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# CHARACTER ANALOGUES OF INFINITE SERIES IDENTITIES RELATED TO GENERALIZED NON-HOLOMORPHIC EISENSTEIN SERIES

Sung Geun Lim

Abstract. In this paper, we derive analogues of a couple of classes of infinite series identities with the confluent hypergeometric functions involving Dirichlet characters.

## 1. Introduction

In [3], the author found character analogues of infinite series identities which originally come from modular transformation formula for generalized Eisenstein series. One of them shows the following symmetric identity( [3], Corollary 3.4);
Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and let $\left(\frac{\cdot}{p}\right)$ be the Legendre symbol modulo $p$, where $p$ is a prime with $p \equiv 1 \pmod 4$. Then, for any integer $M > 0$,

$$\alpha^{2M} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{4M-1}(\left(\frac{\cdot}{p}\right), n) \cos(2\pi n/p) e^{-2\alpha n/p}$$
$$= \beta^{2M} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{4M-1}(\left(\frac{\cdot}{p}\right), n) \cos(2\pi n/p) e^{-2\beta n/p},$$

where

$$\sigma_s(\left(\frac{\cdot}{p}\right), n) = \sum_{d|n} \left(\frac{d}{p}\right) d^s.$$

The study to find this type of character analogues was motivated by the works of B. C. Berndt, A. Dixit and J. Sohn in [2]. For example, a character analogue of Guinand's formula shows the following elegant symmetric identity (Corollary 3.2 in [2]);

$$\sqrt{\alpha} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(n) e^{-2n\alpha/p} = \sqrt{\beta} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(n) e^{-2n\beta/p},$$

where $\sigma_s(n)$ is the sum of the $s$-th powers of the positive divisors of $n$.
In this paper, we establish character analogues of certain classes of infinite series identities which stem from a modular transformation formula for a class of functions related to generalized non-holomorphic Eisenstein series. We start with introducing

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necessary notations and then shall state the principal theorem which shows a modular transformation formula for a large class of functions coming from generalized non-holomorphic Eisenstein series. In fact, the theorem that we shall use in this paper is a twisted version of the theorem in [4] and so some notations are twisted versions of those in [4].

Let $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of integers, real numbers and complex numbers, respectively. Throughout this paper, let the branch of the argument of $z \in \mathbb{C}$ be defined by $-\pi \leq \arg z < \pi$. For any non-negative integer $n$, the rising factorial $(x)_n$ is defined by

$$(x)_n = x(x+1)\cdots(x+n-1),\ n > 0 \text{ and } (x)_0 = 1.$$

Let $\Gamma(s)$ denote the gamma function. It is easy to see that

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{1.1}$$

The confluent hypergeometric function of the first kind ${}_1F_1(\alpha;\beta;z)$ is defined by

$${}_1F_1(\alpha;\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n n!} z^n$$

and the confluent hypergeometric function of the second kind $U(\alpha,\beta,z)$ is defined by

$$U(\alpha,\beta,z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\ {}_1F_1(\alpha;\beta;z) + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} z^{1-\beta}\ {}_1F_1(1+\alpha-\beta;2-\beta;z).$$

The function $U(\alpha,\beta,z)$ can be analytically continued to all values of $\alpha,\ \beta,\ z \in \mathbb{C}$ [6]. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$ be the upper half-plane. For $r_k,\ h_k \in \mathbb{R}$ $(k = 1,2)$, let $\mathbf{r} = (r_1, r_2)$ and $\mathbf{h} = (h_1, h_2)$. Let $e(x) = e^{2\pi i x}$ and let $N$ be a positive integer. For $\tau \in \mathbb{H}$ and $s_1,\ s_2 \in \mathbb{C}$ with $s = s_1 + s_2$, define

$$\begin{aligned}\mathcal{A}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \sum_{Nm+r_1>0} \sum_{n-h_2>0} & \frac{e\left(Nmh_1 + ((Nm+r_1)\tau + r_2)(n-h_2)\right)}{(n-h_2)^{1-s}} \\ & \times U(s_2; s; 4\pi(Nm+r_1)(n-h_2)\mathrm{Im}(\tau))\end{aligned}$$

and

$$\begin{aligned}\bar{\mathcal{A}}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \sum_{Nm+r_1>0} \sum_{n+h_2>0} & \frac{e\left(Nmh_1 - ((Nm+r_1)\bar{\tau} + r_2)(n+h_2)\right)}{(n+h_2)^{1-s}} \\ & \times\ U(s_1; s; 4\pi(Nm+r_1)(n+h_2)\mathrm{Im}(\tau)).\end{aligned}$$

Let

$$\mathcal{H}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \mathcal{A}_N(\tau, s_1, s_2; r, h) + e^{\pi i s}\mathcal{A}_N(\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h})$$

and

$$\bar{\mathcal{H}}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \bar{\mathcal{A}}_N(\tau, s_1, s_2; r, h) + e^{\pi i s}\bar{\mathcal{A}}_N(\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h}).$$

Let

$$\mathbf{H}_N(\tau, \bar{\tau}, s_1, s_2; \mathbf{r}, \mathbf{h}) = \frac{1}{\Gamma(s_1)}\mathcal{H}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) + \frac{1}{\Gamma(s_2)}\bar{\mathcal{H}}_N(\tau, s_1, s_2; r, h).$$

The functions $\mathcal{A}_N, \bar{\mathcal{A}}_N, \mathcal{H}_N, \bar{\mathcal{H}}_N$ and $\mathbf{H}_N$ are twisted versions of those in [4]. In fact, the function $\mathbf{H}_N$ comes from generalized non-holomorphic Eisenstein series and the relation between $\mathbf{H}_N$ and the generalized non-holomorphic Eisenstein series is given in [4]. For $x,\ \alpha \in \mathbb{R}$ and $t \in \mathbb{C}$ with $\operatorname{Re} t > 1$, let

$$\psi(x,\alpha,t) = \sum_{n+\alpha>0} \frac{e(nx)}{(n+\alpha)^t}$$

and let

$$\begin{aligned}\Psi(x,\alpha,t) &= \psi(x,\alpha,t) + e^{\pi i t}\psi(-x,-\alpha,t),\\ \Psi_{-1}(x,\alpha,t) &= \psi(x,\alpha,t-1) + e^{\pi i t}\psi(-x,-\alpha,t-1).\end{aligned}$$

Let $\lambda_N$ denote the characteristic function of the integers modulo $N$. For $x \in \mathbb{R}$, $[x]$ denotes the greatest integer less than or equal to $x$ and $\{x\} = x - [x]$. Let

$$V\tau = \frac{a\tau + b}{c\tau + d}$$

denote a modular transformation with $c > 0$ and $c \equiv 0 \pmod N$ for $\tau \in \mathbb{C}$. Let

$$\mathfrak{R} = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$\mathfrak{H} = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

Put

$$\varrho_N = c\{R_2\} - Nd\left\{\frac{R_1}{N}\right\}.$$

We now state a twisted version of Theorem 3.4 in [4] which we shall use to obtain our results.

THEOREM 1.1. [4]. *Let $Q = \{\tau \in \mathbb{H} \mid \operatorname{Re}\tau > -d/c\}$. Let $s_1, s_2 \in \mathbb{C}$ with $s = s_1+s_2$ and assume that $s$ is not an integer less than or equal to 1. Then, for $\tau \in Q$,*

$$\begin{aligned}&z^{-s_1}\bar{z}^{-s_2}\mathbf{H}_N(V\tau, V\bar{\tau}, s_1, s_2; \mathbf{r}, \mathbf{h})\\ &= \mathbf{H}_N(\tau,\bar{\tau},s_1,s_2;\mathfrak{R},\mathfrak{H}) + \lambda_N(R_1)e(-R_1H_1)(2\pi i)^{-s}e^{-\pi i s_2}\Psi(-H_2,-R_2,s)\\ &\quad -\lambda_N(r_1)e(-r_1h_1)(2\pi i)^{-s}e^{\pi i s_1}z^{-s_1}\bar{z}^{-s_2}\Psi(h_2,r_2,s)\\ &\quad +\lambda_N(H_2)(4\pi \operatorname{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}\Psi_{-1}(H_1,R_1,s)\\ &\quad -\lambda_N(h_2)(4\pi \operatorname{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}z^{s_2-1}\bar{z}^{s_1-1}\Psi_{-1}(h_1,r_1,s)\\ &\quad +\frac{(2\pi i)^{-s}e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}_N(\tau,\bar{\tau},s_1,s_2;\mathfrak{R},\mathfrak{H}),\end{aligned}$$

*where $z = c\tau + d$ and*

$$\begin{aligned}&\mathbf{L}_N(\tau,\bar{\tau},s_1,s_2;\mathfrak{R},\mathfrak{H})\\ &= \sum_{j=1}^{c} e(-H_1(Nj + N[R_1/N] - c) - H_2([R_2] + 1 + [(Njd+\varrho_N)/c] - d))\\ &\quad \times \int_0^1 v^{s_1-1}(1-v)^{s_2-1}\int_C u^{s-1}\frac{e^{-(zv+\bar{z}(1-v))(Nj-N\{R_1/N\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - e(cH_1+dH_2)}\frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - e(-H_2)}\,du\,dv,\end{aligned}$$

*where $C$ is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in a counterclockwise direction so that $u = 0$ is the only zero of*

$$(e^{-(zv+\bar{z}(1-v))u} - e(cH_1+dH_2))(e^u - e(-H_2))$$

*lying inside the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of $u^s$ with $0 < \arg u < 2\pi$.*

Let $B_n(x)$ denote the $n$-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi).$$

The $n$-th Bernoulli number $B_n$, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. Let ${}_2F_1(\alpha, \beta; \gamma; z)$ be a hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n.$$

The function $\frac{1}{\Gamma(\gamma)}\ {}_2F_1(\alpha, \beta; \gamma; z)$ can be analytically continued to all $\alpha,\ \beta,\ \gamma \in \mathbb{C}$ and all $z \in \mathbb{C}$ with $|z| < 1$ ( [1]).

Remark 1.2. Let $s = s_1 + s_2$ be an integer and let $h_1 = h_2 = 0$. By the residue theorem, we find that

$$\begin{aligned}
&\int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(Nj - N\{R_1/N\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - 1} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - 1}\, du \\
&= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((Nj - N\{R_1/N\})/c) \bar{B}_{-s+2-k}((Njd + \varrho_N)/c)}{k!(-s+2-k)!} (-zv - \bar{z}(1-v))^{k-1}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^1 v^{s_1-1}(1-v)^{s_2-1} \int_C u^{-s-1} \frac{e^{-(zv+\bar{z}(1-v))(Nj - N\{R_1/N\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - 1} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - 1}\, dudv \\
&= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((Nj - N\{R_1/N\})/c) \bar{B}_{-s+2-k}((Njd + \varrho_N)/c)}{k!(-s+2-k)!} (-z)^{k-1} \\
&\qquad \times \int_0^1 (1-v)^{s_1-1} v^{s_2-1} \left(1 - \frac{z - \bar{z}}{z} v\right)^{k-1} dv \\
&= 2\pi i \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s)} \sum_{k=0}^{-s+2} \frac{B_k((Nj - N\{R_1/N\})/c) \bar{B}_{-s+2-k}((Njd + \varrho_N)/c)}{k!(-s+2-k)!} \\
&\qquad \times (-z)^{k-1}\ {}_2F_1\left(s_2, 1-k; s; \frac{z - \bar{z}}{z}\right).
\end{aligned}$$

The last equality holds due to the integral representation of the hypergeometric function. Hence we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}_N(\tau, \bar{\tau}, s_1, s_2; \mathfrak{R}, \mathfrak{H}) \\
&= \frac{2\pi i}{\Gamma(s)} \sum_{k=0}^{-s+2} \frac{B_k((Nj - N\{R_1/N\})/c) \bar{B}_{-s+2-k}((Njd + \varrho_N)/c)}{k!(-s+2-k)!} \\
&\qquad \times (-z)^{k-1}\ {}_2F_1\left(s_2, 1-k; s; \frac{z - \bar{z}}{z}\right).
\end{aligned}$$

We now see that $\frac{1}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}_N(\tau,\bar\tau,s_1,s_2;\mathfrak{R},\mathfrak{H})$ vanishes for $s>2$. Let $s=2$ and $s_2=-B$ be a non-positive integer. Then, applying

$$(s_2)_n=\begin{cases}(-1)^n\frac{B!}{(B-n)!}, & n\le B,\\ 0, & n>B\end{cases}$$

and using the binomial expansion

$$(1+x)^B=\sum_{n=0}^{B}\binom{B}{n}x^n,$$

we find

$$\begin{aligned}{}_2F_1\left(s_2,1;2;\frac{z-\bar z}{z}\right)&=\sum_{n=0}^{B}\binom{B}{n}\frac{1}{n+1}\left(\frac{\bar z-z}{z}\right)^n\\ &=\frac{1}{B+1}\frac{z}{\bar z-z}\left(\left(\frac{\bar z}{z}\right)^{B+1}-1\right).\end{aligned}$$

Thus, for $s=2$ and $s_2$ a non-positive integer, after the evaluation of $\mathbf{L}_N(\tau,\bar\tau,s_1,s_2;\mathfrak{R},\mathfrak{H})$, the relevant formula will be valid for all $z\in\mathbb{H}$ by analytic continuation.

## 2. A class of character analogues of infinite series identities

In this section, let $\chi$ be a Dirichlet character of modulus $N$ and $\chi_{\circ}$ be the principal Dirichlet character of modulus $N$. From now on, we set

$$V\tau=\frac{\tau-1}{N\tau-N+1}.$$

The function $\varphi$ denotes Euler's phi function, i.e., $\varphi(N)$ is the number of positive integers up to $N$ that is relatively prime to $N$. Let $\zeta_N=e^{2\pi i/N}$ and let

$$\sigma_t(\chi,n)=\sum_{d\mid n}\chi(d)d^t.$$

Theorem 2.1. *Let $\chi$ be even. For any integers $B\ge 0$, $M\ge 1$ and for $z\in\mathbb{H}$,*

$$\begin{aligned}&z^{-B-2M}\bar z^{B}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(4\pi\mathrm{Im}(z^{-1})/N)^\ell}{(2M+k-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\zeta_N^{n}\sigma_{2M-1}(\bar\chi,n)e\left(\frac{-nz^{-1}}{N}\right)\\ &=\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi\mathrm{Im}(z)/N)^\ell}{(2M+k-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\zeta_N^{-n}\sigma_{2M-1}(\bar\chi,n)e\left(\frac{nz}{N}\right)+\frac12\delta_M(B,z)\nu_\chi(N),\end{aligned}$$

*where*

$$\delta_M(B,z)=\begin{cases}\dfrac{(-1)^{B+1}}{2\pi(B+1)}\dfrac{1}{z-\bar z}\left(\left(\dfrac{\bar z}{z}\right)^{B+1}-1\right), & M=1,\\ 0, & M>1\end{cases}$$

*and*

$$\nu_\chi(N)=\begin{cases}\varphi(N), & \chi=\chi_\circ,\\ 0, & \chi\ne\chi_\circ.\end{cases}$$

*Proof.* Let $s_1 = A \geq 1$, $s_2 = -B \leq 0$ and $s = 2M \geq 2$ for $A, B, M \in \mathbb{Z}$. Put $\mathbf{r} = (k, 0)$ for any integer $k$ with $1 \leq k < N$ and put $\mathbf{h} = (0, 0)$ in Theorem 1.1. Then $\mathfrak{R} = (k, -k)$ and $\mathfrak{H} = (0, 0)$. We see that $\lambda_N(r_1) = \lambda_N(R_1) = \lambda_N(k) = 0$. Put $z = N\tau - N + 1$. Using Remark 1.2, we have

$$\frac{(2\pi i)^{1-s} e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}_N(\tau, \bar{\tau}, s_1, s_2; \mathfrak{R}, \mathfrak{H}) = \begin{cases} \dfrac{(-1)^{B+1}}{2\pi(B+1)} \dfrac{1}{z - \bar{z}} \left( \left( \dfrac{\bar{z}}{z} \right)^{B+1} - 1 \right), & M = 1, \\ 0, & M > 1. \end{cases}$$

Note that $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$. Thus, in Theorem 1.1, the terms with $\lambda_N(h_2)$ and $\lambda_N(H_2)$ are equal to 0. Thus we have

$$z^{-A} \bar{z}^B \mathbf{H}_N(V\tau, V\bar{\tau}, A, -B; \mathbf{r}, \mathbf{h}) = \mathbf{H}_N(\tau, \bar{\tau}, A, -B; \mathfrak{R}, \mathfrak{H}) + \delta_M(B, z). \tag{2.1}$$

Multiplying both sides in (2.1) by $\chi(k)$ and summing over $k$, we find that

$$\begin{aligned} &z^{-A} \bar{z}^B \sum_{k=1}^{N-1} \chi(k) \mathbf{H}_N(V\tau, V\bar{\tau}, A, -B; \mathbf{r}, \mathbf{h}) \\ &= \sum_{k=1}^{N-1} \chi(k) \mathbf{H}_N(\tau, \bar{\tau}, A, -B; \mathfrak{R}, \mathfrak{H}) + \sum_{k=1}^{N-1} \chi(k) \delta_M(B, z). \end{aligned} \tag{2.2}$$

Since $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$,

$$\mathbf{H}_N(V\tau, V\bar{\tau}, A, -B; \mathbf{r}, \mathbf{h}) = \frac{1}{\Gamma(A)} \mathcal{H}_N(V\tau, A, -B; \mathbf{r}, \mathbf{h}).$$

It is easy to see that

$$\begin{aligned} &\mathcal{A}_N(V\tau, A, -B; \mathbf{r}, \mathbf{h}) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e((Nm+k)nV\tau)}{n^{1-2M}} U(-B; 2M; 4\pi(Nm+k)n\mathrm{Im}(V\tau)) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\mathcal{A}_N(V\tau, A, -B; -\mathbf{r}, -\mathbf{h}) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e((Nm+N-k)nV\tau)}{n^{1-2M}} U(-B; 2M; 4\pi(Nm+N-k)n\mathrm{Im}(V\tau)). \end{aligned} \tag{2.4}$$

Using (2.3) and (2.4), we obtain that

$$\begin{aligned} &\sum_{k=1}^{N-1} \chi(k) \mathbf{H}_N(V\tau, V\bar{\tau}, A, -B; \mathbf{r}, \mathbf{h}) \\ &= \frac{1}{\Gamma(A)} \sum_{k=1}^{N-1} \chi(k) \left( \mathcal{A}_N(V\tau, A, -B; \mathbf{r}, \mathbf{h}) + \mathcal{A}_N(V\tau, A, -B; -\mathbf{r}, -\mathbf{h}) \right) \\ &= \frac{2}{\Gamma(A)} \sum_{k=1}^{N-1} \chi(k) \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e((Nm+k)nV\tau)}{n^{1-2M}} U(-B; 2M; 4\pi(Nm+k)n\mathrm{Im}(V\tau)) \\ &= \frac{2}{\Gamma(A)} \sum_{n=1}^{\infty} \frac{1}{n^{1-2M}} \sum_{m=1}^{\infty} \chi(m) e(mnV\tau) U(-B; 2M; 4\pi mn\mathrm{Im}(V\tau)) \\ &= \frac{2}{\Gamma(A)} \sum_{n=1}^{\infty} \chi(n) \sigma_{2M-1}(\bar{\chi}, n) e(nV\tau) U(-B; 2M; 4\pi n\mathrm{Im}(V\tau)). \end{aligned}$$

Recall $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$ and apply (1.1) to obtain

$$U(-B;2M;4\pi n\mathrm{Im}(V\tau)) = (-1)^B(A-1)!\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi n\mathrm{Im}(V\tau))^\ell}{(2M+\ell-1)!}.$$

Since $z = N\tau - N + 1$,

$$V\tau = \frac{\tau-1}{N\tau-N+1} = \frac{1}{N}(1-z^{-1}),$$

$$e(nV\tau) = e^{2\pi i(n(1-z^{-1})/N)} = \zeta_N^n e\left(\frac{-nz^{-1}}{N}\right)$$

and

$$\mathrm{Im}(V\tau) = -\frac{1}{N}\mathrm{Im}(z^{-1}).$$

Hence we obtain

$$\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A,-B;\mathbf{r},\mathbf{h})$$
$$= 2(-1)^B\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(4\pi\mathrm{Im}(z^{-1})/N)^\ell}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\zeta_N^n\sigma_{2M-1}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right).$$

By the same way, we also obtain

$$\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(\tau,\bar{\tau},A,-B;\mathfrak{R},\mathfrak{H})$$
$$= 2(-1)^B\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi\mathrm{Im}(z)/N)^\ell}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\zeta_N^{-n}\sigma_{2M-1}(\bar{\chi},n)e\left(\frac{nz}{N}\right).$$

Put the last two identities into (2.2) and use

$$\sum_{k=1}^{N-1}\chi(k) = \begin{cases}\varphi(N), & \chi=\chi_\circ,\\ 0, & \chi\neq\chi_\circ.\end{cases}$$

to complete the proof. □

THEOREM 2.2. *Let $\chi$ be even and let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. For any integers $B\geq 0$ and $M\geq 1$,*

$$(-1)^B\alpha^M\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\zeta_N^n\sigma_{2M-1}(\bar{\chi},n)e^{-2n\alpha/N}$$
$$= (-1)^M\beta^M\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\beta/N)^\ell}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\zeta_N^{-n}\sigma_{2M-1}(\bar{\chi},n)e^{-2n\beta/N} - \frac{1}{2}\delta_M(B)\nu_\chi(N),$$

*where*

$$\delta_M(B) = \begin{cases}\dfrac{(-1)^B+1}{4(B+1)}, & M=1,\\ 0, & M>1.\end{cases}$$

*Proof.* Put $z = \frac{\pi}{\alpha} i$ in Theorem 2.1. Then

$$z^{-A}\bar{z}^{B} = (-1)^{B}\alpha^{M}(-\beta)^{-M}, \ e\left(\frac{-nz^{-1}}{N}\right) = e^{-2n\alpha/N} \text{ and } e\left(\frac{nz}{N}\right) = e^{-2n\beta/N}.$$

A short calculation shows that

$$\delta_1\left(B, \frac{\pi}{\alpha} i\right) = \frac{(-1)^{B}+1}{4\beta(B+1)}.$$

Multiplying both sides of the identity in Theorem 2.1 by $(-\beta)^{M}$, we complete the proof. □

Let $B = 0$ in Theorem 2.2. If $M \geq 2$ or $\chi \neq \chi_{\mathrm{o}}$, then we have

$$\alpha^{M}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{n}\sigma_{2M-1}(\bar{\chi}, n)e^{-2n\alpha/N} = (-\beta)^{M}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{-n}\sigma_{2M-1}(\bar{\chi}, n)e^{-2n\beta/N},$$

which is given as Corollary 3.3 in [3].
Let $\chi = \chi_{\mathrm{o}}$, $M = 1$ and $\alpha = \beta = \pi$ in Theorem 2.2. If $B$ is even, then

$$\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi_{\mathrm{o}}(n)}{n^{-\ell}}\cos\left(\frac{2n\pi}{N}\right)\sigma_1(\bar{\chi}_{\mathrm{o}}, n)e^{-2n\pi/N} = -\frac{\varphi(N)}{8\pi(B+1)}.$$

Put $B = 0$. Then

$$\sum_{n=1}^{\infty}\chi_{\mathrm{o}}(n)\cos\left(\frac{2n\pi}{N}\right)\sigma_1(\chi_{\mathrm{o}}, n)e^{-2n\pi/N} = -\frac{1}{8\pi}\varphi(N).$$

Let $\chi = \bar{\chi}$ and let $M \geq 2$ or $\chi \neq \chi_{\mathrm{o}}$ in Theorem 2.2. Then, equating the real part and the imaginary part, respectively, we have

$$\begin{aligned}(-1)^{B}\alpha^{M}&\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\alpha/N)^{\ell}}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M-1}(\bar{\chi}, n)e^{-2n\alpha/N}\\ &= (-1)^{M}\beta^{M}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\beta/N)^{\ell}}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M-1}(\bar{\chi}, n)e^{-2n\beta/N}\end{aligned}$$

and

$$\begin{aligned}(-1)^{B}\alpha^{M}&\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\alpha/N)^{\ell}}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M-1}(\bar{\chi}, n)e^{-2n\alpha/N}\\ &= (-1)^{M+1}\beta^{M}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\beta/N)^{\ell}}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M-1}(\bar{\chi}, n)e^{-2n\beta/N}.\end{aligned}$$

Thus we obtain the following two corollaries which include elegant symmetric identities for $\alpha$ and $\beta$.

Corollary 2.3. *Let $\chi$ be even and $\chi = \bar{\chi}$. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and let $B, M$ be integers with $B \geq 0$ and $M \geq 1$. Suppose that $M = 1$ and $\chi = \chi_{\mathrm{o}}$ cannot be considered simultaneously. If $B$ and $M$ have the same parity, then*

$$\alpha^{M}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\alpha/N)^{\ell}}{(2M+\ell-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M-1}(\chi, n)e^{-2n\alpha/N}$$

$$= \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\beta/N},$$

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\alpha/N}$$
$$= -\beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\beta/N}.$$

*If $B$ and $M$ have the different parity, then*

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\alpha/N}$$
$$= -\beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\beta/N},$$

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\alpha/N}$$
$$= \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\beta/N}.$$

Corollary 2.3 contains generalizations of Corollary 3.4 and 3.5 in [3].

Corollary 2.4. *Let $\chi$ be even and $\chi = \bar{\chi}$. Let $B, M$ be integers with $B \geq 0$ and $M \geq 1$. Suppose that $M = 1$ and $\chi = \chi_{\text{o}}$ cannot be considered simultaneously. If $B$ and $M$ have the same parity, then*

$$\sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\pi/N} = 0.$$

*If $B$ and $M$ have the different parity, then*

$$\sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\pi/N} = 0.$$

*Proof.* Let $\alpha = \beta = \pi$ in Corollary 2.3. □

Let $B = 0$ and replace $M$ by $2M$ in the first equation in Corollary 2.4. Then

$$\sum_{n=1}^{\infty} \chi(n) \sin\left(\frac{2\pi n}{N}\right) \sigma_{4M-1}(\chi, n) e^{-2n\pi/N} = 0.$$

Let $B = 0$ and replace $M$ by $2M-1$ in the second equation in Corollary 2.4. Then

$$\sum_{n=1}^{\infty} \chi(n) \cos\left(\frac{2\pi n}{N}\right) \sigma_{4M-3}(\chi, n) e^{-2n\pi/N} = 0.$$

Let $p$ be a prime with $p \equiv 1 \pmod 4$ and let $(\frac{\cdot}{p})$ be the Legendre symbol. Then $(\frac{\cdot}{p})$ is an even character with real values. Thus we can put $\chi = (\frac{\cdot}{p})$ in Theorem 2.1, Theorem 2.2, Corollary 2.3 and Corollary 2.4. For example, if $\chi = (\frac{\cdot}{p})$ in the first identity in Corollary 2.3, we obtain

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/p)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) n^\ell \cos\left(\frac{2\pi n}{p}\right) \sigma_{2M-1}(\left(\frac{\cdot}{p}\right), n) e^{-2n\alpha/p}$$
$$= \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/p)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) n^\ell \cos\left(\frac{2\pi n}{p}\right) \sigma_{2M-1}(\left(\frac{\cdot}{p}\right), n) e^{-2n\beta/p}$$

which gives a generalization of Corollary 3.4 in [3].

For an odd character $\chi$, applying the similar method, we obtain the following theorems and corollaries.

Theorem 2.5. *Let $\chi$ be odd. For any integers $B \geq 0$, $M \geq 1$ and for $z \in \mathbb{H}$,*

$$z^{-B-2M-1}\bar{z}^B \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(4\pi \mathrm{Im}(z^{-1})/N)^\ell}{(2M+k)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^n \sigma_{2M}(\bar{\chi}, n) e\left(\frac{-nz^{-1}}{N}\right)$$
$$= \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi \mathrm{Im}(z)/N)^\ell}{(2M+k)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^{-n} \sigma_{2M}(\bar{\chi}, n) e\left(\frac{nz}{N}\right).$$

*Proof.* Let $s_1 = A \geq 1,\ s_2 = -B \leq 0$ and $s = 2M+1 \geq 3$ for $A, B, M \in \mathbb{Z}$. Since $s \geq 3$, we have, by using Remark 1.2,

$$\frac{1}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}_N(\tau, \bar{\tau}, s_1, s_2; \mathfrak{R}, \mathfrak{H}) = 0.$$

For the other parts of the proof, apply the similar method in the proof of Theorem 2.1. □

Theorem 2.6. *Let $\chi$ be odd and let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $M \geq 1$,*

$$(-1)^B \alpha^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^n \sigma_{2M}(\bar{\chi}, n) e^{-2n\alpha/N}$$
$$= (-\beta)^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^{-n} \sigma_{2M}(\bar{\chi}, n) e^{-2n\beta/N}.$$

Corollary 2.7. *Let $\chi$ be odd and $\chi = \bar{\chi}$. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Let $B,\ M$ be integers with $B \geq 0$, $M \geq 1$. If $B$ and $M$ have the same parity, then*

$$\alpha^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M}(\chi, n) e^{-2n\alpha/N}$$
$$= \beta^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M}(\chi, n) e^{-2n\beta/N}.$$

*If $B$ and $M$ have the different parity, then*

$$\alpha^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M}(\chi, n) e^{-2n\alpha/N}$$

$$= -\beta^{M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\beta/N)^\ell}{(2M+\ell)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\chi,n)e^{-2n\beta/N}.$$

COROLLARY 2.8. *Let $\chi$ be odd and $\chi=\bar{\chi}$. Let $B$, $M$ be integers with $B\geq 0$, $M\geq 1$. If $B$ and $M$ have the same parity, then*

$$\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi/N)^\ell}{(2M+\ell)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{2M}(\chi,n)e^{-2n\pi/N}=0.$$

*If $B$ and $M$ have the different parity, then*

$$\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi/N)^\ell}{(2M+\ell)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\sin\left(\frac{2\pi n}{N}+\frac{\pi}{4}\right)\sigma_{2M}(\chi,n)e^{-2n\pi/N}=0.$$

*Proof.* Let $\alpha=\beta=\pi$ in Corollary 2.7. □

Put $B=0$ in Corollary 2.8. Then

$$\sum_{n=1}^{\infty}\chi(n)\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{4M}(\chi,n)e^{-2n\pi/N}=0$$

and

$$\sum_{n=1}^{\infty}\chi(n)\sin\left(\frac{2\pi n}{N}+\frac{\pi}{4}\right)\sigma_{4M-2}(\chi,n)e^{-2n\pi/N}=0.$$

Let $p$ be a prime with $p\equiv 3 \pmod 4$. Then $\left(\frac{\cdot}{p}\right)$ is an odd character with real values. Thus we also put $\chi=\left(\frac{\cdot}{p}\right)$ in Theorem 2.5, Theorem 2.6, Corollary 2.7 and Corollary 2.8.

## 3. Another class of character analogue of infinite series identities

In this section, we obtain another type of character analogue of infinite series identities. We shall let $s_1\geq 1$, $s_2\geq 1$ and let $s\geq 3$ or $s\geq 4$ for $s=s_1+s_2$. Thus, by Remark 1.2, we see that

$$\frac{1}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}_N(\tau,\bar{\tau},s_1,s_2;\mathfrak{R},\mathfrak{H})=0.$$

THEOREM 3.1. *Let $\chi$ be even. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M$. Then, for $z\in\mathbb{H}$,*

$$z^{-A-1}\bar{z}^{-B-1}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(-4\pi\mathrm{Im}(z^{-1})/N)^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)$$
$$+z^{-A-1}\bar{z}^{-B-1}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(-4\pi\mathrm{Im}(z^{-1})/N)^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{n\bar{z}^{-1}}{N}\right)$$
$$=\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(z)/N)^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{nz}{N}\right)$$

$$+\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(z)/N)^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^n\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-n\bar{z}}{N}\right)$$
$$+\mu_\chi(A,B,M,z),$$

*where*

$$\mu_\chi(A,B,M,z)=\begin{cases}\frac{(2M)!}{(4\pi\mathrm{Im}(z/N))^{2M+1}}(1-z^B\bar{z}^A)\zeta(2M+1)\varphi(N), & \chi=\chi_{\text{o}},\\ 0, & \chi\neq\chi_{\text{o}}.\end{cases}$$

*Proof.* Let $s_1=A+1\geq 1$, $s_2=B+1\geq 1$ and $s=2M+2\geq 4$ with $A,B,M\in\mathbb{Z}$. Put $\mathbf{r}=(k,0)$ for any integer $k$ with $1\leq k<N$ and put $\mathbf{h}=(0,0)$ in Theorem 1.1. Then $\mathfrak{R}=(k,-k)$ and $\mathfrak{H}=(0,0)$. We see that $\lambda_N(r_1)=\lambda_N(R_1)=\lambda_N(k)=0$ and $\lambda(h_2)=\lambda(H_2)=1$. Put $z=N\tau-N+1$. Since

$$\Psi_{-1}(0,k,2M+2)=2\sum_{n=1}^{\infty}\frac{1}{n^{2M+1}}=2\zeta(2M+1),$$

we have

$$\begin{aligned}&z^{-A-1}\bar{z}^{-B-1}\mathbf{H}_N(V\tau,V\bar{\tau},A+1,B+1;\mathbf{r},\mathbf{h})\\ &=\mathbf{H}_N(\tau,\bar{\tau},A+1,B+1;\mathfrak{R},\mathfrak{H})+\frac{2(2M)!}{A!B!}\frac{1-z^B\bar{z}^A}{(4\pi\mathrm{Im}(\tau))^{2M+1}}\zeta(2M+1).\end{aligned}\tag{3.1}$$

Note that, for any $b\in\mathbb{C}$,

$$\sum_{k=1}^{N-1}\chi(k)b=\begin{cases}b\varphi(N), & \chi=\chi_{\text{o}},\\ 0, & \chi\neq\chi_{\text{o}}.\end{cases}$$

Thus, multiplying both sides in (3.1) by $\chi(k)$ and summing over $k$, we find that

$$\begin{aligned}&z^{-A-1}\bar{z}^{-B-1}\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A+1,B+1;\mathbf{r},\mathbf{h})\\ &=\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(\tau,\bar{\tau},A+1,B+1;\mathfrak{R},\mathfrak{H})+2\mu_\chi(A,B,M,z),\end{aligned}\tag{3.2}$$

where

$$\mu_\chi(A,B,M,z)=\begin{cases}\frac{(2M)!}{A!B!}\frac{1-z^B\bar{z}^A}{(4\pi\mathrm{Im}(z/N))^{2M+1}}\zeta(2M+1)\varphi(N), & \chi=\chi_{\text{o}},\\ 0, & \chi\neq\chi_{\text{o}}.\end{cases}$$

To compute $\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A+1,B+1;\mathbf{r},\mathbf{h})$, we shall apply the same method in the proof of Theorem 2.1. Then

$$\begin{aligned}&\sum_{k=1}^{N-1}\chi(k)\mathcal{H}_N(V\tau,A+1,B+1;\mathbf{r},\mathbf{h})\\ &=2\sum_{n=1}^{\infty}\chi(n)\zeta_N^n\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)U(B+1;2M+2;4\pi n\mathrm{Im}(V\tau))\end{aligned}$$

and

$$\sum_{k=1}^{N-1}\chi(k)\bar{\mathcal{H}}_N(V\tau,A+1,B+1;-\mathbf{r},-\mathbf{h})$$

$$= 2\sum_{n=1}^{\infty}\chi(n)\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{n\bar{z}^{-1}}{N}\right)U(A+1;2M+2;4\pi n\mathrm{Im}(V\tau)).$$

Apply Lemma 2.1 in [5] to obtain

$$U(B+1;2M+2;4\pi n\mathrm{Im}(V\tau)) = \frac{1}{B!}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi n\mathrm{Im}(V\tau))^{2M-\ell+1}}$$

and

$$U(A+1;2M+2;4\pi n\mathrm{Im}(V\tau)) = \frac{1}{A!}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi n\mathrm{Im}(V\tau))^{2M-\ell+1}}.$$

Thus we have

$$\begin{aligned}&\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A+1,B+1;\mathbf{r},\mathbf{h})\\ &= \frac{2}{A!B!}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(V\tau))^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)\\ &\quad+\frac{2}{A!B!}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(V\tau))^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{n\bar{z}^{-1}}{N}\right).\end{aligned}$$

By the similar way, we also have

$$\begin{aligned}&\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(\tau,\bar{\tau},A+1,B+1;\mathfrak{R},\mathfrak{H})\\ &= \frac{2}{A!B!}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(\tau))^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{nz}{N}\right)\\ &\quad+\frac{2}{A!B!}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(\tau))^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-n\bar{z}}{N}\right).\end{aligned}$$

Use $\mathrm{Im}(V\tau) = \mathrm{Im}\left(\frac{-z^{-1}}{N}\right)$ and $\mathrm{Im}(\tau) = \mathrm{Im}(\frac{z}{N})$ to complete the proof. □

THEOREM 3.2. *Let $\chi$ be even and let $\alpha,\beta > 0$ with $\alpha\beta = \pi^2$. For $A,B,M \in \mathbb{Z}$, let $A \geq 0$, $B \geq 0$, $M \geq 1$ with $A+B = 2M$. Then*

$$\begin{aligned}&(-1)^B\alpha^{-M}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e^{-2\alpha n/N}\\ &+(-1)^B\alpha^{-M}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e^{-2\alpha n/N}\\ &= (-1)^M\beta^{-M}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e^{-2\beta n/N}\\ &\quad+(-1)^M\beta^{-M}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e^{-2\beta n/N}\\ &\quad+\nu_\chi(B,M,\alpha,\beta),\end{aligned}$$

*where*

$$\nu_\chi(B,M,\alpha,\beta)=\begin{cases}(2M)!((-1)^M\beta^{-M}-(-1)^B\alpha^{-M})\zeta(2M+1)\varphi(N), & \chi=\chi_\circ,\\ 0, & \chi\neq\chi_\circ.\end{cases}$$

*Proof.* Put $z=\frac{\pi}{\alpha}i$ in Theorem 3.1. Apply

$$e\left(\frac{-nz^{-1}}{N}\right)=e\left(\frac{n\bar{z}^{-1}}{N}\right)=e^{-2\alpha n/N},\ e\left(\frac{nz}{N}\right)=e\left(\frac{-n\bar{z}}{N}\right)=e^{-2\beta n/N}$$

and

$$z^{-A-1}\bar{z}^{-B-1}=(-1)^{B+M}\alpha^{M+1}\beta^{-M-1},\ z^B\bar{z}^A=(-1)^{B+M}\alpha^{-M}\beta^M.$$

Multiplying both sides of the identity in Theorem 3.1 by $(-1)^M\left(\frac{4}{M}\right)^{2M+1}\beta^{M+1}$, we obtain the desired result. □

Corollary 3.3. *Let $\chi$ be even with $\chi\neq\chi_\circ$. Let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. For any integer $M\geq 1$,*

$$\alpha^{-M}\sum_{\ell=0}^{M}\binom{M}{\ell}\frac{(2M-\ell)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\alpha n/N}$$
$$=\beta^{-M}\sum_{\ell=0}^{M}\binom{M}{\ell}\frac{(2M-\ell)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\beta n/N}.$$

*Proof.* Let $A=B$ in Theorem 3.2 and use $\zeta_N^n+\zeta_N^{-n}=2\cos(\frac{2\pi n}{N})$. □

Corollary 3.3 shows a fairly good symmetric identity for $\alpha$ and $\beta$. If we put $M=1$ in Corollary 3.3, then

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^3}\cos\left(\frac{2\pi n}{N}\right)\sigma_3(\bar{\chi},n)\left(\frac{1}{\alpha}e^{-2\alpha n/N}-\frac{1}{\beta}e^{-2\beta n/N}\right)$$
$$=-\frac{2}{N}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2}\cos\left(\frac{2\pi n}{N}\right)\sigma_3(\bar{\chi},n)\left(e^{-2\alpha n/N}-e^{-2\beta n/N}\right).$$

Corollary 3.4. *Let $\chi$ be even. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M$. If $B$ and $M$ have the same parity, then*

$$\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\pi n/N}$$
$$=\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\pi n/N}.$$

*Proof.* Put $\alpha=\beta=\pi$ in Theorem 3.2. Then $\nu_\chi(B,M,\alpha,\beta)=0$ for any $\chi$. Apply $\zeta_N^n-\zeta_N^{-n}=2i\sin\left(\frac{2\pi n}{N}\right)$, $(-1)^B=(-1)^M$. □

Corollary 3.4 also gives an elegant symmetric identity for $A$ and $B$.

Corollary 3.5. *Let $\chi$ be even. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M$. If $B$ and $M$ have the different parity, then*

$$\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\pi n/N}$$

$$= -\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\pi n/N}+\nu_{\chi}(M),$$

*where*

$$\nu_{\chi}(M)=\begin{cases}-2(2M)!\zeta(2M+1)\varphi(N), & \chi=\chi_{\circ},\\ 0, & \chi\neq\chi_{\circ}.\end{cases}$$

*Proof.* Put $\alpha=\beta=\pi$ in Theorem 3.2. Use $\zeta_N^n+\zeta_N^{-n}=2\cos\left(\frac{2\pi n}{N}\right)$, $(-1)^B=-(-1)^M$. □

COROLLARY 3.6. *Let $\chi$ be even. For any integer $M\geq 1$,*

$$\sum_{\ell=1}^{4M}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-\ell+1}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{4M+1}(\bar{\chi},n)e^{-2\pi n/N}=0.$$

*Proof.* Put $B=0$ in Corollary 3.4 and replace $M$ by $2M$. □

COROLLARY 3.7. *Let $\chi$ be even. For any integer $M\geq 1$,*

$$\sum_{\ell=1}^{4M-2}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-\ell-1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{4M-1}(\bar{\chi},n)e^{-2\pi n/N}$$
$$=-2\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{4M-1}(\bar{\chi},n)e^{-2\pi n/N}+\nu_{\chi}(2M-1).$$

*Proof.* Put $B=0$ in Corollary 3.5 and replace $M$ by $2M-1$. □

For a prime $p$ with $p\equiv 1 \pmod 4$, we can put $\chi=\left(\frac{\cdot}{p}\right)$ in Theorem 3.1, Theorem 3.2, Corollary 3.3 – Corollary 3.7.
Next we find character analogues of infinite series identities for an odd character $\chi$. In this case, we shall let $s$ be any odd integer greater than 2. The process to obtain the results is similar to the case of $\chi$ even.

THEOREM 3.8. *Let $\chi$ be odd. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M-1$. Then, for $z\in\mathbb{H}$,*

$$z^{-A-1}\bar{z}^{-B-1}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(-4\pi\mathrm{Im}(z^{-1})/N)^{2M-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)$$
$$+z^{-A-1}\bar{z}^{-B-1}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(-4\pi\mathrm{Im}(z^{-1})/N)^{2M-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e\left(\frac{n\bar{z}^{-1}}{N}\right)$$
$$=\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\pi\mathrm{Im}(z)/N)^{2M-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e\left(\frac{nz}{N}\right)$$
$$+\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\pi\mathrm{Im}(z)/N)^{2M-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar{\chi},n)e\left(\frac{-n\bar{z}}{N}\right).$$

*Proof.* Let $s_1=A+1\geq 1$, $s_2=B+1\geq 1$ and $s=2M+1\geq 3$ with $A,B,M\in\mathbb{Z}$. Put $\mathbf{r}=(k,0)$ for any integer $k$ with $1\leq k<N$ and put $\mathbf{h}=(0,0)$ in Theorem 1.1. The basic process of the proof is similar to the proof of Theorem 3.1. The only

noticeable difference arise from the terms with $\lambda(h_2)$ and $\lambda(H_2)$. Direct calculations show that they are vanished by using

$$\begin{aligned}\Psi_{-1}(0,R_1,s) &= \psi(0,k,2M)+e^{\pi i(2M+1)}\psi(0,-k,2M)\\ &= \sum_{n=1}^{\infty}\frac{e(nk)}{n^{2M}}-\sum_{n=1}^{\infty}\frac{e(-nk)}{n^{2M}}=0.\end{aligned}$$

□

THEOREM 3.9. *Let $\chi$ be odd and let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M-1$. Then*

$$\begin{aligned}&(-1)^A\alpha^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar{\chi},n)e^{-2\alpha n/N}\\ &+(-1)^A\alpha^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e^{-2\alpha n/N}\\ &=(-\beta)^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e^{-2\beta n/N}\\ &\quad+(-\beta)^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar{\chi},n)e^{-2\beta n/N}.\end{aligned}$$

*Proof.* Put $z=\frac{\pi}{\alpha}i$ in Theorem 3.8.

$$z^{-A-1}\bar{z}^{-B-1}=(-1)^{B+1}\alpha^{M+1/2}(-\beta)^{-M-1/2}.$$

Multiplying both sides of the identity in Theorem 3.8 by $\left(\frac{4}{N}\right)^{2M}(-\beta)^{M+1/2}$, we obtain the desired result. □

If we put $s_1=B+1$, $s_2=A+1$ in the proof of Theorem 3.8, then the associated Theorem 3.9 is changed to the following identity;

$$\begin{aligned}&(-1)^B\alpha^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar{\chi},n)e^{-2\alpha n/N}\\ &+(-1)^B\alpha^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e^{-2\alpha n/N}\\ &=(-\beta)^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e^{-2\beta n/N}\\ &\quad+(-\beta)^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar{\chi},n)e^{-2\beta n/N}.\end{aligned}\tag{3.3}$$

Note that $A$ and $B$ have the different parity. Adding the identity in Theorem 3.9 and (3.3), we obtain the following theorem.

THEOREM 3.10. *Let $\chi$ be odd and let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M-1$. Then*

$$(-1)^A\alpha^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\bar{\chi},n)e^{-2\alpha n/N}$$

$$
\begin{aligned}
&-(-1)^A\alpha^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\bar\chi,n)e^{-2\alpha n/N}\\
&=(-1)^M\beta^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\bar\chi,n)e^{-2\beta n/N}\\
&\quad+(-1)^M\beta^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\bar\chi,n)e^{-2\beta n/N}.
\end{aligned}
$$

Corollary 3.11. *Let $\chi$ be odd. For $A, B, M \in \mathbb{Z}$, let $A \geq 0$, $B \geq 0$, $M \geq 1$ with $A+B = 2M-1$ and $A-B \equiv 1 \pmod 4$. Then*

$$
\begin{aligned}
&\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{2M}(\bar\chi,n)e^{-2\pi n/N}\\
&=\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\sin\left(\frac{2\pi n}{N}+\frac{\pi}{4}\right)\sigma_{2M}(\bar\chi,n)e^{-2\pi n/N}.
\end{aligned}
$$

*Proof.* Let $\alpha = \beta = \pi$ in Theorem 3.10. Use the fact that $A - B \equiv 1 \pmod 4$ is equivalent to that $A$ and $M$ have the same parity. □

Corollary 3.12. *Let $\chi$ be odd. For any integer $M \geq 1$,*

$$
\begin{aligned}
&\sum_{\ell=1}^{4M-3}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-\ell-2}}\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{4M-2}(\bar\chi,n)e^{-2\pi n/N}\\
&=\sqrt{2}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-2}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{4M-2}(\bar\chi,n)e^{-2\pi n/N}.
\end{aligned}
$$

*Proof.* Let $\alpha = \beta = \pi$. Put $A = 0$ and $B = 2M-1$ in Theorem 3.10. Then

$$
\begin{aligned}
&\sum_{\ell=0}^{2M-1}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\left(\sin\left(\frac{2\pi n}{N}\right)+(-1)^M\cos\left(\frac{2\pi n}{N}\right)\right)\sigma_{2M}(\bar\chi,n)e^{-2\pi n/N}\\
&=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M}}\left(\sin\left(\frac{2\pi n}{N}\right)-(-1)^M\cos\left(\frac{2\pi n}{N}\right)\right)\sigma_{2M}(\bar\chi,n)e^{-2\pi n/N}.
\end{aligned}
$$

Replace $M$ by $2M-1$ and pull out the term with $\ell = 0$ to complete the proof. □

Let $M = 1$ in Corollary 3.12. Then we have

$$
\begin{aligned}
&\frac{4\pi}{N}\sum_{n=1}^{\infty}\frac{\chi(n)}{n}\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_2(\bar\chi,n)e^{-2\pi n/N}\\
&=\sqrt{2}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2}\cos\left(\frac{2\pi n}{N}\right)\sigma_2(\bar\chi,n)e^{-2\pi n/N}.
\end{aligned}
$$

Corollary 3.13. *Let $\chi$ be odd. For any integer $M \geq 1$,*

$$
\begin{aligned}
&\sum_{\ell=1}^{4M-1}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-\ell}}\sin\left(\frac{2\pi n}{N}+\frac{\pi}{4}\right)\sigma_{4M}(\bar\chi,n)e^{-2\pi n/N}\\
&=-\sqrt{2}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{4M}(\bar\chi,n)e^{-2\pi n/N}.
\end{aligned}
$$

*Proof.* Let $\alpha = \beta = \pi$. Put $A = 0$ and $B = 2M - 1$ in Theorem 3.10. Replace $M$ by $2M$ and pull out the term with $\ell = 0$. $\square$

For a prime $p$ with $p \equiv 3 \pmod 4$, we can put $\chi = \left(\frac{\cdot}{p}\right)$ in Theorem 3.8 – Theorem 3.10, Corollary 3.11 – Corollary 3.13.

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**Sung Geun Lim**
Department of Mathematics Education, Mokwon University,
Daejun 35349, Korea
*E-mail*: sglimj@mokwon.ac.kr