Korean J. Math. **19** (2011), No. 1, pp. 25–32

ON THE ES CURVATURE TENSOR IN $g - ESX_n$

In Ho Hwang

Abstract. This paper is a direct continuation of [1]. In this paper we investigate some properties of ES-curvature tensor of $g - ESX_n$, with main emphasis on the derivation of several useful generalized identities involving it. In this subsequent paper, we are concerned with contracted curvature tensors of $g - ESX_n$ and several generalized identities involving them. In particular, we prove the first variation of the generalized Bianchi's identity in $g - ESX_n$, which has a great deal of useful physical applications.

1. Preliminaries

This paper is a direct continuation of our previous paper [1], which will be denoted by I in the present paper. All considerations in this paper are based on our results and symbolism of I([2],[3],[5],[6],[8],[9]). Whenever necessary, these results will be quoted in the text. In this section, we introduce a brief collection of basic concepts, notations, and results of I, which are frequently used in the present paper.

(a) generalized n-dimensional Riemannian manifold X_n.

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformations $x^\nu \to x^{\nu'}$ for which

$$\det\left(\frac{\partial x'}{\partial x}\right) \neq 0. \tag{1.1}$$

In $n - g - UFT$ the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric

Received February 10, 2011. Revised March 5, 2011. Accepted March 10, 2011.
2000 Mathematics Subject Classification: 83E50, 83C05, 58A05.
Key words and phrases: ES-manifold, ES-curvature tensor.
This research was supported by University of Incheon Research Grant, 2009-2010.

part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}. \tag{1.2}$$

where

$$\mathfrak{g} = \det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = \det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = \det(k_{\lambda\mu}) \tag{1.3}$$

In virtue of (1.3) we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{1.4}$$

which together with $h_{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner.There exists a unique tensor $*g^{\lambda\nu}$ satisfying

$$g_{\lambda\mu} {}^*g^{\lambda\nu} = g_{\mu\lambda} {}^*g^{\nu\lambda} = \delta^{\nu}_{\mu}. \tag{1.5}$$

It may be also decomposed into its symmetric part ${}^*h_{\lambda\mu}$ and skew-symmetric part ${}^*k_{\lambda\mu}$:

$${}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}. \tag{1.6}$$

The manifold X_n is connected by a general real connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ with the following transformation rule:

$$\Gamma_{\lambda'}{}^{\nu'}{}_{\mu'} = \frac{\partial x^{\nu'}}{\partial x^{\alpha}} \left(\frac{\partial x^{\beta}}{\partial x^{\lambda'}} \frac{\partial x^{\gamma}}{\partial x^{\mu'}} \Gamma_{\beta}{}^{\alpha}{}_{\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x^{\lambda'} \partial x^{\mu'}} \right). \tag{1.7}$$

It may also be decomposed into its symmetric part $\Lambda_{\lambda}{}^{\nu}{}_{\mu}$ and its skew-symmetric part $S_{\lambda\nu}{}^{\nu}$, called the torsion of $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$:

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \Lambda_{\lambda}{}^{\nu}{}_{\mu} + S_{\lambda\mu}{}^{\nu}; \quad \Lambda_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{(\lambda}{}^{\nu}{}_{\mu)}; \quad S_{\lambda\mu}{}^{\nu} = \Gamma_{[\lambda}{}^{\nu}{}_{\mu]}. \tag{1.8}$$

A connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is said to be Einstein if it satisfies the following system of Einstein's equations:

$$\partial_{\omega} g_{\lambda\mu} - \Gamma_{\lambda}{}^{\alpha}{}_{\omega} g_{\alpha\mu} - \Gamma_{\omega}{}^{\alpha}{}_{\mu} g_{\lambda\alpha} = 0. \tag{1.9}$$

or equivalently

$$D_{\omega} g_{\lambda\mu} = 2 S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}. \tag{1.10}$$

where D_{ω} is the symbolic vector of the covariant derivative with respect to $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. In order to obtain $g_{\lambda\mu}$ involved in the solution for $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ in (1.9), certain conditions are imposed. These conditions may be condensed to

$$S_{\lambda} = S_{\lambda\alpha}{}^{\alpha} = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu} Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0. \tag{1.11}$$

where Y_λ is an arbitrary vector, and

(1.12) $R_{\omega\mu\lambda}{}^{\nu} = 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^{\nu}{}_{\omega]} + \Gamma_{\alpha}{}^{\nu}{}_{[\mu}\Gamma_{|\lambda|}{}^{\alpha}{}_{\omega]}), \qquad R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^{\alpha}.$

If the system (1.10) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form (Hlavatý, 1957)

(1.13) $$\Gamma_\lambda{}^\nu{}_\mu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}.$$

where $U^{\nu}{}_{\lambda\mu} = 2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}$ and $\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}$ are Christoffel symbols defined by $h_{\lambda\mu}$.

(b) Some notations and results

The following quantities are frequently used in our further considerations:

(1.14) $$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}}$$

(1.15) $$K_p = k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2} \cdots k_{\alpha_p]}{}^{\alpha^p}, \quad (p = 0, 1, 2, \cdots).$$

(1.16) $${}^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda, \ {}^{(p)}k_\lambda{}^\nu = k_\lambda{}^\alpha \ {}^{(p-1)}k_\alpha{}^\nu \quad (p = 1, 2, \cdots).$$

In X_n it was proved in [4] that

(1.17) $K_0 = 1, \ K_n = k$ if n is even, and $\mathrm{K_p} = 0$ if p is odd.

(1.18) $$\mathfrak{g} = \mathfrak{h}(1 + K_1 + K_2 + \cdots + K_n)$$

or

$$g = 1 + K_1 + K_2 + \cdots + K_n.$$

(1.19) $$\sum_{s=0}^{n-\sigma} K_s \ {}^{(n-s+p)}k_\lambda{}^\nu = 0 \quad (p = 0, 1, 2, \cdots).$$

We also use the following useful abbreviations for an arbitrary vector Y, for $p = 1, 2, 3, \cdots$:

(1.20) $${}^{(p)}Y_\lambda ={}^{(p-1)} k_\lambda{}^\alpha Y_\alpha.$$

(1.21) $${}^{(p)}Y^\nu ={}^{(p-1)} k^\nu{}_\alpha Y^\alpha.$$

(c) n-dimensional ES manifold ESX_n In this subsection, we display an useful representation of the ES-connection in n-g-UFT.

DEFINITION 1.1. *A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be semi-symmetric if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form*

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda} X_{\mu]} \tag{1.22}$$

for an arbitrary non-null vector X_μ.

A connection which is both semi-symmetric and Einstein is called a ES-connection. An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by $g_{\lambda\mu}$ by means of a ES-connection, is called an n-dimensional ES-manifold. We denote this manifold by $g - ESX_n$ in our further considerations.

THEOREM 1.2. *Under the condition* (1.22)*, the system of equations* (1.10) *is equivalent to*

$$\Gamma_\lambda{}^\nu{}_\mu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2k^\nu_{(\lambda} X_{\mu)} + 2\delta^\nu_{[\lambda} X_{\mu]}. \tag{1.23}$$

Proof. Substituting (1.22) for $S_{\lambda\mu}{}^\nu$ into (1.13), we have the representation (1.23). □

2. The ES curvature tensor in $g - ESX_n$

The n-dimensional ES curvature tensor $R_{\omega\mu\lambda}{}^\nu$ of $g - ESX_n$ is the curvature tensor defined by the ES-connection $\Gamma_\lambda{}^\nu{}_\mu$ under the present conditions. A lengthy, but precise and surveyable tensorial representation of $R_{\omega\mu\lambda}{}^\nu$ in terms of $g_{\lambda\mu}$ and their first two derivatives may be obtained by simply substituting (1.13) for $\Gamma_\lambda{}^\nu{}_\mu$ into (1.12). In this section, we present more concise and useful tensorial representation of $R_{\omega\mu\lambda}{}^\nu$ in terms of $g_{\lambda\mu}$ and the ES vector X_λ, and prove three identities involving it.

THEOREM 2.1. *Under the present conditions, the ES curvature tensor $R_{\omega\mu\lambda}{}^\nu$ of $g - ESX_n$ may be given by*

$$R_{\omega\mu\lambda}{}^\nu = L_{\omega\mu\lambda}{}^\nu + M_{\omega\mu\lambda}{}^\nu + N_{\omega\mu\lambda}{}^\nu \tag{2.1}$$

where

$$L_{\omega\mu\lambda}{}^{\nu} = 2\left(\partial_{[\mu}\left\{ \begin{array}{c} \nu \\ \omega]\lambda \end{array} \right\} + \left\{ \begin{array}{c} \nu \\ \alpha[\mu \end{array} \right\}\left\{ \begin{array}{c} \alpha \\ \omega]\lambda \end{array} \right\}\right) \tag{2.2}$$

$$M_{\omega\mu\lambda}{}^{\nu} = 2(\delta_{\lambda}^{\nu}\partial_{[\mu}X_{\omega]} + \delta_{[\mu}^{\nu}\nabla_{\omega]}X_{\lambda} + \nabla_{[\mu}U^{\nu}{}_{\omega]\lambda}) \tag{2.3}$$

$$N_{\omega\mu\lambda}{}^{\nu} = 2(\delta_{[\omega}^{\nu}X_{\mu]}X_{\lambda} +^{(2)} X_{\lambda}k_{[\mu}{}^{\nu}X_{\omega]}). \tag{2.4}$$

Proof. Substitute (1.13) into (1.12) and make use of (2.2) to obtain

$$\begin{aligned}
R_{\omega\mu\lambda}{}^{\nu} &= 2\partial_{[\mu}\left(\left\{ \begin{array}{c} \nu \\ \omega]\lambda \end{array} \right\} + X_{\omega]}\delta_{\lambda}^{\nu} - \delta_{\omega]}^{\nu}X_{\lambda} + U^{\nu}{}_{\omega]\lambda}\right) \\
&+ 2\left(\left\{ \begin{array}{c} \nu \\ \alpha[\mu \end{array} \right\} + \delta_{\alpha}^{\nu}X_{[\mu} - X_{\alpha}\delta_{[\mu}^{\nu} + U^{\nu}{}_{\alpha[\mu}\right) \\
&\times \left(\left\{ \begin{array}{c} \alpha \\ \omega]\lambda \end{array} \right\} + X_{\omega]}\delta_{\lambda}^{\alpha} - \delta_{\omega]}^{\alpha}X_{\lambda} + U^{\alpha}{}_{\omega]\lambda}\right) \\
&= L_{\omega\mu\lambda}{}^{\nu} + 2\delta_{\lambda}^{\nu}\partial_{[\mu}X_{\omega]} + 2\left(\delta_{[\mu}^{\nu}\partial_{\omega]}X_{\lambda} - \delta_{[\mu}^{\nu}\left\{ \begin{array}{c} \alpha \\ \omega]\lambda \end{array} \right\}X_{\alpha}\right) \\
&+ 2\left(\partial_{[\mu}U^{\nu}{}_{\omega]\lambda} + \left\{ \begin{array}{c} \alpha \\ \lambda[\omega \end{array} \right\}U^{\nu}{}_{\mu]\alpha} + \left\{ \begin{array}{c} \nu \\ \alpha[\mu \end{array} \right\}U^{\alpha}{}_{\omega]\lambda}\right) \\
&+ 2\left(\delta_{[\omega}^{\nu}X_{\mu]}X_{\lambda} - X_{\alpha}\delta_{[\mu}^{\nu}U^{\alpha}{}_{\omega]\lambda} + U^{\nu}{}_{\alpha[\mu}U^{\alpha}{}_{\omega]\lambda}\right).
\end{aligned} \tag{2.5}$$

In virtue of (1.22), the sum of the second, third and fourth terms on the right-hand side of (2.5) is $M_{\omega\mu\lambda}{}^{\nu}$. On the other hand, using (1.22), the first relation of (3.4), and (3.10) in I, we have

$$U^{\nu}{}_{\lambda\mu} = 2k_{(\lambda}{}^{\nu}X_{\mu)} \tag{2.6}$$

$$-X_{\alpha}\delta_{[\mu}^{\nu}U^{\alpha}{}_{\omega]\lambda} = 0 \tag{2.7}$$

$$U^{\nu}{}_{\alpha[\mu}U^{\alpha}{}_{\omega]\lambda} =^{(2)} X_{\lambda}k_{[\mu}{}^{\nu}X_{\omega]}. \tag{2.8}$$

Substituting (2.7) and (2.8) into the fifth term of (2.5), we find that it is equal to $N_{\omega\mu\lambda}{}^{\nu}$. Consequently, our proof of the theorem is completed. □

THEOREM 2.2. *Under the present conditions, the ES curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $g-ESX_n$ is a tensor involved in the following identity:*

$$R_{[\omega\mu\lambda]}{}^{\nu} = 4\delta^{\nu}_{[\lambda}\partial_{\mu}X_{\omega]} \tag{2.9}$$

Proof. The relation (2.1) gives

$$R_{[\omega\mu\lambda]}{}^{\nu} = L_{[\omega\mu\lambda]}{}^{\nu} + M_{[\omega\mu\lambda]}{}^{\nu} + N_{[\omega\mu\lambda]}{}^{\nu} \tag{2.10}$$

On the other hand, in virtue of (2.2),(2.3) and (2.4) we have

$$L_{[\omega\mu\lambda]}{}^{\nu} = M_{[\omega\mu\lambda]}{}^{\nu} = 0, \qquad N_{[\omega\mu\lambda]}{}^{\nu} = 4\delta^{\nu}_{[\mu}\partial_{\omega}X_{\lambda]}. \tag{2.11}$$

Our identity (2.9) is a consequence of (2.10) and (2.11). □

THEOREM 2.3. *(Generalized Ricci identity in $g-ESX_n$) Under the present conditions, the ES curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $g-ESX_n$ satisfies the following identity:*

$$\begin{aligned} 2D_{[\omega}D_{\mu]}T^{\nu_1\cdots\nu_q}_{\lambda_1\cdots\lambda_p} &= -\sum_{\alpha=1}^{p} T^{\nu_1\cdots\nu_{\alpha-1}\epsilon\nu_{\alpha+1}\cdots\nu_p}_{\lambda_1\cdots\lambda_p} R_{\omega\mu\epsilon}{}^{\nu_\alpha} \\ &\quad +\sum_{\beta=1}^{q} T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_{p-1}\epsilon\lambda_{p+1}\cdots\lambda_q} R_{\omega\mu\lambda_\beta}{}^{\epsilon} - 4X_{[\omega}D_{\mu]}T^{\nu_1\cdots\nu_q}_{\lambda_1\cdots\lambda_p} \end{aligned} \tag{2.12}$$

Proof. Making use of (1.22), we see that (2.12) is a direct consequence of Hlavatý's results([7],1957)

$$\begin{aligned} 2D_{[\omega}D_{\mu]}T^{\nu_1\cdots\nu_q}_{\lambda_1\cdots\lambda_p} &= -\sum_{\alpha=1}^{p} T^{\nu_1\cdots\nu_{\alpha-1}\epsilon\nu_{\alpha+1}\cdots\nu_p}_{\lambda_1\cdots\lambda_q} R_{\omega\mu\epsilon}{}^{\nu_\alpha} \\ &\quad +\sum_{\beta=1}^{q} T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_{p-1}\epsilon\lambda_{p+1}\cdots\lambda_q} R_{\omega\mu\lambda_\beta}{}^{\epsilon} + 2S_{\omega\mu}{}^{\alpha} D_\alpha T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_q}. \end{aligned} \tag{2.13}$$

□

THEOREM 2.4. *(Generalized Bianchi's identity in $g-ESX_n$) Under the present conditions, the ES curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $g-ESX_n$ satisfies the following identity:*

$$D_{[\epsilon}R_{\omega\mu]\lambda}{}^{\nu} = -4X_{[\epsilon}L_{\omega\mu]\lambda}{}^{\nu} + O_{[\epsilon\omega\mu]\lambda}{}^{\nu} \tag{2.14}$$

where

$$\begin{aligned} \frac{1}{8}O_{\epsilon\omega\mu\lambda}{}^{\nu} &= \delta^{\nu}_{\lambda}X_\epsilon\partial_\omega X_\mu + X_\epsilon\delta^{\nu}_{\omega}\nabla_\mu X_\lambda \\ &\quad + X_\epsilon\nabla_\omega U^{\nu}{}_{\mu\lambda} + X_\epsilon\delta^{\nu}_{\mu}X_\omega X_\lambda +^{(2)} X_\lambda X_\epsilon k_\omega{}^{\nu}X_\mu \end{aligned} \tag{2.15}$$

Proof. On a manifold X_n to which an Einstein's connection is connected, Hlavatý proved the following identity([7], 1957):

$$D_{[\epsilon} R_{\omega\mu]\lambda}{}^{\nu} = -2S_{[\epsilon\omega}{}^{\beta} R_{\mu]\beta\lambda}{}^{\nu} \tag{2.16}$$

In virtue of (1.22) and (2.1), the identity (2.16) may be written as

$$\begin{aligned} (2.17) D_{[\epsilon} R_{\omega\mu]\lambda}{}^{\nu} &= -2S_{[\epsilon\omega}{}^{\beta} L_{\mu]\beta\lambda}{}^{\nu} - 2S_{[\epsilon\omega}{}^{\beta} M_{\mu]\beta\lambda}{}^{\nu} - 2S_{[\epsilon\omega}{}^{\beta} N_{\mu]\beta\lambda}{}^{\nu} \\ &= -4X_{[\epsilon} L_{\omega\mu]\lambda}{}^{\nu} - 4X_{[\epsilon} M_{\omega\mu]\lambda}{}^{\nu} - 4X_{[\epsilon} N_{\omega\mu]\lambda}{}^{\nu} \end{aligned}$$

In virtue of (2.3), the second relation on the right-hand side of (2.17) may be expressed in the form

$$\begin{aligned} (2.18) \qquad & - \quad 4X_{[\epsilon} M_{\omega\mu]\lambda}{}^{\nu} \\ & = \quad -8(\delta_{\lambda}^{\nu} X_{[\epsilon} \partial_{\mu} X_{\omega]} + X_{[\epsilon} \delta_{\mu}^{\nu} \nabla_{\omega]} X_{\lambda} + X_{[\epsilon} \nabla_{\mu} U^{\nu}{}_{\omega]\lambda]}). \end{aligned}$$

The relation (2.4) enalbes one to write the third term on the right-hand side of (2.7) as follows:

$$-4X_{[\epsilon} N_{\omega\mu]\lambda}{}^{\nu} = -8(X_{[\epsilon} \delta_{\omega}^{\nu} X_{\mu]} X_{\lambda} +^{(2)} X_{\lambda} X_{[\epsilon} k_{\mu}{}^{\nu} X_{\omega]}). \tag{2.19}$$

We now substitute (2.18) and (2.19) into (2.17) and make use of (2.15) to complete the proof of the theorem. □

References

[1] Hwang, I.H., *A study on the recurrence relations and vectors X_λ, S_λ and U_λ in $g - ESX_n$*, Korean J. Math. **18**(2) (2010), 133–139.
[2] Hwang, I.H., *A study on the geometry of 2-dimensional RE-manifold X_2*, J. Korean Math. Soc. **32**(2) (1995), 301–309.
[3] Hwang, I.H., *Three- and Five- dimensional considerations of the geometry of Einstein's g-unified field theory*, Int. J . Theor. Phys. **27**(9) (1988), 1105–1136.
[4] Chung, K.T., *Einstein's connection in terms of ${}^*g^{\lambda\nu}$*, Nuovo Cimento Soc. Ital. Fis. B **27**(X) (1963), 1297–1324.
[5] Datta, D.k., *Some theorems on symmetric recurrent tensors of the second order*, Tensor (N.S.) **15** (1964), 1105–1136.
[6] Einstein, A., *The meaning of relativity*, Princeton University Press, 1950.
[7] Hlavatý, V., *Geometry of Einstein's unified field theory*, Noordhoop Ltd., 1957.
[8] Mishra, R.S., *n-dimensional considerations of unified field theory of relativity*, Tensor (N.S.) **9** (1959), 217–225.
[9] Werde, R.C., *n-dimensional considerations of the basic principles A and B of the unified field theory of relativity*, Tensor (N.S.) **8** (1958), 95–122.

Department of Mathematics
University of Incheon
Incheon 406-772 , Korea
E-mail: ho818@incheon.ac.kr