Korean J. Math. **30** (2022), No. 2, pp. 283–296
http://dx.doi.org/10.11568/kjm.2022.30.2.283

IDENTITIES PRESERVED UNDER EPIS OF PERMUTATIVE POSEMIGROUPS

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Abstract. In 1985, Khan gave some sufficient conditions on semigroup identities to be preserved under epis of semigroups in conjunction with the general semigroup permutation identity. But determination of all identities which are preserved under epis in conjunction with the general permutation identity is an open problem in the category of all semigroups and hence, in the category of all posemigroups. In this paper, we first find some sufficient conditions on an identity to be preserved under epis of posemigroups in conjunction with any nontrivial general permutation identity. We also find some sufficient conditions on posemigroup identities to be preserved under epis of posemigroups in conjunction with the posemigroup permutation identity, not a general permutation identity.

1. Introduction

The determination of all identities which are preserved under epis in conjunction with the general permutation identity is an open problem in the category of all semigroups and hence, in the category of all posemigroups. However, in ([8], Theorem 4.7) Khan gave some sufficient conditions on semigroup identities to be preserved under epis of semigroups in conjunction with the general semigroup permutation identity. In this paper we are able to generalize the results due to Khan to posemigroups.

Also, in ([4], Theorem 3.9), Ahanger, Shah, and Khan partially generalized the results due of Khan ([8], Theorem 4.7) in the category of posemigroups. In this paper we fully generalize the above results of Khan by relaxing the assumption taken by Ahanger, Shah, and Khan [4]. We also find some sufficient conditions on posemigroup identities to be preserved under epis of posemigroups in conjunction with posemigroup permutation identity, not a general permutation identity.

2. Preliminaries

A partially ordered semigroup, briefly posemigroup is a pair $(S, \leq)$ comprising a semigroup S and a partial order $\leq$ on S that is compatible with its binary operation, i.e. for all $s_1, s_2, t_1, t_2 \in S, s_1 \leq t_1$ and $s_2 \leq t_2$ implies $s_1 s_2 \leq t_1 t_2$. We call $(U, \leq_U)$ a

Received September 30, 2021. Revised April 28, 2022. Accepted June 1, 2022.
2010 Mathematics Subject Classification: 06F05, 20M07, 20M10.
Key words and phrases: Posemigroups, Dominion, Zigzag, Identity.
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subposemigroup of a posemigroup $(S, \leq_S)$ if U is subsemigroup of the semigroup S and $\leq_U = \leq_S \cap (U \times U)$.

A posemigroup morphism $f : (S, \leq_S) \to (T, \leq_T)$ is a monotone ($x \leq_S y \Rightarrow f(x) \leq_T f(y)$) semigroup morphism. We shall also denote posemigroups by S, T etc. whenever no explicit mention of the order relation is required.

A class of posemigroups is called a variety of posemigroups if it is closed under taking the products (endowed with componentwise operation and order), morphic images and subposemigroups. It is also possible to discribe posemigroup varieties alternatively with the help of inequalities using a Birkhoff type characterization; we refer to [5] for details.

Let S and T be posemigroups and $f : S \to T$ be a posemigroup morphism. Then f is said be an epimorphism (epi) if for any posemigroup W and any posemigroup morphisms $\alpha, \beta : T \to W$, $\alpha \circ f = \beta \circ f$ implies $\alpha = \beta$. We observe that $f : S \to T$ is necessarily a posemigroup epimorphism if $f : S \to T$ is semigroup epimorphism, where in the latter case we disregard the orders (and hence the monotonocity) and treat S and T as semigroups.

Let U be a subposemigroup of a posemigroup S and $d \in S$. We say that U dominates d if for all $\alpha, \beta : S \to T$ posemigroup morphisms, such that $\alpha(u) = \beta(u)$ for all $u \in U$, one has $\alpha(d) = \beta(d)$. The set of all elements of S that are dominated by U is called the posemigroup dominion of U in S and is denoted by $\widehat{Dom}(U, S)$. One can easily verify that $\widehat{Dom}(U, S)$ is a subposemigroup of S containing U.

An identity $u = v$ is said to be preserved under posemigroup epis if for all posemigroups U and S with U as a subposemigroup of S such that $\widehat{Dom}(U, S) = S$, U satisfies $u = v$ implies S also satisfies $u = v$.

The following characterization of posemigroup dominion is provided by Sohail and Tart called the Zigzag Theorem for posemigeroups and will frequently be used in whatever follows.

Theorem 2.1. ([9], Theorem 5) *Let U be a subposemigroup of a posemigroup S. Then we have $d \in \widehat{Dom}(U, S)$ if and only if $d \in U$ or*

$$\begin{aligned} d &\leq x_1 u_0, & u_0 &\leq u_1 y_1 \\ x_i u_{2i-1} &\leq x_{i+1} u_{2i}, & u_{2i} y_i &\leq u_{2i+1} y_{i+1}, \ 1 \leq i \leq n-1 \\ x_n u_{2n-1} &\leq u_{2n}, & u_{2n} y_n &\leq d \end{aligned} \tag{1}$$

$$\begin{aligned} v_0 &\leq s_1 v_1, & d &\leq v_0 t_1 \\ s_j v_{2j} &\leq s_{j+1} v_{2j+1} & v_{2j-1} t_{j-1} &\leq v_{2j} t_j, \ 1 \leq j \leq m-1 \\ s_m v_{2m} &\leq d, & v_{2m-1} t_m &\leq v_{2m}, \end{aligned} \tag{2}$$

where, $u_0, v_0, \ldots u_{2n}, v_{2m} \in U$; $x_1, y_1, \ldots, x_n, y_n, s_1, t_1, \ldots, s_m, t_m \in S$.

Let us call the above inequalities, posemigroup zigzag inequalities in S over U with value d and length (n, m) and we say that it is of minimal length (n, m) if n and m are the least positive integers.

The next theorems are from [1] and are very important for our investigations.

Theorem 2.2. ([1], Lemma 3.2) *Let* $d \in \widehat{Dom}(U,S) \setminus U$ *and let (1) and (2) be the zigzag inequalities for* d *of minimal length* (n, m), *then* $x_i, y_i \in S \setminus U$ *for* $i = 1, 2, \ldots, m$ *and* $s_j, t_j \in S \setminus U$ *for all* $j = 1, 2, \ldots, m'$.

Theorem 2.3. ([1], Lemma 3.3) *For any* $d \in S \setminus U$ *and for any positive integers* k *and* k' *there exist* $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{k'} \in U$ *and* $d_k, d_{k'} \in S \setminus U$ *such that* $d = u_1u_2 \cdots u_kd_k = d_{k'}v_{k'}v_{k'-1} \cdots v_2v_1$.

Theorem 2.4. ([4], Theorem 2.1) *If* U *is a permutative posemigroup and* S *is any posemigroup containing* U *properly as a subposemigroup such that* $\widehat{Dom}(U,S) = S$. *Then* S *is also permutative.*

Bracketed statements whenever used shall mean the dual to the other statements.

3. Main Results

A semigroup S is said to be permutative if S satisfies a permutation identity.

$$z_1z_2 \cdots z_n = z_{i_1}z_{i_2} \cdots z_{i_n} \ (n \geq 2), \tag{3}$$

where i is any non-trivial permutation of the set $\{1, 2, \ldots, n\}$. A posemigroup S is said to be permutative if it is so as a semigroup.

In order to prove the main theorems of this section we first prove the following.

Lemma 3.1. ([4], Lemma 3.2) *Let* S *be any posemigroup satisfying (3) with* $n \geq 3$, *the following hold:*

(i) For each $j \in \{2, 3, \ldots, n\}$ *such that* $z_{j-1}z_j$ *is not a subword of* $z_{i_1}z_{i_2} \cdots z_{i_n}$, S *satisfies the permutation identity*

$$z_1z_2 \cdots z_{j-1}xyz_j \cdots z_n = z_1z_2 \cdots z_{j-1}yxz_j \cdots z_n.$$

(ii) If $z_{i_n} \neq z_n$, *then* S *also satisfies the permutation identity*

$$z_1z_2 \cdots z_nxy = z_1z_2 \cdots z_nyx.$$

Lemma 3.2. *Let* S *be any permutative posemigroup satisfying a permutation identity (3) with* $n \geq 3$. *Then for each* $j \in \{2, 3, \ldots, n\}$ *such that* $z_{j-1}z_j$ *is not a subword of* $z_{i_1}z_{i_2} \cdots z_{i_n}$, *for all* $r \geq j-1$, $s \geq n-j+1$ *and for all* $u \in S^r$, $v \in S^s$ *we have*

$$ux_1x_2v = ux_2x_1v, \text{ for all } x_1, x_2 \in S.$$

In particular S^k *is medial for all* $k \geq max(j-1, n-j+1)$.

Proof. The proof follows from ([7], Proposition 6.3). □

The next lemma easily follows from Lemma 3.2.

Lemma 3.3. *Let* S *be any permutative posemigroup satisfying a permutation identity (3) with* $n \geq 3$. *If* $j \in \{2, 3, \ldots, n\}$ *such that* $z_{j-1}z_j$ *is not a subword of* $z_{i_1}z_{i_2} \cdots z_{i_n}$. *Then for all* $r \geq j-1$, $s \geq n-j+1$ *and for all* $u \in S^r$, $v \in S^s$, *we have*

$$uz_1z_2 \cdots z_lv = uz_{h_1}z_{h_2} \cdots z_{h_l}v,$$

for all $z_1, z_2, \ldots, z_l \in S$ $(l \geq 2)$, *where* h *is any permutation of the set* $\{1, 2, \ldots, l\}$.

LEMMA 3.4. *Let S be any permutative posemigroup satisfying a permutation identity (3) with $n \geq 3$. If $j \in \{2, 3, \ldots, n\}$ such that $z_{j-1}z_j$ is not a subword of $z_{i_1}z_{i_2}\cdots z_{i_n}$. Then for all $r \geq j-1$, $s \geq n-j+1$ and for all $x \in S^r$, $z \in S^s$ and $y \in S$, $(xyz)^k = x^k y^k z^k$ for all $k \geq 1$.*

Proof. For $k = 1$, the result is vacuously true. We shall prove it for $k > 1$. Now

$$\begin{aligned}(xyz)^k &= (xyz)(xyz)^{k-2}(xyz)\\ &= xyzx^{k-2}y^{k-2}z^{k-2}xyz \text{ (by Lemma 3.3 as } x \in S^r, z \in S^s)\\ &= x^k y^k z^k \text{ (by Lemma 3.3 as } x \in S^r, z \in S^s).\end{aligned}$$

□

In the following results, let U and S be any posemigroups with U as a proper subposemigroup of S such that U satisfies (3) and $\widehat{Dom}(U, S) = S$.

LEMMA 3.5. ([2], Lemma 3.3) *For any $z_1, z_2 \in S$ and $x, y \in S \setminus U$, $xz_1z_2y = xz_2z_1y$.*

COROLLARY 3.6. *For any $x, y \in S \setminus U$, $z_1, z_2, \ldots, z_k \in S$ and for any permutation j of the set $\{1, 2, \ldots, k\}$, we have*

$$xz_1z_2\cdots z_ky = xz_{j_1}z_{j_2}\cdots z_{j_k}y.$$

LEMMA 3.7. *If $i_n \neq n$ in (3), then $xz_1z_2 = xz_2z_1$ for all $z_1, z_2 \in S$ and $x \in S \setminus U$.*

Proof. Since S is permutative, by Lemma 3.1 (ii), S also satisfies the following permutation identity:

$$z_1z_2\cdots z_nst = z_1z_2\cdots z_nts.$$

By Theorem 2.3, for any $x \in S \setminus U$ and for every integer k, we have $x \in S^k$. Therefore the proof of the lemma follows. □

COROLLARY 3.8. *If $i_n \neq n$ in (3), then for any $z_1, z_2, \ldots, z_k \in S$, $x \in S \setminus U$ and for any permutation j of the set $\{1, 2, \ldots, k\}$, we have*

$$xz_1z_2\cdots z_k = xz_{j_1}z_{j_2}\cdots z_{j_k}.$$

PROPOSITION 3.9. *Let U be any permutative posemigroup satisfying permutation identity (3) and let S be any posemigroup with U as a proper subposemigroup of S such that $\widehat{Dom}(U, S) = S$. Then for any $d \in S \setminus U$ there exist $z \in U^r, w \in U^s$ and $x \in S \setminus U$ with $r \geq j-1, s \geq n-j+1$ such that $d^k = z^k x^k w^k$ for all $k \geq 1$.*

Proof. Suppose that U satisfies permutation identity (3). By Theorem 2.4, S also satisfies permutation identity (3). Let $d \in S \setminus U$. By Theorems 2.2 and 2.3 together we can write $d = zxw$ for some $z \in U^r$, $w \in U^s$ and $x \in S \setminus U$. Now for $k \geq 1$, we have

$$\begin{aligned}d^k &= (zxw)^k\\ &= z^k x^k w^k \text{ (by Lemma 3.4 as } z \in U^r, w \in U^s).\end{aligned}$$

□

THEOREM 3.10. *Non-trivial identities I of the following forms are preserved under epis of posemigroups in conjunction with permutation identity (3):*

(i) at least one side of I has no repeated variable;
(ii) $z_1^p = z_2^q,\ p, q > 0$;

(iii) $z_1^p z_2^p \cdots z_l^p = z_1^q z_2^q \cdots z_l^q,\ p, q > 0, l \geq 1$;
(iv) $z_1^p = 0,\ p > 0$.

Proof. Take any posemigroups U and S with U as a proper subposemigroup of S such that $Dom(U, S) = S$. Suppose that U satisfies the permutation identity (3) and any non-trivial identity I. Then by Theorem 2.4, S also satisfies the permutation identity (3). We will show that S satisfies each of the identities (i) to (iv).

(i) Assume U satisfies (i), then by ([3], Theorem 3.1), S also satisfies (i).

(ii) Assume U satisfies (ii). In order to prove that S satisfies (ii), we first prove the following lemma.

Lemma 3.11. *For any $y \in S \setminus U$ and $u \in U$, $y^p = y^q = u^p (= u^q)$.*

Proof. Since $y \in S \setminus U$. Then by Proposition 3.9, $y^p = z^p x^p w^p$ for some $z \in U^r, w \in U^s$ and $x \in S \setminus U$. Let (1) and (2) be the zigzag inequalities for x of minimal length (n, m). Now

$$\begin{aligned}
y^p &= z^p (x_1 u_0)^p w^p \text{ (by zigzag inequalities (1))} \\
&= z^p x_1^p u_0^p w^p \text{ (by Lemma 3.3)} \\
&= z^p x_1^p u_1^p w^p \text{ (as } U \text{ satisfies (ii))} \\
&= z^p (x_1 u_1)^p w^p \text{ (by Lemma 3.3)} \\
&\leq z^p (x_2 u_2)^p w^p \text{ (by zigzag inequalities (1))} \\
&= z^p x_2^p u_2^p w^p \text{ (by Lemma 3.3)} \\
&= z^p x_2^p u_3^p w^p \text{ (as } U \text{ satisfies (ii))} \\
&\vdots \\
&= z^p x_n^p u_{2n-1}^p w^p \\
&= z^p (x_n u_{2n-1})^p w^p \text{ (by Lemma 3.3)} \\
&= z^p u_{2n}^p w^p \\
&= (z u_{2n} w)^p \text{ (by Lemma 3.3)} \\
&= u^p \text{ (as } U \text{ satisfies (ii)).}
\end{aligned} \tag{4}$$

□

On similar lines for any $y \in S \setminus U$ and $u \in U$, we can get

$$y^q \geq u^q. \tag{5}$$

On combining (4) and (5), we get $y^p \leq u^p = u^q \leq y^q$. Similarly, we can show that $y^p \geq u^p = u^q \geq y^q$. Therefor, $y^p = y^q = u^p (= u^q)$, as required.

Now to complete the proof of (ii), take any $z_1, z_2 \in S$. If $z_1 \in S \setminus U$ and $z_2 \in U$, then the result follows by Lemma 3.11. Assume that $z_1, z_2 \in S \setminus U$. Now

$$\begin{aligned}
z_1^p &= u^p \text{ (where } u \in U \text{, by Lemma 3.11 as } z_1 \in S \setminus U) \\
&= u^q \text{ (as } U \text{ satisfies (ii))} \\
&= z_2^q \text{ (by Lemma 3.11 as } z_2 \in S \setminus U),
\end{aligned}$$

as required. This completes the proof of (ii).

(iii) Assume U satisfies (iii). For $j = 1, 2, \ldots, l$, let $z_1^p z_2^p \cdots z_j^p$ be the word in S of length jp. To prove that S satisfies (iii), we use induction on j by assuming that the remaining element $z_{j+1}, \ldots, z_l \in U$. For $j = 0$, the equation (iii) is vacuously satisfied. Assume inductively that the equation (iii) is true for all $z_1, z_2, \ldots, z_{j-1} \in S$ and for all $z_j, z_{j+1}, \ldots, z_l \in U$. We will prove that this assumption also holds for all $z_1, z_2, \ldots, z_j \in S$ and for all $z_{j+1}, \cdots, z_l \in U$. There is no need to consider the case $z_j \in U$. So, assume that $z_j \in S \setminus U$, then by Proposition 3.9,

$$z_j^p = z^p x^p w^p \tag{6}$$

for some $z \in U^r, w \in U^s$ and $x \in S \setminus U$. Let (1) and (2) be the zigzag inequalities for x of minimal length (n, m). For each $k = 1, 2, \ldots, m-1$, Theorems 2.2 and 2.3 together allow us to write $s_k = s_k' b_1^{(k)} b_2^{(k)} \cdots b_{j-2}^{(k)}$ and $t_k = c_{j+1}^{(k)} c_{j+1}^{(k)} \cdots c_l^{(k)} t_k'$, where $b_1^{(k)}, b_2^{(k)}, \ldots, b_{j-2}^{(k)}, c_{j+1}^{(k)}, c_{j+1}^{(k)}, \ldots, c_l^{(k)} \in U$ and $x_k', y_k' \in S \setminus U$.

For each $k = 1, 2, \ldots, m-1$, in whatever follows, we shall be using phrases *expanding t_k, expanding s_k* and *collapsing t_k', collapsing s_k'* to mean that $t_k = c_{j+1}^{(k)} \cdots c_l^{(k)} t_k'$, $s_k = s_k' b_1^{(k)} b_2^{(k)} \cdots b_{j-2}^{(k)}$ and $c_{j+1}^{(k)} \cdots c_l^{(k)} t_k' = t_k$, $s_k' x_k' b_1^{(k)} b_2^{(k)} \cdots b_{j-2}^{(k)}$ $= s_k$, respectively.

For each $k = 1, 2, \ldots, m-1$, consider the product

$$P_k =: z_1^q z_2^q \cdots z_{j-1}^q z^q s_k'^q b_1^{(k)^p} b_2^{(k)^p} \cdots b_{j-1}^{(k)^p} (v_{2k-1} t_k)^p w^p z_{j+1}^p \cdots z_l^p.$$

Lemma 3.12. *For $k = 1, 2, \ldots, m-1$, $P_k \leq P_{k+1}$ and $P_{m-1} \leq z_1^q z_2^q \cdots z_l^q$.*

Proof. Assume that $1 \leq j \leq l$. We essentially prove the lemma when $1 < j < l$. The proof in the cases when $j = l$ and $j = l$ follows by slight modification (see Remark 3.14). Now

$$\begin{aligned}
P_k &= z_1^q z_2^q \cdots z_{j-1}^q z^q s_k'^q b_1^{(k)^p} b_2^{(k)^p} \cdots b_{j-1}^{(k)^p} (v_{2k-1} t_k)^p w^p z_{j+1}^p \cdots z_l^p \\
&\leq z_1^q z_2^q \cdots z_{j-1}^q z^q s_k'^q b_1^{(k)^p} \cdots b_{j-1}^{(k)^p} (v_{2k} t_{k+1})^p w^p z_{j+1}^p \cdots z_l^p \\
&\qquad\qquad\qquad\qquad\text{(by zigzag inequalities (2))} \\
&= w_1 s_k'^q b_1^{(k)^p} b \cdots b_{j-1}^{(k)^p} (v_{2k} t_{k+1})^p w_2 \\
&\qquad\text{(where } w_1 = z_1^q z_2^q \cdots z_{j-1}^q z^q, w_2 = w^p z_{j+1}^p \cdots z_l^p) \\
&= w_1 s_k'^q b_1^{(k)^p} \cdots b_{j-1}^{(k)^p} v_{2k}^p c_{j+1}^{(k+1)^p} \cdots c_l^{(k+1)^p} t_{k+1}'^p w_2 \\
&\qquad\text{(by expanding } t_{k+1} \text{ and Corollary 3.6 as } s_k', t_{k+1}' \in S \setminus U) \\
&= w_1 s_k'^q b_1^{(k)^q} \cdots b_{j-1}^{(k)^q} v_{2k}^q c_{j+1}^{(k+1)^q} \cdots c_l^{(k+1)^q} t_{k+1}'^p w_2 \text{ (as } U \text{ satisfies (iii))} \\
&= w_1 s_k^q v_{2k}^q c_{j+1}^{(k+1)^q} \cdots c_l^{(k+1)^q} t_{k+1}'^p w_2 \text{ (by collapsing } s_k' \text{ and} \\
&\qquad\qquad\qquad\qquad\text{Corollary 3.6 as } s_k', t_{k+1}' \in S \setminus U) \\
&= w_1 (s_k v_{2k})^q c_{j+1}^{(k+1)^q} \cdots c_l^{(k+1)^q} t_{k+1}'^p w_2 \text{ (by Corollary 3.6 as } s_k, t_{k+1}' \in S \setminus U) \\
&\leq w_1 (s_{k+1} v_{2k+1})^q c_{j+1}^{(k+1)^q} \cdots c_l^{(k+1)^q} t_{k+1}'^p w_2 \text{ (by zigzag inequalities (2))} \\
&= w_1 s_{k+1}^q v_{2k+1}^q c_{j+1}^{(k+1)^q} \cdots c_l^{(k+1)^q} t_{k+1}'^p w_2 \\
&\qquad\qquad\qquad\text{(by Corollary 3.6 as } s_{k+1}, t_{k+1}' \in S \setminus U) \\
&= w_1 s_{k+1}'^q b_1^{(k+1)^q} \cdots b_{j-1}^{(k+1)^q} v_{2k+1}^q c_{j+1}^{(k+1)^q} \cdots c_l^{(k+1)^q} t_{k+1}'^p w_2
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\text{(by expanding } s'_{k+1} \text{ and Corollary 3.6 as } s_{k+1}, t'_{k+1} \in S \setminus U)\\
&= w_1 s'^q_{k+1} b_1^{(k+1)^p} \cdots b_{j-1}^{(k+1)^p} v^p_{2k+1} c_{j+1}^{(k+1)^p} \cdots c_l^{(k+1)^p} t'^p_{k+1} w_2 \text{ (as } U \text{ satisfies (iii))}\\
&= w_1 s'^q_{k+1} b_1^{(k+1)^p} \cdots b_{j-1}^{(k+1)^p} v^p_{2k+1} t^p_{k+1} w_2 \text{ (by collapsing } t'_{k+1} \text{ and}\\
&\qquad\qquad\text{Corollary 3.6 as } s'_{k+1}, t'_{k+1} \in S \setminus U)\\
&= w_1 s'^q_{k+1} b_1^{(k+1)^p} \cdots b_{j-1}^{(k+1)^p} (v_{2k+1} t_{k+1})^p w_2\\
&\qquad\qquad\text{(by Corollary 3.6 as } s'_{k+1}, t_{k+1} \in S \setminus U)\\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q s'^q_{k+1} b_1^{(k+1)^p} \cdots b_{j-1}^{(k+1)^p} (v_{2k+1} t_{k+1})^p w^p z^p_{j+1} \cdots z_l^p\\
&\qquad\qquad\text{(since } w_1 = z_1^q z_2^q \cdots z_{j-1}^q z^q, w_2 = w^p z^p_{j+1} \cdots z^p_l)\\
&= P_{k+1}.
\end{aligned}
$$

In particular it shows that

$$
\begin{aligned}
P_{m-1} &\leq w_1 s'^q_m b_1^{(m)^p} \cdots b_{j-1}^{(m)^p} v^p_{2m-1} t^p_m w_2\\
&= w_1 s'^q_m b_1^{(m)^p} \cdots b_{j-1}^{(m)^p} (v_{2m-1} t_m)^p w_2\\
&\qquad\qquad\text{(by Corollary 3.6 as } s'_m, t_m \in S \setminus U)\\
&\leq w_1 s'^q_m b_1^{(m)^p} \cdots b_{j-1}^{(m)^p} v^p_{2m} w_2 \text{ (by zigzag inequalities (2))}\\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q s'^q_m b_1^{(m)^p} \cdots b_{j-1}^{(m)^p} v^p_{2m} w^p z^p_{j+1} \cdots z^p_l\\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q s'^q_m b_1^{(m)^p} \cdots b_{j-1}^{(m)^p} (v_{2m} w)^p z^p_{j+1} \cdots z^p_l\\
&\qquad\qquad\text{(by Lemma 3.3, as } z \in U^r \text{ and } w \in U^s)\\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q s'^q_m b_1^{(m)^q} \cdots b_{j-1}^{(m)^q} (v_{2m} w)^q z^q_{j+1} \cdots z^q_l\\
&\qquad\qquad\text{(as } U \text{ satisfies (iii))}\\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q s^q_m v^q_{2m} w^q z^q_{j+1} \cdots z^q_l \text{ (by collapsing } s'_m \text{ and}\\
&\qquad\qquad\text{Lemma 3.3, as } z \in U^r \text{ and } w \in U^s)\\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q (s_m v_{2m})^q w^q z^q_{j+1} \cdots z^q_l\\
&\qquad\qquad\text{(by Lemma 3.3, as } z \in U^r \text{ and } w \in U^s)\\
&\leq z_1^q z_2^q \cdots z_{j-1}^q z^q x^q w^q z^q_{j+1} \cdots z^q_l \text{ (by zigzag inequalities (2))}\\
&= z_1^q z_2^q \cdots z_j^q z^q_{j+1} \cdots z^q_l,
\end{aligned}
$$

as required. □

Lemma 3.13.

$$z_1^p z_2^p \cdots z_l^p \leq P_1.$$

Proof.

$$
\begin{aligned}
z_1^p z_2^p \cdots z_q \cdots z_l^p &= z_1^p z_2^p \cdots z_{j-1}^p z^p x^p w^p z^p_{j+1} \cdots z^p_l \text{ (by equation (6))}\\
&\leq z_1^p z_2^p \cdots z_{j-1}^p z^p (v_0 t_1)^p w^p z^p_{j+1} \cdots z^p_l \text{ (by zigzag inequalities (2))}\\
&= z_1^p z_2^p \cdots z_{j-1}^p z^p v_0^p c_{j+1}^{(1)^p} \cdots c_l^{(1)^p} t'^p_1 w^p z^p_{j+1} \cdots z^p_l\\
&\qquad\text{(by expanding } t_1 \text{ and Lemma 3.3, as } z \in U^r \text{ and } w \in U^s)\\
&= z_1^p z_2^p \cdots z_{j-1}^p z^p (v_0 c^{(1)}_{j+1})^p c_{j+2}^{(1)^p} \cdots c_l^{(1)^p} t'^p_1 w^p z^p_{j+1} \cdots z^p_l\\
&\qquad\qquad\text{(by Lemma 3.3, as } z \in U^r \text{ and } w \in U^s)
\end{aligned}
$$

$$
\begin{aligned}
&= z_1^q z_2^q \cdots z_{j-1}^q z^q (v_0 c_{j+1}^{(1)})^q c_{j+2}^{(1)^q} \cdots c_l^{(1)^q} t_1'^p w^p z_{j+1}^p \cdots z_l^p \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(as } U \text{ satisfies (iii))} \\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q v_0^q c_{j+1}^{(1)^q} c_{j+2}^{(1)^q} \cdots c_l^{(1)^q} t_1'^p w^p z_{j+1}^p \cdots z_l^p \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 3.3, as } z \in U^r \text{ and } w \in U^s) \\
&\leq z_1^q z_2^q \cdots z_{j-1}^q z^q (s_1 v_1)^q c_{j+1}^{(1)^q} c_{j+2}^{(1)^q} \cdots c_l^{(1)^q} t_1'^p w^p z_{j+1}^p \cdots z_l^p \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by zigzag inequalities (2))} \\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q s_1^q v_1^q c_{j+1}^{(1)^q} \cdots c_l^{(1)^q} t_1'^p w^p z_{j+1}^p \cdots z_l^p \\
&\qquad\qquad\qquad\qquad\qquad \text{(by Corollary 3.6 as } s_1, t_1' \in S \setminus U) \\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q s_1'^q b_1^{(1)^q} \cdots b_{j-1}^{(1)^q} v_1^q c_{j+1}^{(1)^q} \cdots c_l^{(1)^q} t_1'^p w^p z_{j+1}^p \cdots z_l^p \\
&\qquad\qquad \text{(by expanding } s_1 \text{ and Corollary 3.6 as } s_1', t_1' \in S \setminus U) \\
&= z_1^q z_2^q \cdots z_{j-1}^q z^q s_1'^q b_1^{(1)^p} \cdots b_{j-1}^{(1)^p} v_1^p c_{j+1}^{(1)^p} \cdots c_l^{(1)^p} t_1'^p w^p z_{j+1}^p \cdots z_l^p \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(as } U \text{ satisfies (iii))} \\
&= P_1,
\end{aligned}
$$

as required. □

Now, by Lemmas 3.12 and 3.13, we have

$$z_1^p z_2^p \cdots z_l^p \leq P_1 \leq \cdots P_{m-1} \leq z_1^q z_2^q \cdots z_l^q.$$

Similarly, we can show that $z_1^p z_2^p \cdots z_l^p \geq z_1^q z_2^q \cdots z_l^q$. Therefore $z_1^p z_2^p \cdots z_l^p = z_1^q z_2^q \cdots z_l^q$, as required.

Remark 3.14. The proof when $j = l$ or $j = 1$ is obtained by making the following conventions:

(i) the word $z_1^q z_2^q \cdots z_{j-1}^q = 1$,

(ii) the word $b_1^{(i)^p} \cdots b_{j-1}^{(i)^p} = 1 = b_1^{(i)^q} \cdots b_{j-1}^{(i)^q}$ and $s_i' = s_i$ for $i = 1, 2, \ldots, m$.

Dually when $j = 1$,

(i) the word $z_{j+1}^p \cdots z_l^p = 1$,

(ii) the word $c_{j+1}^{(i)^p} \cdots c_l^{(i)^p} = 1 = c_{j+1}^{(i)^q} \cdots c_l^{(i)^q}$ and $t_i' = t_i$ for $i = 1, 2, \ldots, m$.

(iv) Assume U satisfies (iv). Let $z_1 \in S \setminus U$, then by Proposition 3.9, $z_1^p = z^p x^p w^p$ for some $z \in U^r, w \in U^s$ and $x \in S \setminus U$. Let (1) and (2) be the zigzag inequalities for x of minimal length (n, m). Now

$$
\begin{aligned}
z_1^p &\leq z^p (v_0 t_1)^p w^p \text{ (by zigzag inequalities (2))} \\
&= z^p v_0^p t_1^p w^p \text{ (by Lemma 3.3 as } z \in U^r \text{ and } w \in U^s) \\
&= z^p v_1^p t_1^p w^p \text{ (as } U \text{ satisfies (iv))} \\
&= z^p (v_1 t_1)^p w^p \text{ (by Lemma 3.3 as } z \in U^r \text{ and } w \in U^s) \\
&\leq z^p (v_2 t_2)^p w^p \text{ (by zigzag inequalities (2))} \\
&= z^p v_2^p t_2^p w^p \text{ (by Lemma 3.3 as } z \in U^r \text{ and } w \in U^s) \\
&= z^p v_3^p t_2^p w^p \text{ (as } U \text{ satisfies (iv))} \\
&\ \ \vdots
\end{aligned}
$$

$$
\begin{aligned}
&= z^p v_{2m-1}^p t_m^p w^p \\
&= z^p (v_{2m-1} t_m)^p w^p \text{ (by Lemma 3.3 as } z \in U^r \text{ and } w \in U^s) \\
&\le z^p v_{2m}^p w^p \text{ (by zigzag inequalities (2))} \\
&= (z v_{2m} w)^p \text{ (by Lemma 3.3 as } z \in U^r \text{ and } w \in U^s) \\
&= 0 \text{ (as } U \text{ satisfies (iv)).}
\end{aligned}
$$

Thus $z_1^p \le 0$. Similarly, we can show that $z_1^p \ge 0$. Hence $z_1^p = 0$, as required. This completes proof of the Theorem 3.10. □

In the following results U is any proper subposemigroup of a posemigroup S satisfying (3), such that $\widehat{Dom}(U, S) = S$.

LEMMA 3.15. ([4], Lemma 3.4) *If $i_n \neq n$ and $i_2 \neq 2$ then for any $z_1, z_3 \in U$ and $z_2 \in S \setminus U$, $z_1 z_2 z_3 = z_1 z_3 z_2$.*

COROLLARY 3.16. *If $i_n \neq n$ and $i_2 \neq 2$ in (3), then for any $z_1, z_2, \ldots, z_k \in S$ such that $z_q \in S \setminus U$ for some $q \in \{2, 3, \ldots, k\}$ and for any permutation j of the set $\{2, 3, \ldots, k\}$. We have*

$$z_1 z_2 \cdots z_k = z_1 z_{j_2} z_{j_3} \cdots z_{j_k}.$$

Proof. We have,

$$
\begin{aligned}
z_1 z_2 \cdots z_q \cdots z_k &= z_1 z_q z_2 \cdots z_{q-1} z_{q+1} \cdots z_k \text{ (by Lemma 3.15 as } z_q \in S \setminus U) \\
&= z_1 z_q z_{j_2} \cdots z_{j_{l-1}} z_{j_{l+1}} \cdots z_{j_k} \text{ (where } z_q = z_{j_l} \text{, by Corollary 3.8)} \\
&= z_1 z_{j_2} \cdots z_{j_{l-1}} z_q z_{j_{l+1}} \cdots z_{j_k} \text{ (by Lemma 3.15 as } z_q \in S \setminus U) \\
&= z_1 z_{j_2} z_{j_3} \cdots z_{j_k},
\end{aligned}
$$

as required. □

LEMMA 3.17. *If $i_n \neq n$ and $i_2 \neq 2$ in (3), then for all $z_1 \in U$ and $z_2 \in S \setminus U$, $(z_1 z_2)^k = z_1^k z_2^k$ for all positive integers k.*

Proof. For $k = 1$, the result is vacously true. Assume $k > 1$, we have

$$
\begin{aligned}
(z_1 z_2)^k &= z_1 z_2 (z_1 z_2)^{k-1} \\
&= z_1 z_2 z_1^{k-1} z_2^{k-1} \text{ (by Corollary 3.8, as } z_2 \in S \setminus U \\
&= z_1^k z_2^k \text{ (by Corollary 3.16, as } i_n \neq n \text{ and } i_2 \neq 2 \text{ and } z_2 \in S \setminus U.
\end{aligned}
$$

□

LEMMA 3.18. *If $i_n \neq n$ in (3), then for all $z_1 \in S \setminus U$ and $z_2 \in U$, $(z_1 z_2)^k = z_1^k z_2^k$ for all positive integers k.*

Proof. For $k = 1$, the result is vacously true. Assume $k > 1$, we have

$$
\begin{aligned}
(z_1 z_2)^k &= z_1 z_2 (z_1 z_2)^{k-1} \\
&= z_1 z_2 z_1^{k-1} z_2^{k-1} \text{ (by Corollary 3.8, as } i_n \neq n \text{ and } z_1 \in S \setminus U \\
&= z_1^k z_2^k \text{ (by Corollary 3.8, as } i_n \neq n \text{ and } z_1 \in S \setminus U.
\end{aligned}
$$

□

THEOREM 3.19. *A nontrivial identity I is preserved under the epis of posemigroups in conjunction with the permutation identity (3), with $i_n \neq n$ and $i_2 \neq 2$ [$i_1 \neq 1$ and $i_{n-1} \neq n - 1$] if I has one of the following forms:*

(i) $z_1^p z_2^q = z_2^r z_1^s, p, q, r, s > 0$;
(ii) $z_1^p z_2^q = 0, p, q > 0$.

Proof. Let U and S be any posemigroups with U as a proper subposemigroup of S such that $\widehat{Dom}(U, S) = S$. Assume that U satisfies (3) with $i_n \neq n$ and $i_2 \neq 2$ and any nontrivial identity I. By Theorem 2.4, S also satisfies permutation identity (3). We will show that S also satisfies the identities (i) and (ii).

(i) Assume U satisfies (i) and let $z_1, z_2 \in S$. We consider the following cases.

Case a: $z_1 \in S \setminus U$ and $z_2 \in U$. Let (1) and (2) be the zigzag inequalities for z_1 of minimal length (n, m). Now, we prove inductively that

$$z_1^p z_2^p \leq x_k^p z_2^r (u_{2k-1} y_k)^s \tag{7}$$

holds for all $k = 1, 2, \ldots, n-1$. For $k = 1$, we have

$$\begin{aligned} z_1^p z_2^q &\leq (x_1 u_0)^p z_2^q \text{ (by zigzag inequalities (1)} \\ &= x_1^p u_0^p z_2^q \text{ (by Lemma 3.18, as } x_1 \in S \setminus U) \\ &= x_1^p z_2^r u_0^s \text{ (since } U \text{ satisfies (i))} \\ &\leq x_1^p z_2^r (u_1 y_1)^s \text{ (by zigzag inequalities (1)).} \end{aligned}$$

Thus (7) holds for $k = 1$. Assume inductively that it holds for $k = l < n - 1$. We will show that it also holds for $k = l + 1$. Now

$$\begin{aligned} z_1^p z_2^q &\leq x_l^p z_2^r (u_{2l-1} y_l)^s \text{ (by inductive hypothesis)} \\ &= x_l^p z_2^r u_{2l-1}^s y_l^s \text{ (by Corollary 3.6, as } x_l, y_l \in S \setminus U) \\ &= x_l^p u_{2l-1}^p z_2^q y_l^s \text{ (as } U \text{ satisfies (i))} \\ &= (x_l u_{2l-1})^p z_2^q y_l^s \text{ (by Corollary 3.6, as } x_l, y_l \in S \setminus U) \\ &\leq (x_{l+1} u_{2l})^p z_2^q y_l^s \text{ (by zigzag inequalities (1))} \\ &= x_{l+1}^p u_{2l}^p z_2^q y_l^s \text{ (by Corollary 3.6, as } x_{l+1}, y_l \in S \setminus U) \\ &= x_{l+1}^p z_2^r u_{2l}^s y_l^s \text{ (as } U \text{ satisfies (i))} \\ &= x_{l+1}^p z_2^r (u_{2l} y_l)^s \text{ (by Corollary 3.6, as } x_{l+1}, y_l \in S \setminus U) \\ &\leq x_{l+1}^p z_2^r (u_{2l+1} y_{l+1})^s \text{ (by zigzag inequalities (1)),} \end{aligned}$$

as required.

Now to complete the proof of Case a, letting $k = n - 1$ in (7), we have

$$\begin{aligned} z_1^p z_2^q &\leq x_n^p z_2^r (u_{2n-1} y_n)^s \\ &= x_n^p z_2^r u_{2n-1}^s y_n^s \text{ (by Corollary 3.6, as } x_n, y_n \in S \setminus U) \\ &= x_n^p u_{2n-1}^p z_2^q y_n^s \text{ (as } U \text{ satisfies (i))} \\ &= (x_n u_{2n-1})^p z_2^q y_n^s \text{ (by Corollary 3.6, as } x_n, y_n \in S \setminus U) \\ &\leq (u_{2n})^p z_2^q y_n^s \text{ (by zigzag inequalities (1))} \\ &= z_2^r u_{2n}^s y_n^s \text{ (as } U \text{ satisfies (i))} \\ &= z_2^r (u_{2n} y_n)^s \text{ (by Lemma 3.17 as } y_n \in S \setminus U) \\ &\leq z_2^r z_1^s \text{ (by zigzag inequalities (1)).} \end{aligned}$$

Thus $z_1^p z_2^q \leq z_2^r z_1^s$. Similarly, we can show that $z_1^p z_2^q \geq z_2^r z_1^s$. Hence, $z_1^p z_2^q = z_2^r z_1^s$ as required.

Case b: $z_2 \in S \setminus U$ and $z_1 \in U$. Let (1) and (2) be the zigzag inequalities for z_2 of minimal length (n, m). Now, we prove inductively that

$$z_1^p z_2^q \leq (s_k v_{2k-1})^r z_1^s t_k^q \tag{8}$$

holds for all $k = 1, 2, \ldots, m-1$. For $k = 1$, we have

$$\begin{aligned} z_1^p z_2^q &\leq z_1^p (v_0 t_1)^q \text{ (by zigzag inequalities (2))} \\ &= z_1^p v_0^q t_1^q \text{ (by Lemma 3.17 as } t_1 \in S \setminus U) \\ &= v_0^r z_1^s t_1^q \text{ (since } U \text{ satisfies (i))} \\ &\leq (s_1 v_1)^r z_1^s t_1^q \text{ (by zigzag inequalities (2))} \end{aligned}$$

Thus (8) holds for $k = 1$. Assume inductively that it holds for $k = l < m - 1$. We will show that it also holds for $k = l + 1$. Now

$$\begin{aligned} z_1^p z_2^q &\leq (s_l v_{2l-1})^r z_1^s t_l^q \text{ (by inductive hypothesis)} \\ &= s_l^r v_{2l-1}^r z_1^s t_l^q \text{ (by Corollary 3.6, as } s_l, t_l \in S \setminus U) \\ &= s_l^r z_1^p v_{2l-1}^q t_l^q \text{ (since } U \text{ satisfies (i))} \\ &= s_l^r z_1^p (v_{2l-1} t_l)^q \text{ (by Corollary 3.6, as } s_l, t_l \in S \setminus U) \\ &= s_l^r z_1^p (v_{2l} t_{l+1})^q \text{ (by zigzag inequalities (2))} \\ &= s_l^r z_1^p v_{2l}^q t_{l+1}^q \text{ (by Corollary 3.6, as } s_l, t_{l+1} \in S \setminus U) \\ &= s_l^r v_{2l}^r z_1^s t_{l+1}^q \text{ (since } U \text{ satisfies (i))} \\ &= (s_l v_{2l})^r z_1^s t_{l+1}^q \text{ (by Corollary 3.6, as } s_l, t_{l+1} \in S \setminus U) \\ &= (s_{l+1} v_{2l+1})^r z_1^s t_{l+1}^q \text{ (by zigzag inequalities (2))}, \end{aligned}$$

as required.

Now to complete the proof of Case b, letting $k = m - 1$ in (8), we have

$$\begin{aligned} z_1^p z_2^q &\leq (s_m v_{2m-1})^r z_1^s t_m^q \\ &= s_m^r v_{2m-1}^r z_1^s t_m^q \text{ (by Corollary 3.6, as } s_m, t_m \in S \setminus U) \\ &= s_m^r z_1^p v_{2m-1}^q t_m^q \text{ (since } U \text{ satisfies (i))} \\ &= s_m^r z_1^p (v_{2m-1} t_m)^q \text{ (by Corollary 3.6, as } s_m, t_m \in S \setminus U) \\ &\leq s_m^r z_1^p v_{2m}^q \text{ (by zigzag inequalities (2))} \\ &= s_m^r v_{2m}^r z_1^s \text{ (since } U \text{ satisfies (i))} \\ &= (s_m v_{2m})^r z_1^s \text{ (by Lemma 3.18, as } s_m \in S \setminus U) \\ &\leq z_2^r z_1^s \text{ (by zigzag inequalities (2))}. \end{aligned}$$

Thus $z_1^p z_2^q \leq z_2^r z_1^s$. Similarly, we can show that $z_1^p z_2^q \geq z_2^r z_1^s$. Hence $z_1^p z_2^q = z_2^r z_1^s$, as required.

Case c: $z_1, z_2 \in S \setminus U$ and let (1) and (2) be the zigzag inequalities for z_1 of minimal length (n, m). Then

$$\begin{aligned} z_1^p z_2^q &\leq (x_1 u_0)^p z_2^q \text{ (by zigzag inequalities (1)} \\ &= x_1^p u_0^p z_2^q \text{ (by Lemma 3.18)} \\ &= x_1^p z_2^r u_0^s \text{ (by Case b)} \end{aligned}$$

$$
\begin{aligned}
&\leq x_1^p z_2^r (u_1 y_1)^s \text{ (by zigzag inequalities (1))}\\
&= x_1^p z_2^r u_1^s y_1^s \text{ (by Corollary 3.6, as } x_1, y_1 \in S \setminus U)\\
&= x_1^p u_1^p z_2^q y_1^s \text{ (by Case b)}\\
&= (x_1 u_1)^p z_2^q y_1^s \text{ (by Corollary 3.6, as } x_1, y_1 \in S \setminus U)\\
&\leq (x_2 u_2)^p z_2^q y_1^s \text{ (by zigzag inequalities (1))}\\
&= x_2^p u_2^p z_2^q y_1^s \text{ (by Corollary 3.6, as } x_2, y_1 \in S \setminus U)\\
&= x_2^p z_2^r u_2^s y_1^s \text{ (by Case b)}\\
&= x_2^p z_2^r (u_2 y_1)^s \text{ (by Corollary 3.6, as } x_2, y_1 \in S \setminus U)\\
&\leq x_2^p z_2^r (u_3 y_2)^s \text{ (by zigzag inequalities (1))}\\
&\vdots\\
&\leq x_n^p z_2^r (u_{2n-1} y_n)^s\\
&= x_n^p z_2^r u_{2n-1}^s y_n^s \text{ (by Corollary 3.6, as } x_n, y_n \in S \setminus U)\\
&= x_n^p u_{2n-1}^p z_2^q y_n^s \text{ (by Case b)}\\
&= (x_n u_{2n-1})^p z_2^q y_n^s \text{ (by Corollary 3.6, as } x_n, y_n \in S \setminus U)\\
&\leq u_{2n}^p z_2^q y_n^s \text{ (by zigzag inequalities (1))}\\
&= z_2^r u_{2n}^s y_n^s \text{ (by Case b)}\\
&= z_2^r (u_{2n} y_n)^s \text{ (by Lemma 3.17)}\\
&\leq z_2^r z_1^s \text{ (by zigzag inequalities (1)).}
\end{aligned}
$$

Thus $z_1^p z_2^q \leq z_2^r z_1^s$. Similarly, we can show that $z_1^p z_2^q \geq z_2^r z_1^s$. Hence $z_1^p z_2^q = z_2^r z_1^s$, as required. This completes the proof of part (i).

(ii) Assume U satisfies (ii), then for all $u, v \in U$, $u^p v^q = 0$. Let $z_1, z_2 \in S$. Then we have the folowing cases.

Case a: $z_1 \in U$, $z_2 \in S \setminus U$. Let (1) and (2) be the zigzag inequalities for z_2 of minimal length (n, m). Then

$$
\begin{aligned}
z_1^p z_2^q &\leq z_1^p (v_0 t_1)^q \text{ (by zigzag inequalities (2))}\\
&= z_1^p v_0^q t_1^q \text{ (by Lemma 3.17 as } t_1 \in S \setminus U)\\
&= z_1^p v_1^q t_1^q \text{ (since } U \text{ satisfies (ii))}\\
&= z_1^p (v_1 t_1)^q \text{ (by Lemma 3.17 as } t_1 \in S \setminus U)\\
&\leq z_1^p (v_2 t_2)^q \text{ (by zigzag inequalities (2))}\\
&= z_1^p v_2^q t_2^q \text{ (by Lemma 3.17 as } t_2 \in S \setminus U)\\
&= z_1^p v_3^q t_2^q \text{ (since } U \text{ satisfies (ii))}\\
&\vdots\\
&= z_1^p v_{2m-1}^q t_m^q\\
&= z_1^p (v_{2m-1} t_m)^q \text{ (by Lemma 3.17 as } t_m \in S \setminus U)\\
&\leq z_1^p v_{2m}^p \text{ (by zigzag inequalities (2))}\\
&= 0 \text{ (since } U \text{ satisfies (ii)).}
\end{aligned}
$$

Thus $z_1^p z_2^q \leq 0$. Similarly, we can show that $z_1^p z_2^q \geq 0$. Hence $z_1^p z_2^q = 0$, as required.

Case b: $z_1 \in S \setminus U$, $z_2 \in U$. It follows on similar lines as Case a, by applying zigzag inequalities (1) and Lemma 3.18.

Case c: $z_1, z_2 \in S \setminus U$ and let (1)and (2) be the zigzag inequalities for z_1 of minimal length (n, m). Then

$$\begin{aligned}
z_1^p z_2^q &\leq (x_1 u_0)^p z_2^q \text{ (by zigzag inequalities (2))} \\
&= x_1^p u_0^p z_2^q \text{ (by Lemma 3.18 as } x_1 \in S \setminus U) \\
&= x_1^p u_1^p z_2^q \text{ (by Case a)} \\
&= (x_1 u_1)^p z_2^q \text{ (by Lemma 3.18 as } x_1 \in S \setminus U) \\
&\leq (x_2 u_2)^p z_2^q \text{ (by zigzag inequalities (1))} \\
&= x_2^p u_2^p z_2^q \text{ (by Lemma 3.18 as } x_2 \in S \setminus U) \\
&= x_2^p u_3^p z_2^q \text{ (by Case a)} \\
&\vdots \\
&= x_n^p u_{2n-1}^p z_2^q \\
&= (x_m u_{2n-1})^p z_2^q \text{ (by Lemma 3.18 as } x_m \in S \setminus U) \\
&\leq u_{2n}^p z_2^q \text{ (by zigzag inequalities (1))} \\
&= 0 \text{ (by Case a).}
\end{aligned}$$

Thus $z_1^p z_2^q \leq 0$. Similarly, we can show that $z_1^p z_2^q \geq 0$. Hence $z_1^p z_2^q = 0$, as required. This completes proof of the Theorem 3.19. □

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