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ON FUZZY BI-IDEALS IN SEMIGROUPS

Inheung Chon

Abstract. We characterize the fuzzy bi-ideal generated by a fuzzy subset in a semigroup and the fuzzy bi-ideal generated by a fuzzy subset A such that $A \subseteq A^2$ in a semigroup with an identity element. Our work generalizes the characterization of fuzzy bi-ideals by Mo and Wang ([8]).

1. Introduction

Zadeh ([13]) introduced the concept of a fuzzy set for the first time and this concept was applied by Rosenfeld ([9]) to define fuzzy subgroups and fuzzy ideals. Based on this crucial work, Kuroki ([2, 3, 4, 5, 6]) defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. On the other hand, Mo and Wang ([8]) defined some kinds of fuzzy ideals generated by fuzzy subsets in a semigroup with an identity element and Xie ([12]) reproved the results of Mo and Wang using the level subsets. However the fuzzy bi-ideal generated by a fuzzy subset in a semigroup has not yet been defined and studied. In this note we are able to define the fuzzy bi-ideal generated by a fuzzy subset in a semigroup and obtain the same results, as special cases of our main results, that Mo and Wang ([8]) found in a semigroup with an identity element or a regular semigroup.

In section 2 we give some definitions and propositions which will be used in the next section . In section 3 we define the fuzzy bi-ideal generated by a fuzzy subset in a semigroup, define the fuzzy bi-ideal generated by a fuzzy subset A such that $A \subseteq A^2$ in a semigroup with an identity element, and find, as special cases, the fuzzy bi-ideal generated

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by a fuzzy subset A in a semigroup S with an identity element e such that $A(e) \geq A(x)$ for all $x \in S$ and the fuzzy bi-ideal generated by a fuzzy subset in a regular semigroup.

2. Preliminaries

In this section, we give some definitions and propositions which will be used in section 3.

Definition 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy subset* in X. For every $x \in X$, $B(x)$ is called the *membership grade* of x in B. A fuzzy subset in X is called a *fuzzy point* iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$, we denote this fuzzy point by x_α, where the point x is called its *support*. The fuzzy point x_α is said to be contained in a fuzzy subset A, denoted by $x_\alpha \in A$, iff $\alpha \leq A(x)$.

remark. *The crisp set S itself is a fuzzy subset of S such that $S(x) = 1$ for all $x \in S$ (see Lemma 2.4 of [5] or [11]).*

Definition 2.2. A *triangular norm* (briefly *t-norm*) is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying, for each p, q, r, s in [0,1],

(1) $T(p, 1) = p$
(2) $T(p, q) \leq T(r, s)$ if $p \leq r$ and $q \leq s$
(3) $T(p, q) = T(q, p)$
(4) $T(p, T(q, r)) = T(T(p, q), r)$

Definition 2.3. A t-norm $T : [0,1] \times [0,1] \to [0,1]$ is continuous if T is continuous with respect to the usual topologies.

It is well known ([1]) that the function $T_m : [0,1] \times [0,1] \to [0,1]$ defined by $T_m(a, b) = \min(a, b)$, the function $T_p : [0,1] \times [0,1] \to [0,1]$ defined by $T_p(a, b) = ab$, and the fuction $T_M : [0,1] \times [0,1] \to [0,1]$ defined by $T_M(a, b) = \max(a + b - 1, 0)$ are continuous t-norms.

For fuzzy sets U, V in a set X, Liu ([7]) defined $U \circ V$ by

$$(U \circ V)(x) = \begin{cases} \sup\limits_{ab=x} \min(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

Sessa ([10]) generalized this definition by replacing the minimum operation with a t-norm as follows.

Definition 2.4. Let X be a set and let U, V be two fuzzy sets in X. $U \circ V$ is defined by

$$(U \circ V)(x) = \begin{cases} \sup\limits_{ab=x} \; T(U(a), V(b)) & \text{if } \; ab = x \\ 0 & \text{if } \; ab \neq x. \end{cases}$$

We write UV for $U \circ V$ throughout this paper.

Proposition 2.5. *Let $A_1, A_2, \ldots, A_n$ be fuzzy subsets of a set S. Then*

(1) $(A_1 \cup A_2 \cup \cdots \cup A_n)S \subseteq A_1S \cup A_2S \cup \cdots \cup A_nS$.
(2) $S(A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq SA_1 \cup SA_2 \cup \cdots \cup SA_n$.

Proof. (1) Since $S(b) = 1$,

$$\begin{aligned} &[(A_1 \cup A_2 \cup \cdots \cup A_n)S](x) \\ &= \sup_{ab=x} \; T((A_1 \cup A_2 \cup \cdots \cup A_n)(a), S(b)) \\ &= \sup_{ab=x} \max[A_1(a), A_2(a), \ldots, A_n(a)]. \end{aligned}$$

Since $S(b) = 1$,

$$\begin{aligned} &(A_1S \cup A_2S \cup \cdots \cup A_nS)(x) \\ &= \max[\sup_{ab=x} T(A_1(a), S(b)), \ldots, \sup_{ab=x} T(A_n(a), S(b))] \\ &= \max[\sup_{ab=x} A_1(a), \sup_{ab=x} A_2(a), \ldots, \sup_{ab=x} \; A_n(a)]. \end{aligned}$$

Thus $(A_1 \cup A_2 \cup \cdots \cup A_n)S \subseteq A_1S \cup A_2S \cup \cdots \cup A_nS$.
(2) Similarly, we may prove $S(A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq SA_1 \cup SA_2 \cup \cdots \cup SA_n$.

□

Liu ([7]) proved Proposition 2.6, Proposition 2.7, and Proposition 2.8 for the case that the t-norm is a minimum function.

Proposition 2.6. *Let A, B be fuzzy sets in a set X and let x_p, y_q be fuzzy points in X. Then*

(1) $x_p y_q = (xy)_{T(p,q)}$.
(2) $AB = \underset{x_p \in A, y_q \in B}{\cup} x_p y_q$, *where* $(x_p y_q)(z) = \sup\limits_{cd=z} T(x_p(c), y_q(d))$.

Proof. The proof is similar to that of Proposition 1.1 of [7]. □

Proposition 2.7. *Let A be a fuzzy set of a groupoid X. Then the followings are equivalent.*

(1) *A is a fuzzy groupoid, that is,* $A(xy) \geq T(A(x), A(y))$.
(2) *For any* $x_p, y_q \in A$, $x_p y_q \in A$.
(3) $AA \subseteq A$.

Proof. Straightforward from Proposition 2.6. □

Proposition 2.8. *Let A, B, and C be fuzzy sets in a semigroup X and let T be a continuous t-norm. Then* $(AB)C = A(BC)$.

Proof. Straightforward. □

From now on, we assume that every t-norm in this paper is continuous.

3. Fuzzy bi-ideals generated by fuzzy subsets in semigroups

In this section, we define the fuzzy bi-ideal generated by a fuzzy subset in a semigroup and the fuzzy bi-ideal generated by a fuzzy subset A in a semigroup with an identity element such that $A \subseteq AA$. Also we find, as special cases, the fuzzy bi-ideal generated by a fuzzy subset A in a semigroup with an identity element e such that $A(e) \geq A(x)$ for all $x \in S$ and the fuzzy bi-ideal generated by a fuzzy subset in a regular semigroup, which were found originally by Mo and Wang ([8]).

Definition 3.1. Let S be a semigroup. The fuzzy subset H in S is a *fuzzy subsemigroup* of S if $H(xy) \geq T(H(x), H(y))$ for all $x, y \in S$. A fuzzy subsemigroup B of S is called a *fuzzy bi-ideal* of S if $B(xyz) \geq T(B(x), B(z))$.

Definition 3.2. Let S be a semigroup. S is called a *regular semigroup* if for every $x \in S$, there exists $a \in S$ such that $xax = x$.

Theorem 3.3. *Let A be a fuzzy subset in a semigroup S. Then the fuzzy bi-ideal F generated by A is $A \cup A^2 \cup ASA$. That is,* $F(x) = \max[A(x), \sup\limits_{ab=x} T(A(a), A(b)),$
$\sup\limits_{cdb=x} T(A(c), A(b))]$.

Proof. Let $\{J_i : i \in I\}$ be the collection of all fuzzy bi-ideals of S containing A. Since T is a continuous and increasing function,

$$\begin{aligned}(J_iSJ_i)(x) &= \sup_{ab=x} T(J_iS(a), J_i(b)) = \sup_{ab=x} T(\sup_{cd=a} T(J_i(c), S(d)), J_i(b)) \\ &= \sup_{ab=x} T(\sup_{cd=a} J_i(c), J_i(b)) = \sup_{ab=x} \sup_{cd=a} T(J_i(c), J_i(b)) \\ &= \sup_{cdb=x} T(J_i(c), J_i(b)) \leq \sup_{cdb=x} J_i(cdb) = J_i(x)\end{aligned}$$

for each $i \in I$. Thus $ASA \subseteq J_iSJ_i \subseteq J_i$ for each $i \in I$. Since each J_i is a fuzzy semigroup, $(J_iJ_i)(x) = \sup\limits_{ab=x} T(J_i(a), J_i(b)) \leq \sup\limits_{ab=x} J_i(ab) = J_i(x)$. That is, $A^2 \subseteq J_iJ_i \subseteq J_i$ for each $i \in I$. Hence $A \cup A^2 \cup ASA \subseteq \bigcap\limits_{i \in I} J_i$.
By Proposition 2.5,

$$(A \cup A^2 \cup ASA)S(A \cup A^2 \cup ASA) \subseteq (AS \cup A^2S \cup ASAS)(A \cup A^2 \cup ASA).$$

Since $A \subseteq S$ and S is a semigroup,

$$\begin{aligned}&(AS \cup A^2S \cup ASAS)(A \cup A^2 \cup ASA) \\ &\subseteq (AS \cup AS^2 \cup AS^3)(A \cup A^2 \cup ASA) \\ &\subseteq AS(A \cup A^2 \cup ASA) \\ &\subseteq ASA \cup ASA^2 \cup ASASA \\ &\subseteq ASA \cup AS^2A \cup AS^3A = ASA.\end{aligned}$$

Thus $(A \cup A^2 \cup ASA)S(A \cup A^2 \cup ASA) \subseteq A \cup A^2 \cup ASA$. Let $H = A \cup A^2 \cup ASA$. Then $HSH \subseteq H$. Since T is a continuous and increasing function,

$$\begin{aligned} H(xyz) &\geq (HSH)(xyz) = \sup_{ab=xyz} T[HS(a),\ H(b)] \\ &= \sup_{ab=xyz} T[\sup_{cd=a} T(H(c), S(d)), H(b)] = \sup_{ab=xyz} T[\sup_{cd=a} H(c), H(b)] \\ &= \sup_{ab=xyz} \sup_{cd=a} T(H(c), H(b)) = \sup_{cdb=xyz} T(H(c), H(b)) \\ &\geq T(H(x), H(z)). \end{aligned}$$

Since $A \subseteq S$ and S is a semigroup,

$$\begin{aligned} &(A \cup A^2 \cup ASA)(A \cup A^2 \cup ASA) \\ &\subseteq (A \cup AS \cup ASS)(A \cup SA \cup SSA) \\ &\subseteq (A \cup AS)(A \cup SA) \\ &\subseteq A^2 \cup ASA \cup ASA \cup ASSA \\ &\subseteq A \cup A^2 \cup ASA. \end{aligned}$$

By Proposition 2.7, $A \cup A^2 \cup ASA$ is a fuzzy subsemigroup. Thus $H = A \cup A^2 \cup ASA$ is a fuzzy bi-ideal of S containing A. Hence $A \cup A^2 \cup ASA = \bigcap_{i \in I} J_i$.

Since T is continuous and increasing,

$$\begin{aligned} (ASA)(x) &= \sup_{ab=x} T(AS(a), A(b)) = \sup_{ab=x} T(\sup_{cd=a} T(A(c), S(d)), A(b)) \\ &= \sup_{ab=x} T(\sup_{cd=a} A(c), A(b)) = \sup_{ab=x} \sup_{cd=a} T(A(c), A(b)) \\ &= \sup_{cdb=x} T(A(c), A(b)). \end{aligned}$$

Thus $F(x) = \max[A(x), \sup_{ab=x} T(A(a), A(b)), \sup_{cdb=x} T(A(c),\ A(b))]$. □

Example of Theorem 3.3. Let $S = \{a, b, c, d, f\}$. We define a binary operation on S by means of the following table.

$*$	a	b	c	d	f
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
f	d	f	f	a	a

It is straightforward to see that S is a noncommutative semigroup. Let A be a fuzzy set in S such that

$$A(a) = 0.3,\ A(b) = 0.1,\ A(c) = 0.5,\ A(d) = 0.9,\ A(f) = 0.7.$$

Since $A^2(b) = \sup_{xy=b} T(A(x), A(y))$,

$$A^2(b) = \max\ [\ T(A(b), A(b)),\ T(A(c), A(c))\ ] = T(0.5, 0.5).$$

Similarly we may show that

$$A^2(a) = T(0.9, 0.9),\ A^2(c) = T(0.1, 0.5),$$
$$A^2(d) = T(0.5, 0.9),\ A^2(f) = T(0.7, 0.5).$$

Since $(SA)(b) = \sup_{xy=b} T(S(x), A(y)) = \max\ [A(b), A(c)] = 0.5$ and $SA(c) = A(c) = 0.5$,

$$\begin{aligned}(ASA)(b) &= \sup_{xy=b} T(A(x), SA(y)) \\ &= \max\ [T(A(b), SA(b)),\ T(A(c), SA(c))] \\ &= \max\ [T(0.1, 0.5),\ T(0.5, 0.5)] = T(0.5, 0.5).\end{aligned}$$

Similarly we may show that

$$(ASA)(a) = (ASA)(d) = T(0.9, 0.9),\ ASA(c) = T(0.5, 0.5),$$
$$ASA(f) = T(0.7, 0.5).$$

Let $H = A \cup A^2 \cup ASA$. Then

$$H(a) = \max\ [0.3,\ T(0.9, 0.9)],\ H(b) = \max\ [0.1,\ T(0.5, 0.5)],$$
$$H(c) = 0.5,\ H(d) = 0.9,\ H(f) = 0.7.$$

It is easily checked that $H(\alpha\beta) \geq T(H(\alpha),\ H(\beta))$ and $H(\alpha\beta\gamma) \geq T(H(\alpha), H(\gamma))$ for every $\alpha, \beta, \gamma \in S$. That is, H is a fuzzy bi-ideal in S. Let I be a fuzzy bi-ideal containing A. Since $I(a) = I(dd) \geq T(I(d), I(d)) \geq T(A(d), A(d)) = T(0.9, 0.9)$ and $I(a) \geq A(a) = 0.3$, $I(a) \geq H(a)$. Since $I(b) = I(cc) \geq T(I(c), I(c)) \geq T(A(c), A(c)) = T(0.5, 0.5)$ and $I(b) \geq A(b) = 0.1$, $I(b) \geq H(b)$. $I(c) \geq A(c) = 0.5 = H(c)$, $I(d) \geq A(d) = 0.9 = H(d)$, and $I(f) \geq A(f) = 0.9 = H(f)$. That is, $H \subseteq I$. Thus $H = A \cup A^2 \cup ASA$ is the fuzzy bi-ideal generated by A.

Theorem 3.4. *Let A be a fuzzy subset in a semigroup S with an identity element e such that $A \subseteq A^2$ and let a t-norm T be a minimum operation. Then the fuzzy bi-ideal F generated by A is ASA. That is, $F(x) = ASA(x) = \sup_{cdb=x} \min(A(c), A(b))$.*

Proof. Let $\{J_i : i \in I\}$ be the collection of all fuzzy bi-ideals of S containing A. Then $ASA \subseteq \bigcap_{i\in I} J_i$ from the proof of Theorem 3.3. We may show that $(ASA)S(ASA) \subseteq ASA$ by the same way as shown in Theorem 3.3. Let $H = ASA$. Then we may show that $H(xyz) \geq \min(H(x), H(z))$ by the same way as shown in Theorem 3.3. Since S is a semigroup and $A \subseteq S$,

$$(ASA)(ASA) = A(SAAS)A \subseteq A(SSSS)A \subseteq ASA.$$

By Proposition 2.7, ASA is a fuzzy subsemigroup. Thus ASA is a fuzzy bi-ideal of S. Also

$$\begin{aligned} ASA(x) &= \sup_{ab=x} \min(AS(a), A(b)) \\ &= \sup_{ab=x} \min(\sup_{cd=a} \min(A(c), S(d)), A(b)) \\ &\geq \sup_{ab=x} \min(\min(A(a), S(e)), A(b)) \\ &= \sup_{ab=x} \min(A(a), A(b)) = AA(x). \end{aligned}$$

Since $A \subseteq AA$, $A \subseteq ASA$. Thus ASA is a fuzzy bi-ideal of S containing A. Hence $ASA = \bigcap_{i\in I} J_i$. □

Mo and Wang showed that the fuzzy bi-ideal F generated by a fuzzy subsemigroup A such that $A(e) \geq A(x)$ for all $x \in S$ in a semigroup S with an identity element e is represented as $F(x) = \sup_{cdb=x} \min(A(c), A(b))$ (see Theorem 5.3 of [8]). Our Corollary 3.5 seems to be somewhat stronger than Mo and Wang's.

Corollary 3.5. *Let A be a fuzzy subset in a semigroup S with an identity element e such that $A(e) \geq A(x)$ for all $x \in S$ and let a t-norm T be a minimum operation. Then the fuzzy bi-ideal F generated by A is ASA. That is, $F(x) = \sup_{cdb=x} \min(A(c), A(b))$.*

Proof. Since $A(e) \geq A(x)$,

$$\begin{aligned} AA(x) &= \sup_{ab=x} \min(A(a), A(b)) \\ &\geq \min(A(x), A(e)) = A(x) \end{aligned}.$$

That is, $A \subseteq AA$. By Theorem 3.4, the fuzzy ideal generated by A is ASA. □

Corollary 3.6 is due to Mo and Wang (see Theorem 5.4 of [8]). Also Xie showed it again (see Theorem 4.2 of [12]). We obtain it as a special case of our more general approach.

Corollary 3.6. *Let A be a fuzzy subset in a regular semigroup S and let a t-norm T be a minimum operation. Then the fuzzy bi-ideal F generated by A is ASA. That is, $F(x) = \sup_{cdb=x} \min(A(c), A(b))$.*

Proof. Since S is a regular semigroup,

$$\begin{aligned} ASA(x) &= \sup_{ab=x} \min(AS(a), A(b)) \\ &= \sup_{ab=x} \min\,[\sup_{cd=a} \min(A(c), S(d)), A(b)] \\ &= \sup_{ab=x} \min(\sup_{cd=a} (A(c), A(b)) \\ &= \sup_{cdb=x} \min(A(c), A(b)) \\ &\geq \min(A(x), A(x)) = A(x). \end{aligned}$$

That is, $A \subseteq ASA$. We may show that ASA is the smallest fuzzy bi-ideal of S containing A by the same way as shown in Theorem 3.4. From the proof of Theorem 3.3, $ASA(x) = \sup_{cdb=x} \min(A(c), A(b))$. □

References

[1] J.M. Anthony and H. Sherwood, *Fuzzy groups redefined*, J. Math. Anal. Appl. **69** (1979), 124–130.

[2] N. Kuroki, *Fuzzy bi-ideals in semigroups*, Comment. Math. Univ. St. Pauli **28** (1979), 17–21.

[3] N. Kuroki, *On fuzzy ideals and bi-ideals in semigroups*, Fuzzy Sets and Systems **5** (1981), 203–215.
[4] N. Kuroki, *Fuzzy semiprime ideals in semigroups*, Fuzzy Sets and Systems **8** (1982), 71–79.
[5] N. Kuroki, *On fuzzy semigroups*, Inform. Sci. **53** (1991), 203–236.
[6] N. Kuroki, *Fuzzy semiprime quasi-ideals in semigroups*, Inform. Sci. **75** (1993), 201–211.
[7] W.J. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets and Systems **8** (1982), 133–139.
[8] Z.-W. Mo and X.-P. Wang, *Fuzzy ideals generated by fuzzy sets in semigroups*, Inform. Sci. **86** (1995), 203–210.
[9] A. Rosenfeld, *Fuzzy Groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
[10] S. Sessa, *On fuzzy subgroups and fuzzy ideals under triangular norms*, Fuzzy Sets and Systems **13** (1984), 95–100.
[11] X.-P. Wang, *Fuzzy regular subsemigroups in semigroups*, Inform. Sci. **68** (1993), 225–231.
[12] X.-Y. Xie, *Fuzzy ideals in semigroups*, J. Fuzzy Math. **7** (1999), 357–365.
[13] L.A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338–353.

Department of Mathematics
Seoul Women's University
126 Kongnung 2-Dong, Nowon-Gu
Seoul 139-774, South Korea