GENERALIZED QUADRATIC MAPPINGS IN 2d VARIABLES

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ABSTRACT. Let $X, Y$ be vector spaces. It is shown that if an even mapping $f : X \to Y$ satisfies $f(0) = 0$, and
\[ 2(2d-2C_{d-1} - 2d-2 C_d) \sum_{j=1}^{2d} x_j + \sum_{\ell(j) = 0, 1 \sum_{j=1}^{2d} \ell(j) = d} f \left( \sum_{j=1}^{2d} (-1)^{\ell(j)} x_j \right) \]
\[ = 2(2d-1 C_d + 2d-2 C_{d-1} - 2d-2 C_d) \sum_{j=1}^{2d} f(x_j) \]
for all $x_1, \ldots, x_{2d} \in X$, then the even mapping $f : X \to Y$ is quadratic.

Furthermore, we prove the Hyers-Ulam stability of the above functional equation in Banach spaces.

1. Introduction and preliminaries

In 1940, S.M. Ulam [14] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let $X$ and $Y$ be Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Hyers [3] showed that if $\epsilon > 0$ and $f : X \to Y$ such that
\[ \| f(x + y) - f(x) - f(y) \| \leq \epsilon \]
for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that
\[ \| f(x) - T(x) \| \leq \epsilon \]

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for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0,1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon (\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Th.M. Rassias [6] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2p} \|x\|^p$$


A square norm on an inner product space satisfies the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. The Hyers-Ulam stability problem of the quadratic functional equation was proved by Skof [13] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. In [2], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation. Several functional equations have been investigated in [5]–[12].

In this paper, we solve the functional equation

$$2(2d-2C_{d-1} - 2d-2 C_d)f\left(\sum_{j=1}^{2d} x_j\right) + \sum_{\sigma(j)=0,1, \sum_{j=1}^{2d} a(j)=d} f\left(\sum_{j=1}^{2d} (-1)^{\sigma(j)} x_j\right)$$

$$= 2(2d-1C_d + 2d-2 C_{d-1} - 2d-2 C_d) \sum_{j=1}^{2d} f(x_j),$$

(1.1)

and prove the Hyers-Ulam stability of the functional equation (1.1) in Banach spaces.
2. Stability of generalized quadratic mappings in 2d variables

Throughout this section, assume that $X$ and $Y$ are vector spaces.

**Lemma 2.1.** If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (1.1), then the mapping $f : X \to Y$ is quadratic.

**Proof.** Letting $x_1 = x$, $x_2 = y$ and $x_3 = \cdots = x_{2d} = 0$ in (1.1), we get
\[
2(2d-2C_{d-1} - 2d-2 C_d)f(x + y) + 2(2d-2C_{d-1} f(x + y) + 2(2d-2C_{d-1} f(x - y)
= 2(2d-1C_d + 2d-2 C_{d-1} - 2d-2 C_d)(f(x) + f(y))
\]
for all $x, y \in X$. So
\[
2d-2C_{d-1}(f(x+y) + f(x-y)) = (2d-1C_d + 2d-2 C_{d-1} - 2d-2 C_d)(f(x) + f(y))
\]
for all $x, y \in X$. Since $2d-1C_d + 2d-2 C_{d-1} - 2d-2 C_d = 2d-2C_{d-1}$,
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
for all $x, y \in X$. Thus the even mapping $f : X \to Y$ is quadratic. □

From now on, assume that $X$ is a normed vector space with norm $\| \cdot \|$ and that $Y$ is a Banach space with norm $\| \cdot \|$.

For a given mapping $f : X \to Y$, we define
\[
Df(x_1, \cdots, x_{2d}) : = 2(2d-2C_{d-1} - 2d-2 C_d)f \left( \sum_{j=1}^{2d} x_j \right)
+ \sum_{\epsilon(j)=0,1, \sum_{j=1}^{2d} \epsilon(j)=d} f \left( \sum_{j=1}^{2d} (-1)^{\epsilon(j)} x_j \right)
- 2(2d-1C_d + 2d-2 C_{d-1} - 2d-2 C_d) \sum_{j=1}^{2d} f(x_j)
\]
for all $x_1, \cdots, x_{2d} \in X$.

**Theorem 2.2.** Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2d} \to [0, \infty)$ such that
\[
(2.1) \quad \varphi(x_1, \cdots, x_{2d}) : = \sum_{j=1}^{\infty} 9^j \varphi \left( \frac{x_1}{3^j}, \cdots, \frac{x_{2d}}{3^j} \right) < \infty,
\]
\[
(2.2) \quad \| Df(x_1, \cdots, x_{2d}) \| \leq \varphi(x_1, \cdots, x_{2d})
\]
for all $x_1, \ldots, x_{2d} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

\[
\|f(x) - Q(x)\| \leq \frac{1}{18d^3} \varphi(x, x, 0, \ldots, 0)
\]

for all $x \in X$.

Proof. Letting $x_1 = x_2 = x_3 = x$ and $x_4 = \cdots = x_{2d} = 0$ in (2.2), we get

\[
\|2(2d-2)C_{d-1} - 2d-2C_d + 2d-3C_d f(3x) - 6(2d-1)C_d + 2d-2C_{d-1} - 2d-2C_d - 2d-3C_{d-1}) f(x)\|
\leq \varphi(x, x, 0, \ldots, 0)
\]

for all $x \in X$. Since

\[
= 3d-3C_{d-1},
\]

(2.4)

\[
\|2d-3C_{d-1} f(3x) - 18d-3C_{d-1} f(x)\|
\leq \varphi(x, x, 0, \ldots, 0)
\]

for all $x \in X$. So

\[
\|f(x) - 9f\left(\frac{x}{3}\right)\| \leq \frac{1}{2d-3C_{d-1}} \varphi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}, 0, \ldots, 0\right)
\]

for all $x \in X$. Hence

(2.5)

\[
\|9^l f\left(\frac{x}{3^l}\right) - 9^m f\left(\frac{x}{3^m}\right)\|
\leq \sum_{j=l}^{m-1} \frac{9^j}{2d-3C_{d-1}} \varphi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}, 0, \ldots, 0\right)
\]

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.1) and (2.5) that the sequence \(\{9^m f\left(\frac{x}{3^m}\right)\}\) is a Cauchy sequence.
for all $x \in X$. Since $Y$ is complete, the sequence $\{9^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} 9^n f(\frac{x}{3^n})$$

for all $x \in X$.

By (2.1) and (2.2),

$$\|DQ(x_1, \cdots, x_{2d})\| = \lim_{n \to \infty} 9^n \|Df(\frac{x_1}{3^n}, \cdots, \frac{x_{2d}}{3^n})\|$$

for all $x_1, \cdots, x_{2d} \in X$. So $DQ(x_1, \cdots, x_{2d}) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get the inequality (2.3).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (2.3). Then we have

$$\|Q(x) - Q'(x)\| = 9^n \bigg| \bigg| Q\left(\frac{x}{3^n}\right) - Q'\left(\frac{x}{3^n}\right) \bigg| \bigg|$$

for all $x \in X$. So $DQ(x_1, \cdots, x_{2d}) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get the inequality (2.3).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (2.3). Then we have

$$\|Q(x) - Q'(x)\| = 9^n \bigg| \bigg| Q\left(\frac{x}{3^n}\right) - Q'\left(\frac{x}{3^n}\right) \bigg| \bigg|$$

for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of $Q$. \[\square\]

**Corollary 2.3.** Let $p > 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and

$$\|Df(x_1, \cdots, x_{2d})\| \leq \theta \sum_{j=1}^{2d} \|x_j\|^p$$

for all $x_1, \cdots, x_{2d} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{3\theta}{2(3^p - 9)} \|x\|^p$$

for all $x \in X$.

**Proof.** Defining $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} \|x_j\|^p$ in Theorem 2.2, we get the desired result, as desired. \[\square\]
THEOREM 2.4. Let $f : X \rightarrow Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2d} \rightarrow [0, \infty)$ satisfying (2.2) and

\begin{equation}
\tilde{\varphi}(x_1, \cdots, x_{2d}) := \sum_{j=0}^{\infty} \frac{1}{9^j} \varphi(3^j x_1, \cdots, 3^j x_{2d}) < \infty
\end{equation}

for all $x_1, \cdots, x_{2d} \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

\begin{equation}
\|f(x) - Q(x)\| \leq \frac{1}{18_{2d-3} C_{d-1}} \tilde{\varphi}(x, x, 0, \cdots, 0)
\end{equation}

for all $x \in X$.

**Proof.** It follows from (2.4) that

\[
\left\| f(x) - \frac{1}{9} f(3x) \right\| \leq \frac{1}{18_{2d-3} C_{d-1}} \varphi(x, x, 0, \cdots, 0)
\]

for all $x \in X$. Hence

\begin{equation}
\left\| \frac{1}{9^l} f(3^l x) - \frac{1}{9^m} f(3^m x) \right\|
\end{equation}

\[
\leq \sum_{j=l}^{m-1} \frac{1}{9^j \cdot 18_{2d-3} C_{d-1}} \varphi(3^j x, 3^j x, 3^j x, 0, \cdots, 0)
\]

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.7) and (2.9) that the sequence $\left\{ \frac{1}{9^n} f(3^n x) \right\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\left\{ \frac{1}{9^n} f(3^n x) \right\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

\[
Q(x) := \lim_{n \rightarrow \infty} \frac{1}{9^n} f(3^n x)
\]

for all $x \in X$.

By (2.2) and (2.7),

\[
\|DQ(x_1, \cdots, x_{2d})\| = \lim_{n \rightarrow \infty} \frac{1}{9^n} \|Df(3^n x_1, \cdots, 3^n x_{2d})\|
\]

\[
\leq \lim_{n \rightarrow \infty} \frac{1}{9^n} \varphi(3^n x_1, \cdots, 3^n x_{2d}) = 0
\]

for all $x_1, \cdots, x_{2d} \in X$. So $DQ(x_1, \cdots, x_{2d}) = 0$. By Lemma 2.1, the mapping $Q : X \rightarrow Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.9), we get the inequality (2.8).
The rest of the proof is similar to the proof of Theorem 2.2.

**Corollary 2.5.** Let $p < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$
\|f(x) - Q(x)\| \leq \frac{3\theta}{2(9 - 3p)(2d - 3C_{d-1})}\|x\|^p
$$

for all $x \in X$.

**Proof.** Defining $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} \|x_j\|^p$ in Theorem 2.4, we get the desired result, as desired.

**References**

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