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NEW INEQUALITIES VIA BEREZIN SYMBOLS AND RELATED QUESTIONS

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Abstract. The Berezin symbol $\tilde{A}$ of an operator A on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω with the reproducing kernel k_λ is defined by

$$\tilde{A}(\lambda) = \left\langle A\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|} \right\rangle, \ \lambda \in \Omega.$$

The Berezin number of an operator A is defined by

$$ber(A) := \sup_{\lambda \in \Omega} \left| \tilde{A}(\lambda) \right|.$$

We study some problems of operator theory by using this bounded function $\tilde{A}$, including estimates for Berezin numbers of some operators, including truncated Toeplitz operators. We also prove an operator analog of some Young inequality and use it in proving of some inequalities for Berezin number of operators including the inequality $ber\,(AB) \leq ber\,(A)\,ber\,(B)$, for some operators A and B on $\mathcal{H}(\Omega)$. Moreover, we give in terms of the Berezin number a necessary condition for hyponormality of some operators.

1. Introduction

In this paper, we prove some inequalities for Berezin symbols of operators on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω. By using Berezin symbols, we estimate numerical radius and so-called Berezin number of some operators, including truncated Toeplitz operators, positive and some self-adjoint operators (Sections 2 and 3).

Recall that a reproducing kernel Hilbert space (shortly, RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that the evaluation functionals $\varphi_\lambda(f) = f(\lambda), \ \lambda \in \Omega$, are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem, for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The family $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$.

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The prototypical RKHSs are the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in C : |z| < 1\}$ is the unit disc, the Bergman space $L^2_a(\mathbb{D})$, the Dirichlet space $D^2(\mathbb{D})$ and the Fock space $F(\mathbb{C})$. A detailed presentation of the theory of RKHSs and reproducing kernels is given, for instance, in Aronzajn [1], Bergman [5], Malyshev [24], Halmos [13] and Saitoh and Sawano [28]. For other applications of operators, [6, 10, 19, 26, 27] studies can be consulted.

For A a bounded linear operator on $\mathcal{H}$ (i.e., for $A \in \mathcal{B}(\mathcal{H})$, the Banach algebra of all bounded linear operators on $\mathcal{H}$), its Berezin symbol (also called Berezin transform) $\tilde{A}$ is defined on Ω by (see Berezin [3, 4], and also Engliš [8])

$$\tilde{A}(\lambda) := \left\langle A\hat{k}_\lambda(z), \hat{k}_\lambda(z) \right\rangle,$$

where $\hat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of the space $\mathcal{H}$ and the inner product $<\ ,\ >$ is taken in the space $\mathcal{H}$. It is obvious that the Berezin symbol $\tilde{A}$ is a bounded function and $\sup_{\lambda\in\Omega} \left|\tilde{A}(\lambda)\right|$, which is called the Berezin number of operator A (see Karaev [16, 17]), does not exceed $\|A\|$, i.e.,

$$ber(A) := \sup_{\lambda\in\Omega} \left|\tilde{A}(\lambda)\right| \leq \|A\|.$$

It is also clear from the definition of Berezin symbol that the range of the Berezin symbol $\tilde{A}$, which is called the Berezin set of operator A (see Karaev [16, 17]), lies in the numerical range $W(A)$ of operator A, i.e.,

$$Ber(A) := range(\tilde{A}) = \{\tilde{A}(\lambda) : \lambda \in \Omega\} \subset$$
$$\subset W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

which implies that $ber(A) \leq w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H} \text{ and } \|x\| = 1\}$ (numerical radius of operator A). So, many questions, which are well studied for the numerical radius $w(A)$ of operator A, can be naturally asked for the Berezin number $ber(A)$ of operator A. For example, is it true, or under which additional conditions the following are true:

1^0 $ber(A) \geq \frac{1}{2}\|A\|$;

2^0 $ber(A^n) \leq ber(A)^n$ for any integer $n \geq 1$; more generally, if A is not nilpotent, then

$$C_1 ber(A)^n \leq ber(A^n) \leq C_1 ber(A)^n$$

for some constants $C_1, C_2 > 0$;

3^0 $ber(AB) \leq ber(A)\, ber(B)$, where $A, B \in \mathcal{B}(\mathcal{H})$;

If $A = cI$ with $c \neq 0$, then obviously $ber(A) = |c| > \dfrac{|c|}{2} = \dfrac{\|A\|}{2}$. In general, if T_φ is a Toeplitz operator on the Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ over the unit disk $\mathbb{D} = \{z \in C : |z| < 1\}$ with $\varphi \in L^\infty(\partial\mathbb{D})$, then by considering the well-known fact that [8, 32] $\widetilde{T_\varphi}(\lambda) = \widetilde{\varphi}(\lambda)$, $\widetilde{\varphi}$ is the harmonic extension of the function φ into the unit disk $\mathbb{D}$, it is easy to see that $ber(T_\varphi) = \|\varphi\|_\infty = \|T_\varphi\|$. Hence $ber(T_\varphi) \geq \dfrac{\|T_\varphi\|}{2}$. However, it is known that in general the above inequality 1^0 is not satisfied (see Karaev [18]).

In the present paper, we investigate some of above mentioned questions $1^0 - 3^0$. The related results are obtained in [11, 12, 31].

2. Inequalities for Berezin number

Let $\mathcal{H}(\Omega)$ be a RKHS of complex - valued functions on a set Ω. A subset $\mathcal{M}(\Omega)$ in $\mathcal{H}(\Omega)$ is called multiplier for space $\mathcal{H}(\Omega)$ if $\mathcal{M}(\Omega)\cdot\mathcal{H}(\Omega)\subset\mathcal{H}(\Omega)$, i.e., $fg\in\mathcal{H}(\Omega)$ for any $f\in\mathcal{M}(\Omega)$ and $g\in\mathcal{H}(\Omega)$. It follows from the closed graph theorem that the multiplication operator $M_f:\mathcal{H}(\Omega)\to\mathcal{H}(\Omega)$, $M_f g=fg$, is bounded for any $f\in\mathcal{M}(\Omega)$. We set $B(\Omega):=\{g\in\mathcal{H}(\Omega): g \text{ is bounded}\}$.

Our first result gives a lower estimate for the Berezin number of operator A.

Proposition 2.1. *Let $\mathcal{H}\in\mathcal{H}(\Omega)$ be a RKHS such that $B(\Omega)$ is a multiplier of $\mathcal{H}$. Let $\theta\in B(\Omega)$ be a function such that $|\theta(z)|\leq 1$ for all $z\in\Omega$. Let $A\in\{M_\theta\}'$, i.e., $AM_\theta=M_\theta A$, and let*

$$N_{\theta,A}:=M_\theta(I-AM_\theta M_\theta^*).$$

For every $\varepsilon\in(0,1)$, we set $K_{\varepsilon,\theta}:=\{z\in\Omega:|\theta(z)|\leq\varepsilon\}$.
Then

$$ber(A)\geq\sup_{0<\varepsilon<1}\left\|\theta-\tilde{N}_{\theta,A}\right\|_{L^\infty(K_{\varepsilon,\theta})}\varepsilon^{-3}. \tag{1}$$

Proof. Indeed, a standard calculus of Berezin symbol of operator $\tilde{N}_{\theta,A}$ shows that

$$\begin{aligned}\tilde{N}_{\theta,A}(\lambda)&=\left\langle\tilde{N}_{\theta,A}\hat{k}_\lambda,\ \hat{k}_\lambda\right\rangle=\langle M_\theta(I-AM_\theta M_\theta^*)\hat{k}_\lambda,\hat{k}_\lambda\rangle\\&=\langle(I-AM_\theta\, M_\theta^*)\hat{k}_\lambda,\ M_\theta^*\hat{k}_\lambda\rangle=\left\langle\hat{k}_\lambda-AM_\theta\overline{\theta(\lambda)}\hat{k}_\lambda,\ \ \overline{\theta(\lambda)}\hat{k}_\lambda\right\rangle\end{aligned}$$

(since M_θ is a multiplication operator on $\mathcal{H}$)

$$=\theta(\lambda)-|\theta(\lambda)|^2\left\langle AM_\theta\hat{k}_\lambda,\hat{k}_\lambda\right\rangle=\theta(\lambda)-|\theta(\lambda)|^2\langle M_\theta A\ \hat{k}_\lambda,\hat{k}_\lambda\rangle$$

(since $AM_\theta=M_\theta A$)

$$\begin{aligned}&=\theta(\lambda)-|\theta(\lambda)|^2\left\langle A\hat{k}_\lambda,M_\theta^*\hat{k}_\lambda\right\rangle=\theta(\lambda)-|\theta(\lambda)|^2\left\langle A\hat{k}_\lambda,\overline{\theta(\lambda)}\hat{k}_\lambda\right\rangle\\&=\theta(\lambda)-\theta(\lambda)\,|\theta(\lambda)|^2\,\tilde{A}(\lambda)\text{ for all }\lambda\in\Omega.\end{aligned}$$

Hence, $\tilde{N}_{\theta,A}(\lambda)=\theta(\lambda)(1-|\theta(\lambda)|^2\tilde{A}(\lambda))$, and therefore

$$\left|\theta(\lambda)-\tilde{N}_{\theta,A}(\lambda)\right|=\left|\tilde{A}(\lambda)\right||\theta(\lambda)|^3,\quad\forall\lambda\in\Omega.$$

In particular, we have from the latter for each $\lambda\in K_{\varepsilon,\theta}$, that

$$\left|\theta(\lambda)-\tilde{N}_{\theta,A}(\lambda)\right|\leq\varepsilon^3\left|\tilde{A}(\lambda)\right|,$$

and hence

$$\frac{\left\|\theta(\lambda)\ -\tilde{N}_{\theta,A}\right\|_{L^\infty(K_{\varepsilon,\theta})}}{\varepsilon^3}\leq\sup_{\lambda\in K_{\varepsilon,\theta}}\left|\tilde{A}(\lambda)\right|\leq\sup_{\lambda\in\Omega}\left|\tilde{A}(\lambda)\right|=ber(A),$$

which obviously implies (1). □

The Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z < 1|\}$ satisfying

$$\|f\|_2 := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^2 dt \right)^{\frac{1}{2}} < +\infty.$$

The normalized reproducing kernel $\hat{k}_\lambda$ of H^2 is the function $\frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\bar{\lambda} z}$.

Let $\theta \in H^2$ be an inner function, i.e., $|\theta(z)| \leq 1$ for all $z \in \mathbb{D}$ and $|\theta(e^{it})| = 1$ for almost all $t \in [0, 2\pi)$. We set $K_\theta := H^2 \ominus \theta H^2 = (\theta H^2)^\perp$ which is called the model space generated by an inner function θ. Let $\varphi \in L^2 = L^2(\mathbb{T})$, where $\mathbb{T} = \partial\mathbb{D}$ is the unit circle, be a function. The truncated Toeplitz operator acting on K_θ is defined by

$$T_\varphi^\theta f = \mathcal{P}_\theta(\varphi f),$$

where $\mathcal{P}_\theta : L^2 \to K_\theta$ is an orthogonal projection operator (see [29], [30] and also [2], and references therein).

It is easy to show that $\mathcal{P}_\theta = P_\theta P_+$, where $P_\theta : H^2 \to K_\theta$ is an orthogonal projection and $P_+ : L^2 \to H^2$ is the Riesz projection. Since $P_\theta P_+ L^2 = K_\theta$ and $(P_\theta P_+)^* = (P_+ P_\theta P_+)^* = P_+ P_\theta P_+ = P_\theta P_+$, it is enough only to show that $(P_\theta P_+)^2 = P_\theta P_+$.

Indeed, since θ is inner function, the associated analytic Toeplitz operator T_θ is an isometry, which implies that $T_\theta H^2 = \theta H^2$ is a closed subspace. Then $P_{\theta H^2} = T_\theta T_\theta^* = T_\theta T_{\bar{\theta}}$, and hence $P_{K_\theta} = I - T_\theta T_{\bar{\theta}}$. Therefore, by considering that $P_+ T_\theta = T_\theta$, and $T_\theta^* T_\theta = I$, we have:

$$\begin{aligned}
(P_\theta P_+)^2 &= (I - T_\theta T_{\bar{\theta}}) P_+ (I - T_\theta T_{\bar{\theta}}) P_+ \\
&= P_+ - P_+ T_\theta T_\theta^* P_+ - T_\theta T_\theta^* P_+ + T_\theta T_\theta^* P_+ T_\theta T_\theta^* P_+ \\
&= (I - T_\theta T_\theta^*) P_+ - T_\theta T_\theta^* P_+ + T_\theta T_\theta^* T_\theta T_\theta^* P_+ \\
&= (I - T_\theta T_\theta^*) P_+ - T_\theta T_\theta^* P_+ + T_\theta T_\theta^* P_+ = P_\theta P_+,
\end{aligned}$$

and hence $\mathcal{P}_\theta = P_\theta P_+$, as desired.

In general, there is also a very short proof:

$$(P_\theta P_+)^2 = P_\theta P_+ P_\theta P_+ = P_\theta P_\theta P_+ = P_\theta^2 P_+ = P_\theta P_H.$$

Our next result estimates in terms of Berezin numbers the Berezin number of the bounded truncated Toeplitz operator with the symbol in L_2.

THEOREM 2.2. *Let θ be a fixed inner function and $\varphi \in L_2$ be a function such that the corresponding truncated Toeplitz operator T_φ^θ is bounded on K_θ, i.e., $T_\varphi^\theta \in \mathcal{B}(K_\theta)$. Then*

$$ber(T_\varphi^\theta) \geq \frac{ber\left(T_{\left|1-\overline{\theta(\lambda)}\theta\right|^2 \varphi} \right)}{ber\,(P_\theta)}.$$

Proof. By condition, for $\varphi \in L_2$, the corresponding truncated Toeplitz operator T_φ^θ is bounded on K_θ. Then $w(T_\varphi^\theta) < +\infty$, and we have for any $f \in K_\theta$ with $\|f\|_2 = 1$ that

$$\left\langle T_\varphi^\theta f, f \right\rangle = \langle \mathcal{P}_\theta(\varphi f), f \rangle = \langle P_\theta P_+(\varphi f), f \rangle = \langle P_+(\varphi f), f \rangle = \langle T_\varphi f, f \rangle,$$

hence

$$\left\langle T_\varphi^\theta f, f \right\rangle = \langle T_\varphi f, f \rangle, \ \forall f \in K_\theta.$$

By considering that the normalized reproducing kernel $\hat{k}_{\theta,\lambda}(z)$ of the subspace K_θ is

$$\hat{k}_{\theta,\lambda}(z) := \left(\frac{1-|\lambda|}{1-|\theta(\lambda)|^2}\right)^{\frac{1}{2}} \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\bar{\lambda}z},\ \lambda \in \mathbb{D},$$

and $\tilde{T}_f = \tilde{f}$, where $\tilde{f}$ is the harmonic extension of the function $f \in L^1(\mathbb{T})$, we get :

$$\begin{aligned}
\left|\left\langle T_\varphi \hat{k}_{\theta,\lambda}, \hat{k}_{\theta,\lambda}\right\rangle\right| &= \frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\left|\left\langle T_\varphi \frac{1-\overline{\theta(\lambda)}\theta}{1-\bar{\lambda}z}, \frac{1-\overline{\theta(\lambda)}\theta}{1-\bar{\lambda}z}\right\rangle\right| \\
&= \frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\left|\left\langle T_\varphi T_{1-\overline{\theta(\lambda)}\theta}\frac{1}{1-\bar{\lambda}z}, T_{1-\overline{\theta(\lambda)}\theta}\frac{1}{1-\bar{\lambda}z}\right\rangle\right| \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left|\left\langle T^*_{1-\overline{\theta(\lambda)}\theta} T_\varphi T_{1-\overline{\theta(\lambda)}\theta}\frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\bar{\lambda}z}, \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\bar{\lambda}z}\right\rangle\right| \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left|\left\langle T_{\overline{1-\overline{\theta(\lambda)}\theta}\varphi(1-\overline{\theta(\lambda)}\theta)}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left|\left\langle T_{\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left|\tilde{T}_{\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi}(\lambda)\right| \\
&= \frac{\left|\left(\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi\right)^\sim(\lambda)\right|}{1-|\theta(\lambda)|^2},
\end{aligned}$$

for all $\lambda \in \Omega$, hence

$$\sup\left(1-|\theta(\lambda)|^2\right) ber(T^\theta_\varphi) \geq \sup_{\lambda\in\mathbb{D}}\left|\left(\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi\right)^\sim(\lambda)\right| = ber\left(T_{\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi}\right),$$

where $\left(\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi\right)^\sim$ is the harmonic extension of the function $\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi$. Since $\widetilde{P_\theta}(\lambda) = 1-|\theta(\lambda)|^2$, this proves the theorem. □

3. A refinement of Young inequality and its application

The classical Young inequality says that for any two positive real numbers a, b and $0 \leq \nu \leq 1$, we have

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b.$$

Moreover, if $a, b > 0$ and $\nu \leq 0$ or $\nu \geq 1$ then we have the supplementary Young inequality [9]

$$a^\nu b^{1-\nu} \geq \nu a + (1-\nu)b.$$

There are many refinements of Young's inequality in the literature, and we refer the interested reader to [14, 15, 23, 25]. In particular, the following inequality is proved by Manasrah and Kittaneh [25, formula (2.8)] : for any $a, b > 0$ and $\nu \leq 0$ or $\nu \geq 1$, we have

$$a^\nu b^{1-\nu} + a^{1-\nu} b^\nu + 2s_0\left(\sqrt{a}-\sqrt{b}\right)^2 \geq a+b, \tag{2}$$

where $s_0 := \min\{\nu, 1-\nu\}$.

In this section, we give an application of inequality (2) in estimation of Berezin number of the product of two operators. Namely, we prove the following.

THEOREM 3.1. *Let f be a bounded continuous function defined on an interval $J \subset (0, +\infty)$ and $f \geq 0$. If $\nu \leq 0$ or $\nu \geq 1$, then*

$$ber\,(f(T)) \leq ber\,(f^{\nu}(T))\, ber\left(f^{1-\nu}(T)\right)$$

for every self-adjoint operator T on the RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ with spectrum contained in J.

Proof. The proof essentially uses some general arguments of Kian's paper [20], where some Hardy-Hilbert type inequalities for Hilbert space operators are established. Let $a, b > 0$ be arbitrary numbers satisfying the inequality (2). Let $x, y \in J$. Noticing that $f(x) \geq 0$ for all $x \in J$ and putting $a = f(x)$ and $b = f(y)$ in (2), we obtain

$$f^{\nu}(x)f^{1-\nu}(y) + f^{1-\nu}(x)f^{\nu}(y) + 2s_0\left(\sqrt{f(x)} - \sqrt{f(y)}\right)^2 \geq f(x) + f(y), \tag{3}$$

for all $x, y \in J$.

Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$ be any self-adjoint operator with spectrum contained in J. Then passing to the functional calculus in inequality (3), we get for any $y \in J$ that

$$f^{\nu}(T)f^{1-\nu}(y) + f^{1-\nu}(T)f^{\nu}(y) + 2s_0\left(\sqrt{f(T)} - \sqrt{f(y)}\right)^2 \geq f(T) + f(y).$$

Now passing to the Berezin symbol in this inequality, we have for any $\lambda \in \Omega$ and $y \in J$ that

$$\begin{aligned}&\left\langle f^{\nu}(T)\hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle f^{1-\nu}(y) + \left\langle f^{1-\nu}(T)\hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle f^{\nu}(y) + 2s_0\left\langle \left(\sqrt{f(T)} - \sqrt{f(y)}\right)^2 \hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle \\ &\geq \left\langle f(T)\hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle + f(y),\end{aligned}$$

and hence

$$\widetilde{f^{\nu}(T)}(\lambda)f^{1-\nu}(y) + \widetilde{f^{1-\nu}(T)}(\lambda)f^{\nu}(y) + 2s_0\left(\widetilde{f(T)^{\frac{1}{2}}}(\lambda) - \sqrt{f(y)}\right)^2 \geq \widetilde{f(T)}(\lambda) + f(y).$$

Applying the functional calculus once more to the self-adjoint operator T, we get

$$\widetilde{f^{\nu}(T)}(\lambda)f^{1-\nu}(T) + \widetilde{f^{1-\nu}(T)}(\lambda)f^{\nu}(T) + 2s_0\left(\widetilde{f(T)^{\frac{1}{2}}}(\lambda) - \sqrt{f(T)}\right)^2 \geq \widetilde{f(T)}(\lambda) + f(T).$$

Again, taking Berezin symbol in this inequality, we have

$$2\widetilde{f^{\nu}(T)}(\lambda)\widetilde{f^{1-\nu}(T)}(\lambda) + 2s_0\left[\widetilde{f(T)}(\lambda) - \left(\widetilde{f(T)^{1/2}}(\lambda)\right)^2\right] \geq 2\widetilde{f(T)}(\lambda),$$

and hence

$$\widetilde{f^{\nu}(T)}(\lambda)\widetilde{f^{1-\nu}(T)}\,(\lambda) + s_0\widetilde{f(T)}(\lambda) - s_0\left(\widetilde{f(T)^{1/2}}(\lambda)\right)^2 \geq \widetilde{f(T)}(\lambda).$$

Therefore

$$\widetilde{f^{\nu}(T)}(\lambda)\widetilde{f^{1-\nu}(T)}\,(\lambda) - s_0\left(\widetilde{f(T)^{1/2}}(\lambda)\right)^2 \geq (1-s_0)\,\widetilde{f(T)}(\lambda).$$

Since $s_0 < 0$, we obtain that

$$(1-s_0)\,\widetilde{f(T)}(\lambda) \leq \widetilde{f^{\nu}(T)}(\lambda)\widetilde{f^{1-\nu}(T)}\,(\lambda) + |s_0|\left(\widetilde{f(T)^{1/2}}(\lambda)\right)^2$$

for all $\lambda \in \Omega$. Hence

$$(1-s_0)\,ber\,(f(T)) \leq ber\,(f^{\nu}(T))\,ber\left(f^{1-\nu}(T)\right) + |s_0|\,ber\left(\sqrt{f\,(T)}\right)^2,$$

and thus

$$ber\,(f(T)) \leq \frac{1}{1-s_0}ber\,(f^{\nu}(T))\,ber\left(f^{1-\nu}(T)\right) + \frac{|s_0|}{1-s_0}ber\left(\sqrt{f\,(T)}\right)^2.$$

By applying McCarthy inequality, we have from the latter that

$$ber\,(f(T)) \leq \frac{1}{1-s_0}ber\,(f^{\nu}(T))\,ber\left(f^{1-\nu}(T)\right) + \frac{|s_0|}{1-s_0}ber\,(f\,(T)),$$

and hence

$$\left(1-\frac{|s_0|}{1-s_0}\right)ber\,(f\,(T)) \leq \frac{1}{1-s_0}ber\,(f^{\nu}(T))\,ber\left(f^{1-\nu}(T)\right). \tag{4}$$

Since $s_0 = \min\{\nu, 1-\nu\} < 0$, we have $1-s_0 > 1$ and $s_0 + |s_0| = 0$, and hence it follows from (4) that

$$ber\,(f(T)) \leq ber\,(f^{\nu}(T))\,ber\left(f^{1-\nu}(T)\right) \quad (\nu \leq 0 \text{ or } \nu \geq 1),$$

which proves the theorem. □

Notice that since $f(T) = f^{\nu}(T)f^{1-\nu}(T)$, Theorem 3.1 partially solves question 3^0 in Section 1. Also, if $f(t) \equiv t$ on J, then we have

$$ber(T) \leq ber(T^{\nu})ber(T^{1-\nu}).$$

The well known Lowner-Heins inequality says that if $A \geq B \geq 0$ are two operators on a Hilbert space H, then $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$ (see Furuta [9, §3.2]).

The following is an immediate corollary of this inequality.

Proposition 3.2. *If $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ and $A \geq B \geq 0$, then*

$$ber(A^{\alpha}) \geq ber(B^{\alpha})$$

for any $\alpha \in [0,1]$.

Proposition 3.3. *Let $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ be two operators such that $A \geq B > 0$. Then we have:*

(a) $ber(\log A) \geq ber(\log B)$;

(b) If A, B are invertible operators, then

$$ber(A^r) \geq ber(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$$

for all $p \geq 0$ and $r \geq 0$.

Proof. (a) Indeed, if $A \geq B > 0$, then $A^{\alpha} \geq B^{\alpha} > 0$ for any $\alpha \in [0,1]$ by the Lowner-Heinz inequality [9], and consequently

$$\frac{A^{\alpha}-I}{\alpha} \geq \frac{B^{\alpha}-I}{\alpha}.$$

Hence by tending $\alpha \to 0^+$, we have that $\log A \geq \log B$. On the other hand, by virtue of Theorem 2 in [9, §3.2.3], the last inequality is equivalent to $A^r \geq (A^{r/2}B^pA^{r/2})^{\frac{r}{p+r}}$ for all $p \geq 0$ and $r \geq 0$. This implies that

$$\tilde{A^r}(\lambda) \geq \left[(A^{r/2}B^pA^{r/2})^{r/p+r}\right]^{\sim}(\lambda)$$

for all $\lambda \in \Omega$. So, by taking supremum, we get the desired result. □

To prove our next results, we need several well-known lemmas which are respectively the simple consequences of the classical Jensen and Young inequalities (see [14]); spectral theorem for positive operators and Jensen's inequality (see Kittaneh [21,23]); and the generalized mixed Schwarz inequality (see Kittaneh [21]).

Lemma 3.4. *For $a, b > 0$, $0 \leq \alpha \leq 1$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

(a) $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq [\alpha a^r + (1-\alpha) b^r]^{\frac{1}{r}}$ for $r \geq 1$;

(b) $ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}$ for $r \geq 1$.

Lemma 3.5 (McCarty inequality). *Let $A \in \mathcal{B}(\mathcal{H})$, $A \geq 0$ and let $x \in \mathcal{H}$ be any unit vector. Then*
(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$;
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $0 < r \leq 1$.

Lemma 3.6. *Let $A \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any unit vectors.*
(i) If $0 \leq \alpha \leq 1$, then

$$|\langle Ax, y\rangle|^2 \leq \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle.$$

(ii) If f, g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)\,g(t) = t$ $(t \geq 0)$, then

$$|\langle Ax, y\rangle| \leq \|f(|A|)\,x\|\,\|g(|A^*|)\,y\|;$$

here $|A| := (A^*A)^{1/2}$ denotes the so-called modul of operator A.

We set

$$\mathcal{F} = \{(f, g) \in C[0, \infty) \times C[0, \infty) : f, g \geq 0 \text{ and } f(t)\,g(t) = t \ \ (t \geq 0)\}.$$

Clearly, $(t^\alpha, t^{1-\alpha}) \in \mathcal{F}$ for any $\alpha \in [0, 1]$, and therefore $\mathcal{F}$ is nonempty.

The following corollary of this lemma gives in particular a refinement of the general inequality

$$ber(A) \leq \sup_{\lambda \in \Omega} \left\|A\widehat{k}_\lambda\right\|. \tag{5}$$

Corollary 3.7. *Let $A \in \mathcal{B}(\mathcal{H}(\Omega))$ be an operator.*
(i) If $0 \leq \alpha \leq 1$, then $ber(A)^2 \leq ber\left(|A|^{2\alpha}\right) ber\left(|A^*|^{2(1-\alpha)}\right)$.
(ii) $ber(A) \leq \inf_{(f,g)\in\mathcal{F}} \sup_{\lambda\in\Omega} \left(\left\|f(|A|)\,\widehat{k}_\lambda\right\| \left\|g(|A^*|)\,\widehat{k}_\lambda\right\|\right)$.

Proof. (i) Put $x = y = \widehat{k}_\lambda$ in Lemma 3.6 (i) and take the supremum over $\lambda \in \Omega$, we obtain the desired inequality. The proof of (ii) is quite similar to the proof of (i). □

Since $(t,1) \in \mathcal{F}$, it is clear from Corollary 3.7 (ii) that

$$\begin{aligned} ber\,(A) &\leq \sup_{\lambda\in\Omega} \left\| |A|\,\widehat{k}_\lambda \right\| = \sup_{\lambda\in\Omega} \left\langle |A|\,\widehat{k}_\lambda, |A|\,\widehat{k}_\lambda \right\rangle^{1/2} \\ &= \sup_{\lambda\in\Omega} \left\langle |A|^*\,|A|\,\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{1/2} = \sup_{\lambda\in\Omega} \left\langle |A|^2\,\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{1/2} \\ &= \sup_{\lambda\in\Omega} \left\langle A^*A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{1/2} = \sup_{\lambda\in\Omega} \left\| A\widehat{k}_\lambda \right\|. \end{aligned}$$

So, we have that

$$ber\,(A) \leq \inf_{(f,g)\in\mathcal{F}} \sup_{\lambda\in\Omega} \left(\left\| f\,(|A|)\,\widehat{k}_\lambda \right\| \left\| g\,(|A^*|)\,\widehat{k}_\lambda \right\| \right) \leq \sup_{\lambda\in\Omega} \left\| A\widehat{k}_\lambda \right\|,$$

which is a refinement of the inequality (5).

Now we are in a position to state our next results.

THEOREM 3.8. *If $A \in \mathcal{B}(\mathcal{H}(\Omega))$, then*

$$ber^{2r}\,(A) \leq \frac{1}{2} \left[\left(ber\,\left(|A|^2\right) ber\,\left(|A^*|^2\right) \right)^{\frac{r}{2}} + ber^r\,\left(A^2\right) \right]$$

for any $r \geq 1$.

Proof. The following refinement of the Cauchy-Schwarz inequality proved by Dragomir [7]:

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - \langle x,e\rangle\,\langle e,y\rangle| + |\langle x,e\rangle\,\langle e,y\rangle| \geq |\langle x,y\rangle|\,, \tag{6}$$

for all $x, y, e \in \mathcal{H}$ and $\|e\| = 1$. From inequality (6), we conclude that

$$\frac{1}{2}\left(\|x\|\,\|y\| + |\langle x,y\rangle|\right) \geq |\langle x,e\rangle\,\langle e,y\rangle|\,.$$

Putting $e = \widehat{k}_\lambda$, $x = A\widehat{k}_\lambda$ and $y = A^*\widehat{k}_\lambda$ in the above inequality and using Lemma 3.4 (i), we have

$$\begin{aligned} \left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 &\leq \frac{1}{2} \left(\left\| A\widehat{k}_\lambda \right\| \left\| A^*\widehat{k}_\lambda \right\| + \left| \left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \right) \\ &\leq \left(\frac{ \left\| A\widehat{k}_\lambda \right\|^r \left\| A^*\widehat{k}_\lambda \right\|^r + \left| \left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r }{2} \right)^{1/r} \\ &= \left(\frac{ \left\langle A^*A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{r}{2}} \left\langle AA^*\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{r}{2}} + \left| \left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r }{2} \right)^{1/r}, \end{aligned}$$

whence

$$\left| \widetilde{A}\,(\lambda) \right|^{2r} \leq \frac{1}{2} \left(\widetilde{A^*A}\,(\lambda)^{\frac{r}{2}}\, \widetilde{AA^*}\,(\lambda)^{\frac{r}{2}} + \left| \widetilde{A^2}\,(\lambda) \right|^r \right),$$

and hence

$$\left| \widetilde{A}\,(\lambda) \right|^{2r} \leq \frac{1}{2} \left(\widetilde{|A|^2}\,(\lambda)^{\frac{r}{2}}\, \widetilde{|A^*|^2}\,(\lambda)^{\frac{r}{2}} + \left| \widetilde{A^2}\,(\lambda) \right|^r \right) \tag{7}$$

for all $\lambda \in \Omega$.

Taking the supremum over $\lambda \in \Omega$ in inequality (7), we obtain the desired inequality. □

Proposition 3.9. *Let $A \in \mathcal{B}(\mathcal{H}(\Omega))$ and $(f,g) \in \mathcal{F}$. Then*

$$ber^{2r}(A) \le \frac{1}{2}\left[\left(ber\left(|A|^2\right) ber\left(|A^*|^2\right)\right)^{\frac{r}{2}} + ber\left(\frac{1}{p} f^{pr}\left(\left|A^2\right|\right) + \frac{1}{q} g^{qr}\left(\left|A^{*2}\right|\right)\right)\right] \tag{8}$$

for all $r \ge 1$, $p \ge q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $qr \ge 2$.

Proof. Let $\lambda \in \Omega$ be any number. Then we have:

$$\begin{aligned}
\left|\left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r &\le \left\|f\left(\left|A^2\right|\right)\widehat{k}_\lambda\right\|^r \left\|g\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda\right\|^r \text{ (by Lemma 3.6 } (ii)\text{)}\\
&= \left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{r}{2}} \left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{r}{2}}\\
&\le \frac{1}{p}\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{pr}{2}} + \frac{1}{q}\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{qr}{2}} \text{ (by Lemma 3.4 } (ii)\text{)}\\
&\le \frac{1}{p}\left\langle f^{pr}\left(\left|A^2\right|\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle g^{qr}\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \text{ (by Lemma 3.5 } (i)\text{)}\\
&= \left\langle\left(\frac{1}{p} f^{pr}\left(\left|A^2\right|\right) + \frac{1}{q} g^{qr}\left(\left|\left(A^2\right)^*\right|\right)\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle.
\end{aligned}$$

Taking the supremum over $\lambda \in \Omega$ in this inequality, we have

$$ber^r\left(A^2\right) \le ber\left(\frac{1}{p} f^{pr}\left(\left|A^2\right|\right) + \frac{1}{q} g^{qr}\left(\left|A^{*2}\right|\right)\right). \tag{9}$$

Now it follows from the inequalities (7) and (9) the desired inequality (8). □

Notice that inequality (8) induces several Berezin number inequalities as special cases. For example, if we take $f(t) = t^\alpha$, $g(t) = t^{1-\alpha}$ and $p = q = 2$ in inequality (8), then we get the next result.

Corollary 3.10. *If $A \in \mathcal{B}(\mathcal{H}(\Omega))$, then*

$$ber^{2r}(A) \le \frac{1}{2}\left[\left(ber\left(|A|^2\right) ber\left(|A^*|^2\right)\right)^{\frac{r}{2}} + ber\left(\left|A^2\right|^{2\alpha r} + \left|A^{*2}\right|^{2(1-\alpha)r}\right)\right]$$

for all $r \ge 1$ and $0 \le \alpha \le 1$. In particular, when $r = 1$ and $\alpha = \frac{1}{2}$, we have

$$ber^2(A) \le ber\left(A^2\right)$$

for any self-adjoint operator $A \in \mathcal{B}(\mathcal{H}(\Omega))$.

In conclusion of this section, recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is hyponormal if $[A^*, A] := A^*A - AA^* \ge 0$. The following is immediate from McCarty inequality (see Lemma 3.5) and arithmetic-geometric mean formula.

Proposition 3.11. *We have:*

$$\sup_{A \ge 0} \frac{ber\left(A^{1/2}\right)}{1 + ber(A)} \le \frac{1}{2}.$$

An immediate corollary of this proposition is the following.

Corollary 3.12. *If $A \in \mathcal{B}(\mathcal{H})$ is an hyponormal operator, then*

$$\frac{ber\left([A^*, A]^{1/2}\right)}{1 + ber\left([A^*, A]\right)} \le \frac{1}{2}.$$

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