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A TURÁN-TYPE INEQUALITY FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

Wali Mohammad Shah and Sooraj Singh*

Abstract. Let $f(z)$ be an entire function of exponential type τ such that $\|f\| = 1$. Also suppose, in addition, that $f(z) \neq 0$ for $\Im z > 0$ and that $h_f(\frac{\pi}{2}) = 0$. Then, it was proved by Gardner and Govil [Proc. Amer. Math. Soc., 123(1995), 2757-2761] that for $y = \Im z \leq 0$

$$\|D_\zeta[f]\| \leq \frac{\tau}{2}(|\zeta| + 1),$$

where $D_\zeta[f]$ is referred to as polar derivative of entire function $f(z)$ with respect to ζ. In this paper, we prove an inequality in the opposite direction and thereby obtain some known inequalities concerning polynomials and entire functions of exponential type.

1. Introduction and Historical Background

An entire function $f(z)$ is said to be an entire function of exponential type τ, if it is of order less than 1 or it is of order 1 and type less than or equal to τ. The indicator function $h_f(\theta)$ of f is defined by

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}.$$

It is important to note that if $f(z)$ is an entire function of exponential type τ, then the indicator function $h_f(\theta) \leq \tau$, for all $\theta : 0 \leq \theta < 2\pi$.
Also, define a norm called supremum norm or Chebyshev norm denoted by $\|f\|$ as

$$\|f\| = \sup_{-\infty<x<\infty} |f(x)|.$$

A classical result of Bernstein (for references, see [1, p.206] and [6, p.513]) states that if $f(z)$ is an entire function of exponential type τ such that $|f(x)| \leq M$ on the real axis, then

$$\|f'\| \leq M\tau. \tag{1}$$

As a refinement of (1), Boas [2] proved the following result for a special class of functions of exponential type.

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∗ Corresponding author.

Theorem 1.1. *Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \leq 1$ on the real axis. Also suppose, in addition, that $f(z) \neq 0$ for $\Im z > 0$ and that $h_f(\frac{\pi}{2}) = 0$. Then*

$$\|f'\| \leq \frac{\tau}{2}. \tag{2}$$

On the other hand, Rahman [4] proved the following:

Theorem 1.2. *Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \leq 1$ on the real axis,$h_f(\frac{-\pi}{2}) = \tau$, $h_f(\frac{\pi}{2}) \leq 0$ and $f(z) \neq 0$ for $y = \Im z < 0$. Then for all real x*

$$|f'(x)| \geq \frac{\tau}{2}. \tag{3}$$

For an entire function f of exponential type τ and for any complex number ζ, Rahman and Schmeisser [5] defined a function $D_\zeta[f]$ as

$$D_\zeta[f(z)] = \tau f(z) + i(1-\zeta)f'(z).$$

In the literature, the function $D_\zeta[f]$ is referred to as polar derivative of entire function f of exponential type τ with respect to ζ. Clearly

$$\lim_{\zeta \to \infty} \frac{D_\zeta[f(z)]}{\zeta} = -if'(z).$$

Therefore, $D_\zeta[f]$ as defined above, is a generalization of the ordinary derivative $f'(z)$ of $f(z)$.

As an extension of Theorem 1.1 to polar derivative, Gardner and Govil [3] proved the following result:

Theorem 1.3. *Let $f(z)$ be an entire function of exponential type τ such that $\|f\| = 1$. Also suppose, in addition, that $f(z) \neq 0$ for $\Im z > 0$ and that $h_f(\frac{\pi}{2}) = 0$. Then for $|\zeta| \geq 1$*

$$\|D_\zeta[f]\| \leq \frac{\tau}{2}(|\zeta| + 1). \tag{4}$$

2. Results and Discussion

In this paper, we extend Theorem 1.2 to the so called polar derivative of entire functions of exponential type and obtain some known Turán-type inequalities. In fact, we prove

Theorem 2.1. *Let $f(z)$ be an entire function of exponential type τ such that $\|f\| = 1$, $f(z) \neq 0$ for $y = \Im z \leq 0$, $h_f(\frac{-\pi}{2}) = \tau$ and $h_f(\frac{\pi}{2}) \leq 0$. Then for $|\zeta| \geq 1$*

$$\|D_\zeta[f]\| \geq \frac{\tau}{2}(|\zeta| - 1). \tag{5}$$

The bound is attained for the functions of the form $f(z) = \left[\frac{e^{iz}-1}{2}\right]^\tau$.

Proof. Since f is an entire function of exponential type τ, $h_f(\frac{-\pi}{2}) = \tau$, $h_f(\frac{\pi}{2}) \leq 0$ and $f(z) \neq 0$ for $\Im z \leq 0$, therefore by a result due to Gardner and Govil [3, Lemma 5], we have for $\Im z \leq 0$

$$|f(z)| \geq |g(z)|,$$

where $g(z) = e^{i\tau z}\overline{f(\bar{z})}$.
Hence for any α with $|\alpha| > 1$, we have

$$g(z) - \alpha f(z) \neq 0$$

for $\Im z \leq 0$. Now, f is an entire function of exponential type τ such that $h_f(\frac{-\pi}{2}) = \tau$ and $h_f(\frac{\pi}{2}) \leq 0$. Also, $|f(z)| \geq |g(z)|$ for $\Im z \leq 0$. Therefore, by a result due to Gardner and Govil [3, Lemma 7], we have for $|\alpha| > 1$

$$h_{g(z)-\alpha f(z)}(\frac{-\pi}{2}) = \tau.$$

Also, $F(z) = g(z) - \alpha f(z)$ being a linear combination of two entire functions of exponential type τ is an entire function of exponential type τ.
Now $F(z)$ is an entire function of exponential type τ having no zeros in the closed lower half-plane, that is, $\Im z \leq 0$ and $h_F(\frac{-\pi}{2}) = \tau$. Therefore, by using a result due to Gardner and Govil [3, Lemma 2], we get for $\Im z \leq 0$ and $|\zeta| \geq 1$

$$D_\zeta[F(z)] \neq 0.$$

This gives for $\Im z \leq 0$, $|\alpha| > 1$ and $|\zeta| \geq 1$

$$D_\zeta[g(z) - \alpha f(z)] \neq 0. \tag{6}$$

It follows from (6) that for $\Im z \leq 0$ and $|\zeta| \geq 1$

$$|D_\zeta[f(z)]| \geq |D_\zeta[g(z)]| \tag{7}$$

where $g(z) = e^{i\tau z}\overline{f(\bar{z})}$.
Because if otherwise, then we can choose some $z_0 \in \Im z \leq 0$ which does not satisfy this inequality and

$$|D_\zeta[f(z_0)]| < |D_\zeta[g(z_0)]|.$$

We take $\alpha = \frac{D_\zeta[g(z_0)]}{D_\zeta[f(z_0)]}$, so that $|\alpha| > 1$ and for this α, we get

$$D_\zeta[g(z_0) - \alpha f(z_0)] = 0$$

contradicting (6). Hence inequality (7) holds true.
Now, we have $g(z) = e^{i\tau z}\overline{f(\bar{z})}$. On differentiating both sides, we get

$$g'(z) = e^{i\tau z}\overline{f'(\bar{z})} + i\tau e^{i\tau z}\overline{f(\bar{z})} = e^{i\tau z}(\overline{f'(\bar{z})} + i\tau\overline{f(\bar{z})}).$$

This gives for $y = \Im z$

$$|g'(z)| = e^{-\tau y}|f'(\bar{z}) - i\tau f(\bar{z})|$$

Therefore for real x, we have

$$|g'(x)| = |f'(x) - i\tau f(x)|. \tag{8}$$

Also, $g(z) = e^{i\tau z}\overline{f(\bar{z})}$ implies $f(z) = e^{i\tau z}\overline{g(\bar{z})}$. Therefore, we get

$$|f'(x)| = |g'(x) - i\tau g(x)|. \tag{9}$$

Now, for $|\zeta| \geq 1$

$$\begin{aligned} |D_\zeta[f(z)]| + |D_\zeta[g(z)]| &= |\tau f(z) + i(1-\zeta)f'(z)| + |\tau g(z) + i(1-\zeta)g'(z)| \\ &= |\tau f(z) + if'(z) - i\zeta f'(z)| + |\tau g(z) + ig'(z) - i\zeta g'(z)| \\ &= |\zeta f'(z) - f'(z) + i\tau f(z)| + |\zeta g'(z) - g'(z) + i\tau g(z)| \\ &\geq |\zeta||f'(z)| - |f'(z) - i\tau f(z)| + |\zeta||g'(z)| - |g'(z) - i\tau g(z)| \end{aligned}$$

By using (8) and (9), we get for real x and $|\zeta| \geq 1$

$$\begin{aligned} |D_\zeta[f(x)]| + |D_\zeta[g(x)]| &\geq |\zeta||f'(x)| - |f'(x) - i\tau f(x)| + |\zeta||g'(x)| - |g'(x) - i\tau g(x)| \\ &= |\zeta||f'(x)| - |f'(x) - i\tau f(x)| + |\zeta||f'(x) - i\tau f(x)| - |f'(x)| \\ &= (|\zeta| - 1)(|f'(x)| + |f'(x) - i\tau f(x)|) \\ &\geq (|\zeta| - 1)(|f'(x) - f'(x) + i\tau f(x)|) \\ &= (|\zeta| - 1)\tau|f(x)|. \end{aligned}$$

This in particular gives

$$\|D_\zeta[f]\| + \|D_\zeta[g]\| \geq (|\zeta| - 1)\tau\|f\| = (|\zeta| - 1)\tau. \tag{10}$$

From inequality (7), we can easily deduce that for real $x, -\infty < x < \infty$ and $|\zeta| \geq 1$

$$|D_\zeta[f(x)]| \geq |D_\zeta[g(x)]|.$$

Equivalently

$$\|D_\zeta[f]\| \geq \|D_\zeta[g]\|. \tag{11}$$

Combining (10) with (11), we get

$$\begin{aligned} 2\|D_\zeta[f]\| &\geq \|D_\zeta[f]\| + \|D_\zeta[g]\| \\ &\geq (|\zeta| - 1)\tau. \end{aligned}$$

From this, the desired result follows. □

Remark 2.2. On dividing both sides of inequality (5) by $|\zeta|$ and letting $|\zeta| \to \infty$, we get Theorem 1.2.

Remark 2.3. If $p(z)$ is a polynomial of degree n such that $p(z) \neq 0$ for $|z| \geq 1$, then $f(z) := p(e^{iz})$ is an entire function of exponential type less than or equal to n, such that $f(z) \neq 0$ for $\Im z \leq 0$. Furthermore

$$D_\zeta[f(z)] = D_{\zeta e^{iz}}[p(e^{iz})].$$

Hence, if we choose $\beta = \zeta e^{iz}$, then Theorem 2.1 can be clearly seen as a generalization of the following sharp extension of Turán's inequality to the polar derivative of a polynomial due to Shah [7]

Theorem 2.4. *Let $p(z)$ be a polynomial of degree n such that all the zeros of $p(z)$ lie in $|z| < 1$. Then for $|\beta| \geq 1$*

$$\max_{|z|=1} |D_\beta p(z)| \geq \frac{n}{2}(|\beta| - 1) \max_{|z|=1} |p(z)|. \tag{12}$$

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Wali Mohammad Shah
Department of Mathematics, Central University of Kashmir,
Jammu and Kashmir, India.
E-mail: wmshah@rediffmail.com

Sooraj Singh
Department of Mathematics, Central University of Kashmir,
Jammu and Kashmir, India.
E-mail: soorajsingh@cukashmir.ac.in