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SOFT INTERSECTION AND SOFT UNION k-IDEALS OF HEMIRINGS AND THEIR APPLICATIONS

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Abstract. The main aim of this paper is to discuss two different types of soft hemirings, soft intersection and soft union. We discuss applications and results related to soft intersection hemirings or soft intersection k-ideals and soft union hemirings or soft union k-ideals. The deep concept of k-closure, intersection and union of soft sets, $\wedge$-product and $\vee$-product among soft sets, upper β-inclusion and lower β-inclusion of soft sets is discussed here. Many applications related to soft intersection-union sum and soft intersection-union product of sets are investigated in this paper. We characterize k-hemiregular hemirings by the soft intersection k-ideals and soft union k-ideals.

1. Introduction

H. S. Vandiver [25] tell about the principle idea of semirings as a common generalization of rings and distributive lattices. Semirings are use in the fields of pure as well as applied mathematics. Several similar structures, and results related to semirings have been described by different researchers(See [9, 26, 27]). Semirings with additive inverse have been discussed by Karvellas [13]. Kaplansky [10], Petrich [19], Goodearl [10], Reutenauer [20] and Fang [15] have also discussed of semirings. Semirings are made for resolving problems in many fields of applied mathematics and information sciences. Hemirings, a semiring with zero, satisfying the commutative property of addition, have been discussed by Xueling Ma Jianming Zhan [16] in their research.

Ideals are use in studying semirings, group theory and rings. In the same way, ideals take a main role in different results and properties associated with hemirings. There are generalization of ideals but Henriksen, in [11], describes a special type of ideals which is k-ideals. La Torre, in [14], discussed many properties and results of k-ideals in hemirings. Some further generalizations of ideals such as h-ideals, and m-ideals can be looked over in [2, 4, 7, 8, 18, 23, 24]. Hemirings are applicable in automata

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and formal languages. Kanak Ray Chowdhury and others, in [6], discussed about the regular semirings. Basic concepts related to k-subhemirings are studied by Shabir et al., in [22–24].

Molodtsov [17] give the concepts of soft sets in the latest history of mathematics. Ali and others in [3] offer more new operations on soft sets. Presently, the studies and researches about algebraic structures of soft sets are expanded in different fields of mathematics like soft rings [1], soft group [5], soft semirings [7], soft BCK/BCI-Algebras [12], soft intersection near-rings [21] and so on. Soft sets were studied by many researchers but Xueling Ma and Jianming Zhan [16] studied soft set as an approximate function. They also discussed the intersection and union of soft sets, $\vee$-product , $\wedge$-product of different soft sets, Upper β-inclusion of different soft set. Jianming Zhan and others, in [28], introduced the basic ideas of soft union h-ideals of hemirings and investigated some characteristics of h-hemiregular hemirings by soft union h-ideals.

In this article, we study two different types of soft hemirings; soft intersection and soft union hemirings. We give applications and outcomes related to soft intersection hemirings, soft intersection k-ideals, soft union hemirings and soft union k-ideals. We deal with intersection and union of soft sets. Several applications which are linked with soft intersection-union sum, soft intersection-union product of soft sets are discussed here. We characterize k-hemiregular hemirings by means of soft intersection k-ideals, soft union k-ideals.

In order to present our research work in an organized way, in Section 2, we discuss some basic definitions which will be used in our further course of work. In Section 3, we deal with the idea of soft intersection hemirings and soft union hemirings, soft intersection k-ideals and derive some related properties. In Section 4, we discuss the characterizations of k-hemiregular hemirings by means of soft intersection k-ideals. We also discuss the charaterstics of soft union k-ideals of hemirings. The conclusion of the paper is presented in the final Section 5.

2. Preliminaries

Definition 2.1. [7]. Suppose $(S, +, \cdot)$ be the semiring having zero and $(S, +)$ be the commutative semigroup, then $(S, +, \cdot)$ is is said to be hemiring .

Definition 2.2. [7]. Suppose S is a hemiring and $\emptyset \neq P \subseteq S$. If the closure law holds in P with respect to $+$ and $\cdot$, then P is called a subhemiring of S .

Definition 2.3. [4]. Consider $(S, +, \cdot)$ is a hemiring, $\emptyset \neq Q \subseteq S$, so Q is a left ideal (right ideal) of S if $(\mathbf{T_1})$ $r^{/} + r^{//} \in Q$ for all $r^{/}, r^{//} \in Q$, and $(\mathbf{T_2})$ $SQ \subseteq Q (QS \subseteq Q)$. If Q is the left ideal and the right ideal of S, so Q be two-sided or simply an ideal of S.

Definition 2.4. [7]. Let X is a subhemiring (left ideal, right ideal, ideal) of S. If for any $t \in S$, and $r, w \in X, t + r = w$ implies $t \in X$, then X is a k-subhemiring (left k-ideal, right k-ideal, k-ideal) of S, respectively .

Definition 2.5. [16]. Let $X \subseteq S$, so k-closure of X, indicated by $\overline{X}$, is defined as

$$\overline{X} = \{t \in S | t + r = w \text{ for some } r, w \in X\}.$$

From here onward, we shall assume that S is a hemiring, R a universal set, T the set of parameters, $F(R)$ the power set of R and $X, Y, Z \subseteq T$.

Definition 2.6. [16]. A soft set of R, denoted by g_X, be the function defined as

$$g_X : T \to F(R) \text{ such that } g_X(m) = \emptyset \text{ if } m \notin X.$$

The soft set g_X is also said to be a approximate function. A soft set over R will be defined by the set of ordered pairs $g_X = \{(m, g_X(m)) | m \in T, g_X(m) \in F(R)\}$. So that soft set is a parametrized family of subsets of the universal set R. Therefore, the power set, $F(R)$, is the set of all soft sets over R.

Definition 2.7. [16]. Suppose $g_X, g_Y \in F(R)$, Then:

1. The intersection of g_X and g_Y, represented by $g_X \widetilde{\cap} g_Y$, is defined by $g_X \widetilde{\cap} g_Y = g_{X \widetilde{\cap} Y}$, where $g_{X \widetilde{\cap} Y}(m) = g_X(m) \cap g_Y(m), \forall m \in T$.
2. The union of g_X and g_Y, represented by $g_X \widetilde{\cup} g_Y$, is define by $g_X \widetilde{\cup} g_Y = g_{X \widetilde{\cup} Y}$, where $g_{X \widetilde{\cup} Y}(m) = g_X(m) \cup g_Y(m), \forall m \in T$.

Definition 2.8. [16]. Let $g_X, g_Y \in F(R)$, so $\wedge$-product and $\vee$-product of g_X and g_Y, denoted respectively by $g_X \wedge g_Y$ and $g_X \vee g_Y$, are respectively defined by $g_{X \wedge Y}(i, z) = g_X(i) \cap g_Y(z)$, and $g_{X \vee Y}(i, z) = g_X(i) \cup g_Y(z)$ for all $i, z \in T$.

Definition 2.9. Referring to [16, 28].

1. Let g_X be the soft set over R and $\beta \subseteq R$, so upper β-inclusion of g_X, represented by $R(g_X, \beta)$, be defined as $R(g_X; \beta) = \{t \in X | g_X(t) \supseteq \beta\}$.
2. Let g_X be the soft set over R and $\beta \subseteq R$, so lower β-inclusion of g_X, represented as $L(g_X, \beta)$, be defined as $L(g_X; \beta) = \{t \in X | g_X(t) \subseteq \beta\}$.

Definition 2.10. [28]. Let $g_X, g_Y \in F(R)$, therefore g_X is the soft subset of g_Y denoted by $g_X \widetilde{\subseteq} g_Y$ and is defined by $g_X(u) \widetilde{\subseteq} g_Y(u)\ \forall\ u \in T$.

Definition 2.11. [28]. Suppose $E \subseteq S$, then the soft characteristic function of the complement of E, indicated by S_{E^c}, is defined by

$$S_{E^c}(t) = \begin{cases} \emptyset \text{ if } t \in E, \\ R \text{ if } t \in S \backslash E. \end{cases}$$

Proposition 2.12. [28]. *Consider $L, M \subseteq S$. Therefore:*

1. $M \subseteq L \Rightarrow S_{L^c} \widetilde{\subseteq} S_{M^c}$,
2. $S_{L^c} \widetilde{\cup} S_{M^c} = S_{L^c R M^c}$.

3. Soft Intersection hemirings, Soft Intersection k-ideals and Soft Union hemirings

In this section, we work on the basic and important idea of soft intersection hemirings, soft interstion k-ideals and soft union hemirings and look over some of their associated properties.

Definition 3.1. Suppose g_S is a soft set over R. If
(SH$_1$) $g_S(m+u) \supseteq g_S(m) \cap g_S(u)$,
(SH$_2$) $g_S(mu) \supseteq g_S(m) \cap g_S(u)$,
(SH$_3$) $g_S(m) \supseteq g_S(r) \cap g_S(w)$ with $m+r=w \ \forall \ m,u, \in S$ for some $r,w \in S$,
then g_S be the soft intersection hemiring of S over R.

Example 3.2. Suppose $R = S = Z_4 = \{0,1,2,3\}$ is a hemiring of non-negative integers modulo 4. If we define the soft set g_S over R by $g_S(0) = \{0,1,2,3\} = g_S(2), g_S(1) = g_S(3) = \{0,2\}$, then g_S be the soft intersection hemiring of S over R.

Definition 3.3. Suppose g_S is a soft set over R. If
(C$_1$) $g_S(m+u) \supseteq g_S(m) \cap \ g_S(u)$
(C$_2$) $g_S(m) \supseteq g_S(r) \cap \ g_S(w)$ with $m+r=w \ \forall \ m,u \in S$, for some $r,w \in S$
(C$_3$) $g_S(mu) \supseteq g_S(u)$ for all $m,u \in S$,
then g_S is called a soft intersection left k-ideal over R. Similarly, soft intersection right k-ideal can be defined.
If the soft set over R be a soft intersection-left k-ideal and soft intersection-right k-ideal of S over R, then it is said to be a soft intersection k-ideal of S over R.

Example 3.4. Suppose $R = Z^+$ be a universal set and $S = Z_4$ be the set of parameters. Define a soft set g_S by

$$g_S(0) = \{m|m \in Z^+\}, g_S(1) = \{4m|m \in Z^+\}, g_S(2) = \{2m|m \in Z^+\} \text{ and } g_S(3) = \{3m|m \in Z^+\},$$

so g_S be the soft intersection-k-ideal of S over R.

Definition 3.5. Suppose g_S is the soft set over R. If
(SR$_1$) $g_S(m+u) \subseteq g_S(m) \cup g_S(u)$,
(SR$_2$) $g_S(mu) \subseteq g_S(m) \cup g_S(u)$,
(SR$_3$) $g_S(m) \subseteq g_S(r) \cup g_S(w)$ with $m+r=w$ for all $m,u, \in S$. for some $r,w \in S$,
so the soft set g_S over R is said to be a soft union hemiring of S over R.

Example 3.6. Consider $S = Z_6 = \{0,1,2,3,4,5\}$ is a hemiring of non-negative integers modulo 6. Suppose that $R = Z_5 = \{0,1,2,3,4\}$ be a universal set. Define the soft set g_S over R by

$$g_S(0) = \{1\}, g_S(2) = g_S(4) = \{1,2,3\}, g_S(1) = g_S(5) = \{1,2,3,4\}, g_S(3) = \{1,4\},$$

then g_S be a soft union hemiring of S over R.

EXAMPLE 3.7. Suppose $R = \left\{ \begin{pmatrix} \mathrm{m} & \mathrm{m} \\ \mathrm{m} & 0 \end{pmatrix} | m \in Z_5 \right\}$ be the set of 2×2 matrices with Z_5 as a universal set. Suppose $S = Z_6 = \{0,1,2,3,4,5\}$ is a hemiring of non-negative integers modulo 6. Define the soft set g_S over R as

$$g_S(0) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}, g_S(2) = g_S(4) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix} \right\}$$

$$g_S(1) = g_S(5) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix} \right\}, g_S(3) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix} \right\}.$$

Then, g_S be the soft union hemiring of S over R.

REMARK 3.8. If $g_S(t) = \emptyset$ for all $t \in S$, so g_S be a soft union-hemiring of S over R. We will represent this type of soft union-hemiring as $\widetilde{\eta}$. Also we see that $\widetilde{\eta} = S_{S^c}$.

DEFINITION 3.9. Let $l_S, h_S \in F(R)$, then

1. soft intersection -union sum, $l_S \oplus h_S$, be defined as

$$(l_S \oplus h_S)(t) = \begin{cases} \bigcap\limits_{t+r_1+w_1=r_2+w_2} (l_S(r_1) \cup l_S(r_2) \cup h_S(w_1) \cup h_S(w_2)) \\ R \text{ if it cannot be expressed as } t + r_1 + w_1 = r_2 + w_2. \end{cases}$$

2. Soft Intersection-union product, $l_S \diamond m_s$, is defined by

$$(l_S \diamond m_s)(t) = \begin{cases} \bigcap\limits_{t+\sum_{p=1}^{n} r_p w_p = \sum_{q=1}^{m} r_q^{/} w_q^{/}} (l_S(r_p) \cup l_S(r_q^{/}) \cup h_S(w_p) \cup h_S(w_q^{/})) \\ \forall p = 1,2,3,\cdots,n; q = 1,2,3,\cdots,m. \\ R \text{ if it cannot be expressed as } t + \sum\limits_{p=1}^{n} r_p w_p = \sum\limits_{q=1}^{m} r_q^{/} w_q^{/} \end{cases}$$

PROPOSITION 3.10. *Suppose g_{S_1} and g_{S_2} are two soft intersection hemirings of S_1 and S_2 over R respectively, then $g_{S_1} \wedge g_{S_2}$ is a soft intersection hemiring of $S_1 \times S_2$ over R.*

Proof. Let g_{S_1} and g_{S_2} be two soft intersection hemirings of S_1 and S_2 over R, respectively.

1. Let $(v_1, c_1), (v_2, c_2) \in S_1 \times S_2$, then

$$\begin{aligned} g_{S_1 \wedge S_2}((v_1, c_1) + (v_2, c_2)) &= g_{S_1 \wedge S_2}(v_1 + v_2, c_1 + c_2) \\ &= g_{S_1}(v_1 + v_2) \cap g_{S_2}(c_1 + c_2) \\ &\supseteq (g_{S_1}(v_1) \cap g_{S_1}(v_2)) \cap (g_{S_2}(c_1) \cap g_{S_2}(c_2)) \\ &= (g_{S_1}(v_1) \cap g_{S_2}(c_1)) \cap (g_{S_1}(v_2) \cap g_{S_2}(c_2)) \\ &= g_{S_1 \wedge S_2}(v_1, c_1) \cap g_{S_1 \wedge S_2}(v_2, c_2). \end{aligned}$$

2. Let, $(v_1, c_1), (v_2, c_2) \in S_1 \times S_2$, then

$$\begin{aligned}
g_{S_1 \wedge S_2}((v_1, c_1)(v_2, c_2)) &= g_{S_1 \wedge S_2}(v_1 v_2 + c_1 c_2, v_1 c_2 + c_1 v_2) \\
&= g_{S_1}(v_1 v_2 + c_1 c_2) \cap g_{S_2}(v_1 c_2 + c_1 v_2) \\
&\supseteq (g_{S_1}(v_1 v_2) \cap g_{S_1}(c_1 c_2)) \cap (g_{S_2}(v_1 c_2) \cap g_{S_2}(c_1 v_2)) \\
&\supseteq g_{S_1}(v_1) \cap g_{S_1}(v_2) \cap g_{S_1}(c_1) \cap g_{S_1}(c_2) \cap g_{S_2}(v_1) \cap \\
&\quad g_{S_2}(c_2) \cap g_{S_2}(c_1) g_{S_2}(v_2) \\
&= g_{S_1}(v_1) \cap g_{S_1}(c_1) \cap g_{S_2}(v_1) \cap g_{S_2}(c_1) \cap g_{S_1}(v_2) \cap \\
&\quad g_{S_1}(c_2) \cap g_{S_2}(v_2) \cap g_{S_2}(c_2) \\
&= g_{S_1 \wedge S_2}(v_1, c_1) \cap g_{S_1 \wedge S_2}(v_2, c_2).
\end{aligned}$$

3. Let $(j_1, j_2), (w_1, w_2), (v_1, v_2) \in S_1 \times S_2$ be such that $(v_1, v_2) + (j_1, j_2) = (w_1, w_2)$, and so $v_1 + j_1 = w_1$ and $v_2 + j_2 = w_2$. Then we get

$$\begin{aligned}
g_{S_1 \wedge S_2}(v_1, v_2) &= g_{S_1}(v_1) \cap g_{S_2}(v_2) \\
&\supseteq (g_{S_1}(j_1) \cap g_{S_1}(w_1)) \cap (g_{S_2}(j_2) \cap g_{S_2}(w_2)) \\
&= (g_{S_1}(j_1) \cap g_{S_2}(w_2)) \cap (g_{S_1}(w_1) \cap g_{S_2}(w_2)) \\
&= g_{S_1 \wedge S_2}(j_1, j_2) \cap g_{S_1 \wedge S_2}(w_1, w_2).
\end{aligned}$$

Therefore, $g_{S_1 \wedge S_2}$ is a soft intersection hemiring of $S_1 \times S_2$ over R. □

PROPOSITION 3.11. *Let g_{S_1} and g_{S_2} are two soft intersection-k-ideals of S_1 and S_2 over R respectively, then $g_{S_1} \wedge g_{S_2}$ is a soft intersection-k-ideal of $S_1 \times S_2$ over R.*

REMARK 3.12. Note that $g_{S_1} \vee g_{S_2}$ is not the soft intersection hemiring or soft intersection-k-ideal of $S_1 \times S_2$ over R.

EXAMPLE 3.13. Consider $R = S_3$, the symmetric group, be the universal set, $S_1 = Z_5 = \{0, 1, 2, 3, 4\}$ and $S_2 = Z_2 = \{0, 1\}$ be two hemirings of non-negative integers modulo 5 and modulo 2 respectively. Define two soft sets g_{S_1} and g_{S_2} over R by

$$g_{S_1}(0) = S_3, g_{S_1}(1) = g_{S_1}(4) = \{(1), (12), (132)\}, g_{S_2}(0) = S_3$$

$$g_{S_1}(2) = g_{S_1}(3) = \{(12), (123), (132)\}, g_{S_2}(1) = \{(1), (12), (132)\}.$$

It is clear that g_{S_1} and g_{S_2} are two soft intersection hemirings over R. However, we have

$$\begin{aligned}
g_{S_1 \vee S_2}((3, 1) + (1, 0)) &= g_{S_1 \vee S_2}(4, 1) \\
&= g_{S_1}(4) \cup g_{S_2}(1) \\
&= \{(1), (12), (132)\}
\end{aligned}$$

but

$$\begin{aligned} g_{S_1 \vee S_2}(3,1) \cap g_{S_1 \vee S_2}(1,0) &= (g_{S_1}(3) \cup g_{S_2}(1)) \cap (g_{S_1}(1) \cup g_{S_2}(0)) \\ &= \{(1),(12),(123),(132)\} \cap S_3 \\ &= \{(1),(12),(123),(132)\}. \end{aligned}$$

This implies that

$$g_{S_1 \vee S_2}((3,1)+(1,0)) \not\supseteq g_{S_1 \vee S_2}(3,1) \cap g_{S_1 \vee S_2}(1,0).$$

Hence, $g_{S_1 \vee S_2}$ is not a soft intersection-hemiring nor a soft intersection-k-ideal over R.

Theorem 3.14. *Consider l_S and m_S are two soft intersection hemirings of S over R so $l_S \widetilde{\cap} m_S$ be the soft intersection hemiring of S over R.*

Proof. Let l_S and m_S are two soft intersection hemirings of S over R.

1. Let $v, c \in S$, then

$$\begin{aligned} (l_S \widetilde{\cap} m_S)(v+c) &= l_S(v+c) \cap m_S(v+c) \\ &\supseteq (l_S(v) \cap l_S(c)) \cap (m_S(v) \cap m_S(c)) \\ &= (l_S(v) \cap m_S(v)) \cap (l_S(c) \cap m_S(c)) \\ &= (l_S \widetilde{\cap} m_S)(v) \cap (l_S \widetilde{\cap} m_S)(c). \end{aligned}$$

2. Let $v, c \in S$. Then, we get

$$\begin{aligned} (l_S \widetilde{\cap} m_S)(vc) &= l_S(vc) \cap m_S(vc) \\ &\supseteq (l_S(v) \cap l_S(c)) \cap (m_S(v) \cap m_S(c)) \\ &= (l_S(v) \cap m_S(v)) \cap (l_S(c) \cap m_S(c)) \\ &= (l_S \widetilde{\cap} m_S)(v) \cap (l_S \widetilde{\cap} m_S)(c). \end{aligned}$$

3. Let $j, w, v \in S$ such that $v + j = w$, then

$$\begin{aligned} (l_S \widetilde{\cap} m_S)(v) &= l_S(v) \cap m_S(v) \\ &\supseteq (l_S(j) \cap l_S(w)) \cap (m_S(j) \cap m_S(w)) \\ &= (l_S(j) \cap m_S(j)) \cap (l_S(w) \cap m_S(w)) \\ &= (l_S \widetilde{\cap} m_S)(j) \cap (l_S \widetilde{\cap} m_S)(w). \end{aligned}$$

Therefore $l_S \widetilde{\cap} m_S$ be the soft intersection-hemiring of S over R. □

Similarly, we can derive the next theorem:

Theorem 3.15. *Suppose l_S and m_S are two soft intersection-k-ideals of S over R, therefore $l_S \widetilde{\cap} m_S$ be the soft intersection-k-ideal of S over R.*

Theorem 3.16. *Suppose h_S is soft set over R, so h_S be the soft union hemiring of S over R iff it satisfies **(SR_3)** and **(SR_4)** $(h_S \oplus h_S) \widetilde{\supseteq} h_S$ and **(SR_5)** $(h_S \diamond h_S) \widetilde{\supseteq} h_S$.*

Proof. Suppose h_S be the soft union-hemiring of S over R. Consider $m \in S$. If $(h_S \oplus h_S)(m) = R$, then it is clear that $(h_S \oplus h_S)(m) \supseteq h_S(m)$. So

$$(h_S \oplus h_S)\widetilde{\supseteq} h_S.$$

Otherwise, consider $m + j_1 + w_1 = j_2 + w_2$ for some $j_1, w_1, j_2, w_2 \in S$, then

$$\begin{aligned}
(h_S \oplus h_S)(m) &= \bigcap_{m+j_1+w_1=j_2+w_2} (h_S(j_1) \cup h_S(j_2) \cup h_S(w_1) \cup h_S(w_2)) \\
&\supseteq \bigcap_{m+j_1+w_1=j_2+w_2} (h_S(j_1+w_1) \cup h_S(j_2+w_2)) \\
&\supseteq \bigcap_{m+j_1+w_1=j_2+w_2} (h_S(m)) \\
&= h_S(m).
\end{aligned}$$

Hence,

$$(h_S \oplus h_S)\widetilde{\supseteq} h_S.$$

Let $m \in S$. If $(h_S \diamond h_S)(m) = R$, then it is clear that $(h_S \diamond h_S)(m) \supseteq h_S(m)$. So $(h_S \diamond h_S)\widetilde{\supseteq} h_S$. Otherwise, consider

$$m + \sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{t} j_q^{/} w_q^{/} \text{ for all } p = 1, 2, 3, \cdots, n; q = 1, 2, 3, \cdots, t.$$

So

$$\begin{aligned}
(h_S \diamond h_S)(m) &= \bigcap_{m+\sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{t} j_q^{/} w_q^{/}} (h_S(j_p) \cup h_S(j_q^{/}) \cup h_S(w_p) \cup h_S(w_q^{/})) \\
&\supseteq \bigcap_{m+\sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{t} j_q^{/} w_q^{/}} (h_S(\sum_{p=1}^{n} j_p w_p) \cup h_S(\sum_{q=1}^{m} j_q^{/} w_q^{/})) \\
&\supseteq \bigcap_{m+\sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{t} j_q^{/} w_q^{/}} (h_S(m)) = h_S(m)
\end{aligned}$$

Hence

$$(h_S \diamond h_S)\widetilde{\supseteq} h_S.$$

Conversely, Suppose that the conditions (SR_3), (SR_4) and (SR_5) hold, then

$$\begin{aligned}
h_S(m+c) &\subseteq (h_S \oplus h_S)(m+c) \\
&= \bigcap_{m+c+j_1+w_1=j_2+w_2} (h_S(j_1) \cup h_S(j_2) \cup h_S(w_1) \cup h_S(w_2)) \\
&\subseteq h_S(m) \cup h_S(c) \cup h_S(0) \\
&= h_S(m) \cup h_S(c).
\end{aligned}$$

Thus $(\mathbf{SR}_1)$ holds.

$$\begin{aligned} h_S(mc) &\subseteq (h_S \diamond h_S)(mc) \\ &= \bigcap_{mc+\sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{t} j_q^{/} w_q^{/}} (h_S(j_p) \cup h_S(j_q^{/}) \cup h_S(w_p) \cup h_S(w_q^{/})) \\ \forall p &= 1,2,3,.....,n; q = 1,2,3,.....,t. \\ &\subseteq h_S(m) \cup h_S(c) \cup h_S(0) \\ &= h_S(m) \cup h_S(c). \end{aligned}$$

Thus $(\mathbf{SR}_2)$ hold.
Hence, l_s is a soft union hemiring of S over R. □

PROPOSITION 3.17. *Suppose $\emptyset \neq X \subseteq S$, then X be a k-subhemiring of S iff the soft subset h_S represented as*

$$h_S(d) = \begin{cases} \gamma & if\ d \in S \setminus X \\ \delta & if\ d \in X. \end{cases}$$

Proof. Consider X is a k-subhemiring of S. Let $v, c \in S$.
(1) If $v, c \in X$, then $vc, v + w \in X$.
So

$$h_S(v + c) = h_S(vc) = h_S(v) = h_S(c) = \delta,$$

and then

$$h_S(v + c) \subseteq h_S(v) \cup h_S(c) \text{ and } h_S(vc) \subseteq h_S(v) \cup h_S(c).$$

(2) If either one of v and c does not belong to X, then $v + c \in X$ or $v + c \notin X$ and $vc \in X$ or $vc \notin X$.
In any case,

$$h_S(v + c) \subseteq h_S(v) \cup h_S(c) = \gamma \text{ and } h_S(vc) \subseteq h_S(v) \cup h_S(c) = \gamma$$

(3) Now, let $j, w, v \in S$ such that $v + j = w$.
(i) If $j, w \in X$, then $v \in X$, and so $h_S(v) = h_S(j) \cup h_S(w) = \delta$.
(ii) If $j \notin X$ or $w \notin X$, then $h_S(v) \subseteq h_S(j) \cup h_S(w) = \gamma$.
Then, h_S is an soft union hemiring of S over R.
Conversely, suppose h_S be the soft union hemiring of S over R.
(1) Suppose $v, c \in X$, then $h_S(v + c) \subseteq h_S(v) \cup h_S(c) = \delta$
and $h_S(vc) \subseteq h_S(v) \cup h_S(c) = \delta$,

$$\Longrightarrow v + c, vc \in X.$$

(2) Suppose $v \in S$ and $j, w \in X$ such that $v + j = c$.
then $h_S(v) = h_S(j) \cup h_S(w) = \delta$.
So, $v \in X$. Hence X is a k-subhemiring of S. □

LEMMA 3.18. *1. Let h_S is soft set over R and $\gamma \subseteq R$ such that $\gamma \in I_n(h_S)$.*
If h_S is a soft union hemiring of S over R, so $L(h_S; \gamma)$ is a k-subhemiring of S.

2. *Suppose h_S is soft set over R, and $L(h_S;\gamma)$ a lower k-subhemiring of h_S for each $\gamma \subseteq R$, $I_n(h_S)$ an ordered set by inclusion so h_S be the soft union hemiring of S over R.*

Proof. 1. As $h_S(v) = \gamma$ for some $v \in S$, so $\emptyset \neq L(h_S;\gamma) \subseteq S$. Suppose $v, c \in L(h_S;\gamma)$, then $h_S(v) \subseteq \gamma$ and $h_S(c) \subseteq \gamma$. Now

$$h_S(v + c) \subseteq h_S(v) \cup h_S(c) \subseteq \gamma \cup \gamma = \gamma,$$

$$h_S(vc) \subseteq h_S(v) \cup h_S(c) \subseteq \gamma \cup \gamma = \gamma.$$

This implies that $v + c, vc \in L(h_S;\gamma)$. Now, Suppose $v \in S$ and $j, w \in L(h_S;\gamma)$ with $v + j = w$. Then, $h_S(j) \subseteq \gamma$ and $h_S(w) \subseteq \gamma$ and

$$h_S(v) \subseteq h_S(j) \cup h_S(w) = \gamma \cup \gamma = \gamma/$$

This implies that $v \in L(h_S;\gamma)$. Therefore, $L(h_S;\gamma)$ is a k-subhemiring of S.

2. Suppose $v, c \in S$ such that $h_S(v) = \gamma_1$ and $h_S(c) = \gamma_2$, where $\gamma_1 \subseteq \gamma_2$.

Then $v \in L(h_S;\gamma_1)$ and $c \in L(h_S;\gamma_2)$, and so $v \in L(h_S;\gamma_2)$.

We know that $L(h_S;\gamma)$ is a k-subhemiring of S, $\forall\ \gamma \subseteq R$. Thus, $v + c \in L(h_S;\gamma_2)$ and $vc \in L(h_S;\gamma_2)$.

Therefore

$$h_S(v + c) \subseteq \gamma_2 = \gamma_1 \cup \gamma_2 = h_S(v) \cup h_S(c),$$

$$h_S(vc) \subseteq \gamma_2 = \gamma_1 \cup \gamma_2 = h_S(v) \cup h_S(c).$$

Now, suppose $v, j, w \in S$ with $v + j = w$ such that $h_S(v) = \gamma_1$and $h_S(c) = \gamma_2$, where $\gamma_1 \subseteq \gamma_2$.

Then, $j \in L(h_S;\gamma_1)$ and $w \in L(h_S;\gamma_2)$, and so $j \in L(h_S;\gamma_2)$. As $L(h_S;\gamma)$ is a k-subhemiring of S for each $\gamma \subseteq R$, so $v \in L(h_S;\gamma_2)$. Then, $h_S(v) \subseteq \gamma_2 = \gamma_1 \cup \gamma_2 = h_S(v) \cup h_S(c)$.

Hence. h_S is a soft union hemiring of S over R.

□

4. k-hemiregular hemirings via soft intersection-k-ideals and soft union-k-ideals

In this section, we discuss the characterizations of k-hemiregular hemirings by means of soft intersection k-ideals. We also discuss the properties of soft union k-ideals of hemirings.

Definition 4.1. [22]. Suppose S is a hemiring, if for each $r \in S$, $\exists\ r_1, r_2 \in S$ such that $t + tr_1r = tr_2t$, then S is said to be k-hemiregular.

Lemma 4.2. [22]. *If H and M, are the right and the left k-ideal of S respectively, then $\overline{HM} \subseteq H \cap M$.*

Lemma 4.3. [22]. *A hemiring S be a k-hemiregular if and only if for any right k-ideal H and for any left k-ideal M, $\overline{HM} = H \cap M$.*

Definition 4.4. Let $l_S, m_S \in F(R)$, then the soft k-sum, $(l_S +_k m_S)$, and soft k-product, $(l_S \circ_k m_S)$, are respectively defined by

$$(l_S +_k m_S)(v) = \begin{cases} \bigcup\limits_{v+j_1+w_1=j_2+w_2} (l_S(j_1) \cap l_S(j_2) \cap m_S(w_1) \cap m_S(w_2)) \\ \emptyset \text{ if } v \text{ cannot be expressed as } v + j_1 + w_1 = j_2 + w_2, \end{cases}$$

and

$$(l_S \circ_k m_S)(v) = \begin{cases} \bigcup\limits_{v+j_1w_1=j_2w_2} (l_S(j_1) \cap l_S(j_2) \cap m_S(w_1) \cap m_S(w_2)) \\ \emptyset \text{ if } v \text{ cannot be expressed as } v + j_1w_1 = j_2w_2. \end{cases}$$

Lemma 4.5. *Let l_S and m_S be soft intersection right k-ideals and soft intersection left k-ideals of S over R respectively, then $l_S \circ_k m_S \widetilde{\subseteq} l_S \widetilde{\cap} m_S$.*

Proof. If $(l_S \circ_k m_S)(v) = \emptyset$, then it is clear that $l_S \circ_k m_S \widetilde{\subseteq} l_S \widetilde{\cap} m_S$. Otherwise, we have

$$\begin{aligned} (l_S \circ_k m_S)(v) &= \bigcup_{v+j_1w_1=j_2w_2} (l_S(j_1) \cap l_S(j_2) \cap m_S(w_1) \cap m_S(w_2)), \\ &\subseteq \bigcup_{v+j_1w_1=j_2w_2} (l_S(j_1w_1) \cap l_S(j_2w_2) \cap m_S(j_1w_1) \cap m_S(j_2w_2)), \\ &\subseteq \bigcup_{v+j_1w_1=j_2w_2} (l_S(v) \cap m_S(v)), \\ &= l_S(v) \cap m_S(v), \\ &= \left(l_S \widetilde{\cap} m_S\right)(v). \end{aligned}$$

This implies that

$$l_S \circ_k m_S \widetilde{\subseteq} l_S \widetilde{\cap} m_S$$

□

Definition 4.6. [16]. Let $H \subseteq S$, then S_H is soft characteristic function of H defined as

$$S_H(m) = \begin{cases} R \ \ if \ v \in H, \\ \emptyset \ \ if \ v \notin H. \end{cases}$$

Proposition 4.7. *Suppose $H, M \subseteq S$, so the following statements are satisfied:*

1. $H \subseteq M \Rightarrow S_H \widetilde{\subseteq} S_M$.
2. $S_H \widetilde{\cap} S_M = S_{H \cap M}$.
3. $S_H \circ_k S_M = S_{\overline{HM}}$.

Proof. 1. Let $H \subseteq M$. Suppose $i \in S_H$, then $i \in H$ or $i \notin H$. Since $H \subseteq M$, so $i \in M$ or $i \notin M$. This implies $i \in S_M$, which again implies $S_H \widetilde{\subseteq} S_M$.
2. Suppose $i \in S_H \widetilde{\cap} S_M$. Since $\left(S_H \widetilde{\cap} S_M\right)(i) = S_H(i) \cap S_M(i)$, there are two cases to be discussed.

(a) If $i \in H$ and $i \in M$, then

$$\left(S_H \widetilde{\cap} S_M\right)(i) = R.$$

(b) If $i \notin H$ and $i \notin M$, then

$$\left(S_H \widetilde{\cap} S_M\right)(i) = \emptyset.$$

Therefore, if $i \in H \cap M$, then $\left(S_H \widetilde{\cap} S_M\right)(i) = R$. If $i \notin H \cap M$, then

$$\left(S_H \widetilde{\cap} S_M\right)(i) = \emptyset.$$

Now, if $i \in H \cap M$, then by definition of characteristic function of $H \cap M$, we get

$$\left(S_H \widetilde{\cap} S_M\right)(i) = R.$$

If $i \notin H \cap M$, then by definition of characteristic function of $H \cap M$, we have

$$\left(S_H \widetilde{\cap} S_M\right)(i) = \emptyset.$$

Hence

$$S_H \widetilde{\cap} S_M = S_{H \cap M}.$$

3. Let $i \in (S_H \circ_k S_M)$, then

$$(S_H \circ_k S_M)(i) = \begin{cases} \bigcup\limits_{i+j_1w_1=j_2w_2} (S_H(j_1) \cap S_H(j_2) \cap S_M(w_1) \cap S_M(w_2)) \\ \text{if } i \text{ can be expressed as } i + j_1w_1 = j_2w_2 \\ \emptyset \text{ if } i \text{ cannot be expressed as } i + j_1w_1 = j_2w_2 \end{cases}$$

Now, let $i \in S_{\overline{HM}}$, then

$$S_{\overline{HM}}(i) = \begin{cases} R & \text{if } i \in \overline{HM} \\ \emptyset & \text{if } i \notin \overline{HM}. \end{cases}$$

If $i \in \overline{HM}$, then $i + j_1w_1 = j_2w_2$. If $i \notin \overline{HM}$, then i cannot be expressed as $i + j_1w_1 = j_2w_2$. Hence

$$S_H \circ_k S_M = S_{\overline{HM}}.$$

□

THEOREM 4.8. *Suppose S be the hemiring, Then the following statements are equivalent :*

1. S is k-hemiregular,
2. $h_S \circ_k m_S \widetilde{\subseteq} h_S \widetilde{\cap} m_S$ for any soft intersection right k-ideal h_S and soft intersection left k-ideal m_S of S over R, respectively.

Proof. (1) ⇒ (2).
Suppose S is k-hemiregular hemiring. Let h_S be the soft intersection right k-ideal over R and m_S be the soft intersection left k-ideal of S over R. If $(h_S \circ_k m_S)(o) = \emptyset$, then it is clear that $h_S \circ_k m_S \widetilde{\subseteq} h_S \widetilde{\cap} m_S$. Otherwise, we have

$$(h_S \circ_k m_S)(o) = \bigcup_{o+j_1w_1=j_2w_2} (h_S(j_1) \cap h_S(j_2) \cap m_S(w_1) \cap m_S(w_2)).$$

Since h_S is soft intersection right k-ideal over R, m_S is the soft intersection left k-ideal of S over R. So we get

$$\begin{aligned}(h_S \circ_k m_S)(o) &\subseteq \bigcup_{o+j_1w_1=j_2w_2}(h_S(j_1w_1)\cap h_S(j_2w_2)\cap m_S(j_1w_1)\cap m_S(j_2w_2))\\ &\subseteq \bigcup_{o+j_1w_1=j_2w_2}(h_S(o)\cap m_S(o))\\ &= h_S(o)\cap m_S(o)\\ &= \left(h_S\widetilde{\cap}m_S\right)(o)\end{aligned}$$

We conclude that

$$h_S \circ_k m_S \widetilde{\subseteq} h_S\widetilde{\cap}m_S$$

Suppose $o\in S$, then $\exists\ j, j^/ \in S$ such that

$$o+ojo=oj^/o.$$

Since S be k-hemiregular. Then we have

$$\begin{aligned}(h_S \circ_k m_S)(o) &= \bigcup_{o+j_1w_1=j_2w_2}(h_S(j_1)\cap h_S(j_2)\cap m_S(w_1)\cap m_S(w_2))\\ &\supseteq h_S(oj)\cap h_S(oj^/)\cap m_S(o)\\ &\supseteq h_S(o)\cap m_S(o)\\ &= \left(h_S\widetilde{\cap}m_S\right)(o)\end{aligned}$$

This implies that

$$h_S \circ_k m_S \widetilde{\supseteq} h_S\widetilde{\cap}m_S$$

Hence

$$h_S \circ_k m_S = h_S\widetilde{\cap}m_S.$$

(2) $\Rightarrow$ (1)
Let E be the right k-ideal, T be the left k-ideal of S. Furthermore, we see that S_R is a soft intersection right k-ideal over R, S_T be the soft intersection left k-ideal of S over R. Suppose $o\in E\cap T$, then,

$$S_{\overline{ET}}(o)=(S_E\circ_k S_T)(o)=(S_E\widetilde{\cap}S_T)(o)=S_{E\cap T}(o)=R$$

and so $o\in\overline{ET}$. Then $E\cap T\subseteq\overline{ET}$, but $\overline{ET}\subseteq E\cap T$ is true always. Thus,

$$E\cap T=\overline{ET}.$$

Hence S is k-hemiregular. □

Definition 4.9. Consider h_S be the soft set over R. Then, h_S is the soft union left k-ideal of S over R if it satisfies the given conditions:

($\mathbf{T}_1$) $h_S(o+c)\subseteq h_S(o)\cup h_S(c)$,
($\mathbf{T}_2$) $h_S(o)\subseteq h_S(j)\cup h_S(w)$ with $o+j=w\ \forall\ o,c,\in S$, for some $j,w\in S$,
($\mathbf{T}_3$) $h_S(oc)\supseteq h_S(c)$, $\forall\ o,c\in S$.

Similarly, soft union right k-ideal of S over R can be defined. If soft set over R is soft union left k-ideal as well as soft union right k-ideal of S over R, then this soft set over R is said to be a soft union k-ideal of S.

EXAMPLE 4.10. Consider $S = Z_6 = \{0, 1, 2, 3, 4, 5\}$ is hemiring of non-negative integers modulo 6, let $R = Z_6$. Define a soft set h_S over R by $h_S(0) = \{1\}, h_S(2) = h_S(4) = \{1, 2\}$, $h_S(1) = h_S(5) = \{1, 2, 3\}$ and $h_S(3) = \{1, 3\}$, then h_S be the soft union-k-ideal of S over R.

THEOREM 4.11. *Suppose h_S is the soft set over R, so h_S be a soft union-left (right) k-ideal of S over R iff it satisfies ($\mathbf{T}_2$) and ($\mathbf{SR}_4$)$(\widetilde{\gamma} \oplus h_S)\widetilde{\supseteq} h_S((h_S \oplus \widetilde{\gamma})\widetilde{\supseteq} h_S)$ and ($\mathbf{SR}_6$) $(\widetilde{\gamma} \diamond h_S)\widetilde{\supseteq} h_S((h_S \diamond \widetilde{\gamma})\widetilde{\supseteq} h_S)$.*

Proof. Suppose h_S is soft union-left k-ideal of S over R. Suppose $o \in S$. If $(\widetilde{\gamma} \oplus h_S)(o) = R$, so we can see that $(\widetilde{\gamma} \oplus h_S)(o) \supseteq h_S(o)$. So $(\widetilde{\gamma} \oplus h_S)\widetilde{\supseteq} h_S$. Otherwise, consider $o + j_1 + w_1 = j_2 + w_2$ for some $j_1, w_1, j_2, w_2 \in S$. Then

$$\begin{aligned}
(\widetilde{\gamma} \oplus h_S)(o) &= \bigcap_{o+j_1+w_1=j_2+w_2} (\widetilde{\gamma}(j_1) \cup \widetilde{\gamma}(j_2) \cup h_S(w_1) \cup h_S(w_2)) \\
&\supseteq \bigcap_{o+j_1+w_1=j_2+w_2} (\emptyset \cup h_S(j_1 + w_1) \cup h_S(j_2 + w_2)) \\
&\supseteq \bigcap_{o+j_1+w_1=j_2+w_2} (h_S(o)) \\
&= h_S(o).
\end{aligned}$$

Hence

$$(\widetilde{\gamma} \oplus h_S)\widetilde{\supseteq} h_S$$

Thus, ($\mathbf{SR}_4$) holds.

Let $o \in S$. If $(\widetilde{\gamma} \diamond h_S)(o) = R$, then it is clear that $(\widetilde{\gamma} \diamond h_S)(o) \supseteq h_S(o)$. Then $(\widetilde{\gamma} \diamond h_S)\widetilde{\supseteq} h_S$. Otherwise, consider

$$o + \sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{m} j_q^{/} w_q^{/} \forall p = 1, 2, 3, ..., n; q = 1, 2, 3, ..., m.$$

Thus

$$\begin{aligned}
(\widetilde{\gamma} \diamond h_S)(o) &= \bigcap_{o+\sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{m} j_q^{/} w_q^{/}} (\widetilde{\gamma}(j_p) \cup \widetilde{\gamma}(j_q^{/}) \cup h_S(w_p) \cup h_S(w_q^{/})) \\
&\supseteq \bigcap_{o+\sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{m} j_q^{/} w_q^{/}} (\emptyset \cup h_S(\sum_{p=1}^{n} j_p w_p) \cup h_S(\sum_{q=1}^{m} j_q^{/} w_q^{/})) \\
&\supseteq \bigcap_{o+\sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{m} j_q^{/} w_q^{/}} (h_S(o)) \quad \text{(by } (\mathbf{T}_2)) \\
&= h_S(o)
\end{aligned}$$

Hence

$$(\widetilde{\gamma} \diamond h_S)\widetilde{\supseteq} h_S$$

Thus ($\mathbf{SR}_6$) holds.

Conversely, suppose that the conditions ($\mathbf{T}_2$) and ($\mathbf{SR}_4$) and ($\mathbf{SR}_6$) hold.

So

$$\begin{aligned} h_S(o+c) &\subseteq (\widetilde{\gamma} \oplus h_S)(o+c) \\ &= \bigcap_{o+c+j_1+w_1=j_2+w_2} (\widetilde{\gamma}(j_1) \cup \widetilde{\gamma}(j_2) \cup h_S(w_1) \cup h_S(w_2)) \\ &\subseteq \emptyset \cup h_S(o) \cup h_S(c) \\ &= h_S(o) \cup h_S(c). \end{aligned}$$

Thus, $(\mathbf{T}_1)$ holds.

$$\begin{aligned} h_S(oc) &\subseteq (\widetilde{\gamma} \diamond h_S)(oc) \\ &= \bigcap_{oc+\sum_{p=1}^{n} j_p w_p = \sum_{q=1}^{m} j_q^{/} w_q^{/}} (\widetilde{\gamma}(j_p) \cup \widetilde{\gamma}(j_q^{/}) \cup h_S(w_p) \cup h_S(w_q^{/})) \\ \forall p &= 1,2,3,.....,n; q = 1,2,3,.....,m. \\ &\subseteq \widetilde{\gamma}(o) \cup h_S(c) \\ &= \emptyset \cup h_S(c) = h_S(c). \end{aligned}$$

Therefore, $(\mathbf{T}_3)$ holds. Thus, h_S is a soft union-left k-ideal of S over R.
Similarly, we can show that the above statement is true for soft union-right k-ideals. □

PROPOSITION 4.12. *Suppose h_S and m_S be two soft union-left or right k-deals of S over R, so $h_S \widetilde{\cup} m_S$ be the soft union-left (right) k-ideal of S over R.*

Proof. Consider h_S and m_S are two soft union-left k-deals of S over R. For any $o, c \in S$,

$$\begin{aligned} (h_S \widetilde{\cup} m_S)(o+c) &= h_S(o+c) \cup m_S(o+c) \subseteq h_S(o) \cup h_S(c) \cup m_S(o) \cup m_S(c) \\ &= (h_S(o) \cup m_S(o)) \cup (h_S(c) \cup m_S(c)) = (h_S \widetilde{\cup} m_S)(o) \cup (h_S \widetilde{\cup} m_S)(c) \end{aligned}$$

Then $(\mathbf{T}_1)$ holds. Now, let $o, j, w \in S$ with $o + j = w$, then

$$\begin{aligned} (h_S \widetilde{\cup} m_S)(o) &= h_S(j) \cup m_S(w) \subseteq (h_S(j) \cup h_S(w)) \cup (m_S(j) \cup m_S(w)) \\ &= ((h_S(j) \cup m_S(j)) \cup (h_S(w) \cup m_S(w)) = (h_S \widetilde{\cup} m_S)(j) \cup (h_S \widetilde{\cup} m_S)(w) \end{aligned}$$

Then $(\mathbf{T}_2)$ holds.
Now

$$(h_S \widetilde{\cup} m_S)(oc) = h_S(oc) \cup m_S(oc) \subseteq h_S(c) \cup m_S(c) = (h_S \widetilde{\cup} m_S)(c)$$

Then $(\mathbf{T}_3)$ holds.
Hence, $h_S \widetilde{\cup} m_S$ is a soft union-left k-deal of S over R.
Similarly we can prove that $h_S \widetilde{\cup} m_S$ be the soft union-right k-deal of S over R. □

PROPOSITION 4.13. *Consider l_S and m_S be two soft union-left (right) k-deals of S over R, therefore $l_S \diamond m_S$ is soft union-left(right) k-ideal of S over R.*

Proof. Consider l_S and m_S be two soft union-left k-deals of S over R. Suppose $v, c \in S$, so

$$\begin{aligned}
(l_S \diamond m_S)(v) \cup (l_S \diamond m_S)(c) &= \bigcap_{v+\sum\limits_{p=1}^{n} j_p w_p = \sum\limits_{q=1}^{m} j_q^{/} w_q^{/}} (l_S(j_p) \cup l_S(j_q^{/}) \cup m_S(w_p) \cup m_S(w_q^{/})) \cup \\
&\quad \bigcap_{c+\sum\limits_{p=1}^{o} e_p f_p = \sum\limits_{q=1}^{s} e_q^{/} f_q^{/}} (l_S(e_p) \cup l_S(e_q^{/}) \cup m_S(f_p) \cup m_S(f_q^{/})) \\
&= \bigcap_{v+\sum\limits_{p=1}^{n} j_p w_p = \sum\limits_{q=1}^{m} j_q^{/} w_q^{/}} \ \bigcap_{c+\sum\limits_{p=1}^{o} e_p f_p = \sum\limits_{q=1}^{s} e_q^{/} f_q^{/}} (l_S(j_p) \cup l_S(j_q^{/}) \cup m_S(w_p) \cup \\
&\quad m_S(w_q^{/}) \cup l_S(e_p) \cup l_S(e_q^{/}) \cup m_S(f_p) \cup m_S(f_q^{/})) \\
&\supseteq \bigcap_{v+c\sum\limits_{p=1}^{v} v_p c_p = \sum\limits_{q=1}^{c} v_q^{/} c_q^{/}} (l_S(v_p) \cup l_S(v_q^{/}) \cup m_S(c_p) \cup m_S(c_q^{/}) \\
(v_p c_p &= j_p w_p + e_p f_p, v_q^{/} c_q^{/} = j_q^{/} w_q^{/} + e_q^{/} f_q^{/}) \\
&= (l_S \diamond m_S)(v + c)
\end{aligned}$$

Thus, $(\mathbf{T}_1)$ holds.

$$\begin{aligned}
(l_S \diamond m_S)(c) &= \bigcap_{c+\sum\limits_{p=1}^{n} j_p w_p = \sum\limits_{q=1}^{m} j_q^{/} w_q^{/}} (l_S(j_p) \cup l_S(j_q^{/}) \cup m_S(w_p) \cup m_S(w_q^{/})) \\
&= \bigcap_{vc+\sum\limits_{p=1}^{n} (vj_p) w_p = \sum\limits_{q=1}^{m} (vj_q^{/}) w_q^{/}} (l_S(j_p) \cup l_S(j_q^{/}) \cup m_S(w_p) \cup m_S(w_q^{/}) \\
&\supseteq \bigcap_{vc+\sum\limits_{p=1}^{n} (vj_p) w_p = \sum\limits_{q=1}^{m} (vj_q^{/}) w_q^{/}} (l_S(vj_p) \cup l_S(vj_q^{/}) \cup m_S(w_p) \cup m_S(w_q^{/}) \\
&= (l_S \diamond m_S)(vc)
\end{aligned}$$

Therefore, $(\mathbf{T}_3)$ holds. Consider $v, j, w \in S$ with $v + j = w$, then similarly we can check

$$(l_S \diamond m_S)(j) \cup (l_S \diamond m_S)(w) \supseteq (l_S \diamond m_S)(v).$$

Therefore, $(\mathbf{T}_2)$ holds. Thus, $l_S \diamond m_S$ is soft union-left k-ideal of over R.
Similarly, we can prove that $l_S \diamond m_S$ is an soft union-right k-ideal of over R. □

THEOREM 4.14. *Suppose l_S is soft union-right k-deal of S over R and h_S is soft union-left k-deal of S over R, then $l_S \diamond h_S \widetilde{\supseteq} l_S \widetilde{\cup} h_S$.*

Proof. Let h_S be the soft union-right k-ideal of S over R, then

$$h_S \diamond m_S \widetilde{\supseteq} h_S \diamond \widetilde{\gamma} \widetilde{\supseteq} h_S$$

which implies that $h_S \diamond m_S \widetilde{\supseteq} h_S$. Also, when m_S is soft union-left k-deal of S over R, so

$$h_S \diamond m_S \widetilde{\supseteq} \widetilde{\gamma} \diamond m_S \widetilde{\supseteq} m_S$$

which implies that $h_S \diamond m_S \widetilde{\supseteq} m_S$. Hence,

$$h_S \diamond m_S \widetilde{\supseteq} h_S \widetilde{\cup} m_S.$$

□

5. Conclusion

In this paper we worked on two different types of soft hemirings, that is, soft intersection and soft union. Many applications and results related to soft intersection hemirings or soft intersection k-ideals and soft union hemirings or soft union k-ideals were discussed. The deep concept of k-closure also used in this paper. We also defined about $\wedge$-product and $\vee$-product among soft sets. Many applications related to soft intersection-union sum and soft intersection-union product of sets were also discussed. We characterized k-hemiregular hemirings by soft intersection k-ideals and soft union k-ideals. This work can be extended for different types of ideals in the future. The theory developed can be used in applied fields such as decision making and data analysis.

Declarations

Data availability:
The data used to support the findings of this study are available from the corresponding author upon request.

Competing interests:
The authors declare that they have no competing interests.

Authors' contributions:
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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