X}(A \ominus \alpha)(x) = \textstyle\bigvee_{y\in X}((A(y) \ominus \alpha) \ominus d_X(x,y))\\ &= \textstyle\bigvee_{y\in X}((A(y) \ominus d_X(x,y)) \ominus \alpha = \mathcal{J}_{d_X}(A)(x) \ominus \alpha.\end{aligned}$$

(J2) For each $A_i \in L^X$, by Lemma 2.3(2), $\mathcal{J}_{d_X}(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \mathcal{J}_{d_X}(A_i)$.

(J3) For each $A \in L^X$, $\mathcal{J}_{d_X}(A)(x) = \bigvee_{y\in X}(A(y) \ominus d_X(x,y)) \ge A(x) \ominus d_X(x,x) = A(x)$.

For each $A \in L^X$,

$$\begin{aligned}&\mathcal{J}_{d_X}(\mathcal{J}_{d_X}(A))(x) = \textstyle\bigvee_{y\in X}(\mathcal{J}_{d_X}(A)(y) \ominus d_X(x,y))\\ &= \textstyle\bigvee_{y\in X}(\bigvee_{z\in X}(A(z) \ominus d_X(y,z)) \ominus d_X(x,y))\\ &= \textstyle\bigvee_{y\in X}(\bigvee_{z\in X}((A(z) \ominus d_X(y,z)) \ominus d_X(x,y))) \text{ (by Lemma 2.3 (2))}\\ &= \textstyle\bigvee_{y,z\in X}(A(z) \ominus (d_X(y,z) \oplus d_X(x,y))) \text{ (by Lemma 2.3 (7))}\\ &= \textstyle\bigvee_{z\in X}(A(z) \ominus \bigwedge_{y\in X}(d_X(y,z) \oplus d_X(x,y))) \text{ (by Lemma 2.3 (2))}\\ &= \textstyle\bigvee_{z\in X}(A(z) \ominus d_X(x,z)) = \mathcal{J}_{d_X}(A)(x).\end{aligned}$$

Hence $\mathcal{J}_{d_X}$ is an $\ominus$-upper approximation operator.

(2) (H1) $\mathcal{H}_{d_X}(\alpha \oplus A)(y) = \bigwedge_{x\in X}((\alpha \oplus A)(x) \oplus d_X(x,y)) = \alpha \oplus \bigwedge_{x\in X}(A(x) \oplus d_X(x,y)) = \alpha \oplus \mathcal{H}_{d_X}(A)(y)$.

(H2) $\mathcal{H}_{d_X}(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{x\in X}(\bigwedge_{i\in\Gamma} A_i(x) \oplus d_X(x,y)) = \bigwedge_{i\in\Gamma}(\bigwedge_{x\in X}(A_i(x) \oplus d_X(x,y)) = \bigwedge_{i\in\Gamma} \mathcal{H}_{d_X}(A_i)(y)$.

(H3) $\mathcal{H}_{d_X}(A)(y) = \bigvee_{x\in X}(A(x) \oplus d_X(x,y)) \le A(y) \oplus d_X(y,y) = A(y)$.

For all $B \in L^X, z \in X$,

$$\begin{aligned}&\mathcal{H}_{d_X}(\mathcal{H}_{d_X}(A))(z) = \textstyle\bigwedge_{x\in X}(\mathcal{H}_{d_X}(A)(x) \oplus d_X(x,z))\\ &= \textstyle\bigwedge_{x\in X}(\bigwedge_{y\in X}(A(y) \oplus d_X(y,x)) \oplus d_X(x,z))\\ &= \textstyle\bigwedge_{y\in X}(A(y) \oplus \bigwedge_{x\in X}(d_X(y,x) \oplus d_X(x,z)))\\ &= \textstyle\bigwedge_{y\in X}(A(y) \oplus d_X(y,z)) = \mathcal{H}_{d_X}(A)(z).\end{aligned}$$

Hence $\mathcal{H}_{d_X}$ is an $\oplus$-lower approximation operator.

(3) Since $\mathcal{J}_{d_X}(\alpha_X)(x) = \bigvee_{y\in X}(\alpha_X(y) \ominus d_X(x,y)) \le \alpha$, by (J3), $\mathcal{J}_{d_X}(\alpha_X) = \alpha_X$. For each $x, z \in X$,

$$\mathcal{J}_{d_X}(d_X(x,-))(z) = \bigvee_{y\in X}(d_X(x,y) \ominus d_X(z,y)) = d_X(x,z).$$

For each $A \in L^X$ and $\alpha \in L$,

$$\begin{array}{l} \alpha \oplus \mathcal{J}_{d_X}(A)(x) = \alpha \oplus \bigvee_{y\in X}(A(y) \ominus d_X(x,y)) \\ \geq \bigvee_{y\in X}(\alpha \oplus (A(y) \ominus d_X(x,y))) \\ \geq \bigvee_{y\in X}((\alpha \oplus A)(y) \ominus d_X(x,y)) \text{ (by Lemma 2.3 (10))} \\ = \mathcal{J}_{d_X}(\alpha \oplus A)(x). \end{array}$$

For $A \leq B$, $\mathcal{J}_{d_X}(A) \leq \mathcal{J}_{d_X}(B)$.

(4) Since $\mathcal{H}_{d_X}(\alpha_X)(y) = \bigwedge_{x\in X}(\alpha_X(x) \oplus d_X(x,y)) \geq \alpha$, $\mathcal{H}_{d_X}(\alpha_X) = \alpha_X$. For $x, z \in X$,

$$\mathcal{H}_{d_X}(d_X(x,-))(z) = \bigwedge_{y\in X}(d_X(x,y) \oplus d_X(y,z)) = d_X(x,z).$$

For each $A \in L^X$ and $\alpha \in L$,

$$\begin{array}{l} \mathcal{H}_{d_X}(A \ominus \alpha)(z) = \bigwedge_{x\in X}((A \ominus \alpha)(x) \oplus d_X(x,z)) \\ \geq \bigwedge_{x\in X}((A(x) \oplus d_X(x,z)) \ominus \alpha) \\ \geq \bigwedge_{x\in X}(A(x) \oplus d_X(x,z)) \ominus \alpha \\ = \mathcal{H}_{d_X}(A)(z) \ominus \alpha. \end{array}$$

(5) By (J2), for each $A \in L^X$,

$$\begin{array}{l} \bigvee\{B(y) \mid \mathcal{J}_{d_X}(B)(x) \leq A(x)\} = \bigvee\{B(y) \mid \bigvee_{y\in Y}(B(y) \ominus d_X(x,y)) \leq A(x)\} \\ = \bigwedge_{x\in X}(d_X(x,y) \oplus A(x)) = \mathcal{H}_{d_X}(A)(y). \end{array}$$

For all $B \in L^X, x \in X$,

$$\begin{array}{l} \mathcal{J}_{d_X}(\alpha \ominus B)(x) = \bigvee_{y\in X}((\alpha \ominus B(y)) \ominus d_X(x,y)) \\ = \bigvee_{y\in X}((\alpha \ominus (B(y) \oplus d_X(x,y)) \text{ (by Lemma 2.3 (7))} \\ = \alpha \ominus \bigvee_{y\in X}((B(y) \oplus d_X(x,y)) \\ = \alpha \ominus \mathcal{H}_{d_X^{-1}}(B)(x) \end{array}$$

(6) By (H2), for each $B \in L^X$,

$$\begin{array}{l} \bigwedge\{A(x) \mid \mathcal{H}_{d_X}(A)(y) \geq B(y)\} \\ = \bigwedge\{A(x) \mid \bigwedge_{x\in X}(A(x) \oplus d_X(x,y)) \geq B(y)\} \\ = \bigvee_{y\in Y}(B(y) \ominus d_X(x,y)) = \mathcal{J}_{d_X}(B)(x). \end{array}$$

(7) For each $B \in L^X$,

$$\begin{array}{l} \mathcal{H}_{d_X}(\mathcal{J}_{d_X}(B))(z) = \bigwedge_{x\in X}(\mathcal{J}_{d_X}(B)(x) \oplus d_X(x,z)) \\ = \bigwedge_{x\in X}(\bigvee_{y\in X}(B(y) \ominus d_X(x,y)) \oplus d_X(x,z)) \\ \geq \bigwedge_{x\in X}\bigvee_{y\in X}((B(y) \ominus d_X(x,y)) \oplus d_X(x,z)) \\ \geq \bigwedge_{x\in X}\bigvee_{y\in X}(B(y) \ominus (d_X(x,y) \ominus d_X(x,z))) \text{ (by Lemma 2.3 (10))} \\ \geq \bigvee_{y\in X}(B(y) \ominus \bigvee_{x\in X}(d_X(x,y) \ominus d_X(x,z))) \\ \geq \bigvee_{y\in X}(B(y) \ominus d_X(z,y)) = \mathcal{J}_{d_X}(B)(z), \end{array}$$

$$\begin{array}{l} \mathcal{J}_{d_X}(\mathcal{H}_{d_X}(B))(x) = \bigvee_{y\in X}(\mathcal{H}_{d_X}(B)(y) \ominus d_X(x,y)) \\ = \bigvee_{y\in X}(\bigwedge_{z\in X}(B(z) \oplus d_X(z,y)) \ominus d_X(x,y)) \\ \leq \bigvee_{y\in X}\bigwedge_{z\in X}((B(z) \oplus d_X(z,y)) \ominus d_X(x,y)) \\ \leq \bigvee_{y\in X}\bigwedge_{z\in X}(B(z) \oplus (d_X(z,y) \ominus d_X(x,y)) \text{ (by Lemma 2.3 (10))} \\ \leq \bigwedge_{z\in X}(B(z) \oplus \bigvee_{y\in X}(d_X(z,y) \ominus d_X(x,y)) \\ = \bigwedge_{z\in X}(B(z) \oplus d_X(x,z)) = \mathcal{H}_{d_X}(B)(x). \end{array}$$

(8) (O1) Since $\alpha_X(x) \oplus d_X(x,y) \geq \alpha_X(y)$, $\alpha_X \in \tau_{d_X}$.

(O2) If $A_i \in \tau_{d_X}$ for all $i \in I$, $\bigvee_{i\in I} A_i(x) \oplus d_X(x,y) \geq \bigvee_{i\in I}(A_i(x) \oplus d_X(x,y)) \geq \bigvee_{i\in I} A_i(y)$ and $\bigwedge_{i\in I} A_i(x) \oplus d_X(x,y) = \bigwedge_{i\in I}(A_i(x) \oplus d_X(x,y)) \geq \bigwedge_{i\in I} A_i(y)$. Hence $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau_{d_X}$.

(O3) If $A \in \tau_{d_X}$ and $\alpha \in L$, then $d_X(x,y) \oplus (A(x) \ominus \alpha) \oplus \alpha \geq d_X(x,y) \oplus A(x) \geq A(y)$. Thus $d_X(x,y) \oplus (A(x) \ominus \alpha) \geq A(y) \ominus \alpha$. So $A \ominus \alpha \in \tau_{d_X}$. Easily, $\alpha \oplus A \in \tau_{d_X}$.

Since $d_X(x,-)(y) \oplus d_X(y,z) \geq d_X(x,-)(z)$, $d_X(x,-) \in \tau_{d_X}$. Since

$$\begin{array}{l}(\alpha \ominus d_X(-,x))(y) \oplus d_X(y,z) \oplus d_X(z,x) \\ \geq (\alpha \ominus d_X(-,x))(y) \oplus d_X(y,x) \geq \alpha,\end{array}$$

$(\alpha \ominus d_X(-,x))(y) \oplus d_X(y,z) \geq \alpha \ominus d_X(z,x)$, that is, $(\alpha \ominus d_X(-,x)) \in \tau_{d_X}$.

For $A \in \tau_{d_X}$, $A = \bigwedge_{x\in X}(A(x) \oplus d_X(x,-)) = \mathcal{H}_{d_X}(A) \in \tau_{d_X}$ and $A = \bigvee_{x\in X}(A(x) \ominus d_X(-,x)) = \mathcal{J}_{d_X}(A) \in \tau_{d_X}$.

□

THEOREM 3.5. (1) *Let $\mathcal{H} : L^X \to L^X$ be an $\oplus$-lower approximation operator iff there exist a distance function $d_{\mathcal{H}}$ on X such that*

$$\mathcal{H}(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_{\mathcal{H}}(x,y)).$$

(2) *If L satisfies a double negative law, then $\mathcal{J} : L^X \to L^X$ be an $\ominus$-upper approximation operator iff there exist a distance function $d_{\mathcal{J}}$ on X such that*

$$\mathcal{J}(B)(x) = \bigvee_{y\in X}(B(y) \ominus d_{\mathcal{J}}(x,y)).$$

Proof. (1) ($\Rightarrow$) Put $d_{\mathcal{H}} : X \times X \to L$ as $d_{\mathcal{H}}(x,y) = \mathcal{H}(0_x)(y)$ where $0_x(x) = 0$ and $0_x(y) = 1$ for $x \neq y \in X$.

(M1) $d_{\mathcal{H}}(x,x) = \mathcal{H}(0_x)(x) \leq 0_x(x) = 0$.

(M2) Since $A = \bigwedge_{y\in X}(A(y) \oplus 0_y)$ and $\mathcal{H}(0_x) = \bigwedge_{y\in X}(\mathcal{H}(0_x)(y) \oplus 0_y)$,

$$\begin{array}{l}\bigwedge_{y\in X}(d_{\mathcal{H}}(x,y) \oplus d_{\mathcal{H}}(y,z)) \\ = \bigwedge_{y\in X}(\mathcal{H}(0_x)(y) \oplus \mathcal{H}(0_y)(z)) \text{ (by (H2))} \\ = \mathcal{H}(\bigwedge_{y\in X}(\mathcal{H}(0_x)(y) \oplus 0_y)(z)) = \mathcal{H}(\mathcal{H}(0_x))(z) \\ = \mathcal{H}(0_x)(z) = d_{\mathcal{H}}(x,z).\end{array}$$

Hence $d_{\mathcal{H}}$ is a distance function. Moreover,

$$\begin{array}{l}\mathcal{H}(A)(y) = \mathcal{H}(\bigwedge_{x\in X}(A(x) \oplus 0_x))(y) \\ = \bigwedge_{x\in X}(A(x) \oplus \mathcal{H}(0_x)(y)) \\ = \bigwedge_{x\in X}(A(x) \oplus d_{\mathcal{H}}(x,y))).\end{array}$$

($\Leftarrow$) It follow from Theorem 3.4(2).

(2) ($\Rightarrow$) Put $d_{\mathcal{J}} : X \times X \to L$ as $d_{\mathcal{J}}(x,y) = N(\mathcal{J}(N(0_y))(x)$ where $0_x(x) = 0$ and $0_x(y) = 1$ for $x \neq y \in X$.

(M1) Since $\mathcal{J}(N(0_x) \geq N(0_x$, $d_{\mathcal{J}}(x,x) = N(\mathcal{J}(N(0_x))(x) \leq N(N(0_x)(x) = 0$.

(M2) Since $A = \bigwedge_{y\in X}(A(y)\oplus 0_y)$, $N(A) = \bigwedge_{y\in X}(N(A)(y)\oplus 0_y)$, by Lemma 2.3(11), $A = \bigvee_{y\in X}(N(0_y)\ominus N(A)(y))$,

$$\begin{array}{l} N(\bigwedge_{y\in X}(d_{\mathcal{J}}(x,y)\oplus d_{\mathcal{J}}(y,z))) \\ = \bigvee_{y\in X}(N(N\mathcal{J}(N(0_y))(x)\oplus N\mathcal{J}(N(0_z))(y)))\ (\text{ by (H2)}) \\ = \bigvee_{y\in X}(\mathcal{J}(N(0_y))(x)\ominus N(\mathcal{J}(N(0_z))(y)))\ (\text{ by (H2)}) \\ = \mathcal{J}(\bigvee_{y\in X}((N(0_y))(x)\ominus N(\mathcal{J}(N(0_z))(y))))\ (\text{ by (H2)}) \\ = \mathcal{J}(\mathcal{J}(N(0_z)))(x) = \mathcal{J}(N(0_z))(x) \\ = N(d_{\mathcal{J}}(x,z)). \end{array}$$

Hence $d_{\mathcal{J}}$ is a distance function. Moreover,

$$\begin{array}{l} \mathcal{J}(B)(x) = \mathcal{J}(\bigvee_{y\in Y}(N(0_y)(x)\ominus N(B)(y)) \\ = \bigvee_{y\in Y}(\mathcal{J}(N(0_y))(x)\ominus N(B)(y)) \\ = \bigvee_{y\in Y}(B(y)\ominus N(\mathcal{J}(N(0_y)))(x)) \\ = \bigvee_{y\in Y}(B(y)\ominus d_{\mathcal{J}}(x,y))). \end{array}$$

($\Leftarrow$) It follow from Theorem 3.4(1).

□

Example 3.6. Let $X = \{x, y, z\}$ be a set and $(L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice with

$$x\oplus y = 1\wedge(x+y),\ x\ominus y = (x-y)\vee 0.$$

Define $d^1_X, d_X : X\times X\to L$ as

$$d^1_X = \begin{pmatrix} 0 & 1 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{2} & 0 \end{pmatrix}, d_X = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{2} & 0 \end{pmatrix}$$

Since $d^1_X(x,z)\oplus d^1_X(z,y) = \frac{1}{4}+\frac{1}{2}\not\geq d^1_X(x,y) = 1$, d^1_X is not a distance function. Since $d_X \uplus d_X = d_X$ from Remark 2.5(5), d_X is a distance function.

By Theorem 3.4(8), we obtain an Alexandrov topology $\tau_{d_X} = \{\mathcal{H}_{d_X}(C) \mid C\in L^X\} = \{\mathcal{J}_{d_X}(D) \mid D\in L^X\}$ where

$$\begin{array}{l} \mathcal{H}_{d_{\tau_X}}(C) = \bigwedge_{x\in X}(C(x)\oplus d_{\tau_X}(x,-)) \\ = \begin{pmatrix} C(x)\wedge(C(y)+\frac{1}{2})\wedge(C(z)+\frac{3}{4}) \\ (C(x)+\frac{3}{4})\wedge C(y)\wedge(C(z)+\frac{1}{2}) \\ (C(x)+\frac{1}{4})\wedge(C(y)+\frac{1}{4})\wedge C(z) \end{pmatrix} \end{array}$$

$$\begin{array}{l} \mathcal{J}_{d_{\tau_X}}(D) = \bigvee_{x\in X}(D(x)\ominus d_{\tau_X}(-,x)) \\ = \begin{pmatrix} D(x)\vee(D(y)-\frac{3}{4})\vee(D(z)-\frac{1}{4}) \\ (D(x)-\frac{1}{2})\vee D(y)\vee(D(z)-\frac{1}{4}) \\ (D(x)-\frac{3}{4})\vee(D(y)-\frac{1}{2})\vee D(z) \end{pmatrix} \end{array}$$

The pair $(\mathcal{H}_{d_X}(A), \mathcal{J}_{d_X}(A))$ is an $(\oplus,\ominus)$-fuzzy rough set for $A\in L^X$.

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