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STATISTICAL CONVERGENCE IN PARTIAL METRIC SPACES

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Abstract. Let X be a partial metric space generated by a partial metric p. In this paper, we introduce the notions of statistical convergence and strongly Cesàro summability in partial metric spaces. Also, we investigate the relations between the statistical convergence and strongly Cesàro summability.

1. Introduction

The density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_K(j),$$

whenever the limit exists, where χ_K is the characteristic function of K. If $\{x_n\}$ is a sequence, which satisfies a property P for all n except a set of natural density zero, then we say that $\{x_n\}$ satisfies P for "almost all n ", and we abbreviate this by "a.a. n." Statistical convergence of sequences of real or complex numbers was introduced by Steinhaus in [17] and Fast in [7]. A sequence $\{x_n\}$ of real or complex numbers is said to be statistically convergent to the number a, and denoted by $st - \lim x_n = a$, if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - a| \geq \varepsilon\}) = 0$, or equivalently there exists a subset $K \subset \mathbb{N}$ with $\delta(K) = 1$ and n_0 such that for any $k \in K$, $k > n_0$ we have $|x_k - a| < \varepsilon$ (see e.g. [3,12,15]). It is known that any convergent sequence is statistically convergent, but not conversely. A sequence $\{x_n\}$ of real or complex numbers is said to be statistically Cauchy if for each $\varepsilon > 0$ there is a positive integer $N = N(\varepsilon)$ such that $\delta(\{k \in \mathbb{N} : |x_k - x_N| \geq \varepsilon\}) = 0$. Basic properties of the statistical convergence and of some related summability methods were established in [1, 16].

Over almost 70 years since its inception, the concept of statistical convergence has been studied in the context of numerous mathematical disciplines including: the summability theory [1,3,4], number theory, trigonometric series theory [18], probability [6], measure theory [12], optimization [14], and approximation theory [8].

In the approximation theory context, the authors of [8] extended the definition of statistical convergence from the number sequences to the sequences of elements from some function spaces to use this convergence for proving generalizations of Korovkin

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and Weierstrauss type approximation theorems. In this study, we will first give definitions of statistically convergent sequence and strongly Cesàro convergent sequence in partial metric spaces. We will then state and prove a theorem that shows the relationship between these concepts. In section 2, we introduce necessary concepts. In section 3, we introduce and prove some base results of statistical convergence in a partial metric space. In section 4, we deal with the strongly Cesàro summability in a partial metric space.

2. Preliminaries

Partial metric spaces were originally developed by S. Matthews [11] in 1994 to provide a mechanism to generalize a metric space. If (X, p) is a partial metric space, then $p(x, x)$ is not necessary zero as $x \in X$. Partial metric spaces as defined has now found vast applications in topological structures in the study of computer science, information science and in biological sciences. Banach contraction principle is a fundamental result in fixed point theory in a complete metric space and the same has been extended in many directions like inviting broader class of mappings or by taking more generalized domain or by making a combination of both. First, we recall some definitions of partial metric space and some their properties.

Definition 2.1. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

(P1) If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (indistancy implies equality),
(P2) $0 \leq p(x, x) \leq p(x, y)$ (nonnegativity and small self-distances),
(P3) $p(x, y) = p(y, x)$ (symmetry), and
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ (triangularity).

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

Each partial metric space gives rise to a metric space with the additional notion of nonzero self-distance introduced. Also, a partial metric space is a generalization of a metric space; indeed, if $p(x, x) = 0$ is imposed, then the above axioms reduce to their metric counterparts. Thus, a metric space can be defined to be a partial metric space in which each self-distance is zero.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls

$$\{B_p(x, \varepsilon), x \in X, \varepsilon > 0\},$$

where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^* : X \times X \longrightarrow \mathbb{R}$ given by

$$p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

The following examples of partial metric spaces can be found in the literature (see e.g. [11, 13]).

EXAMPLE 2.2. Let X denote the set of all intervals $[a, b]$ for any real numbers $a \leq b$. Define $p : X \times X \to [0, \infty)$ such that

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}.$$

Then (X, p) is a partial metric space.

EXAMPLE 2.3. Let $X = \mathbb{R}$ and $p(x, y) = 2^{max\{x,y\}}$ for all $x, y \in X$. Then, (X, p) is a partial metric space.

EXAMPLE 2.4. Let $X = \mathbb{R}^+$ and $p : X \times X \to \mathbb{R}^+$ given by $p(x, y) = max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a partial metric space.

EXAMPLE 2.5. For a given positive integer n, let Ψ denotes the collection of all real polynomials like $f(t) = a_0 + a_1 t + ... + a_n t^n, a_i \in \mathbb{R}$ with degree $\leq n$. If $f_1, f_2 \in \Psi$, let

$$p(f_1, f_2) = \max_{0 \leq i \leq n} \{a_i, b_i\},$$

where a_i, b_i are coefficients of the polynomials f_1, f_2, respectively. Then (Ψ, p) is a partial metric space.

DEFINITION 2.6. Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ is bounded if there exists a real number $M > 0$ such that $p(x_n, x_m) \leq M$ for all $n, m \in \mathbb{N}$,

(ii) $\{x_n\}$ converges to a point $x \in X$ if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$,

(iii) $\{x_n\}$ is called a Cauchy sequence if there exists $l \geq 0$ such that for each $\varepsilon > 0$ there exists k_0 such that for all $n, m > k_0$, $|p(x_n, x_m) - l| < \varepsilon$.

Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(,)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$. For example, if $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}, p(x_n, x) = x = p(x, x)$ for each $x \geq 1$ and so, e.g., $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$.

Suppose $\{x_n\}$ is a sequence in a partial metric space (X, p), and we define $\mathcal{L}(x_n)$ to be the set of limit points of $\{x_n\}$. For example, in $\mathbb{R}$ with the usual partial metric, the sequence $\{\frac{1}{n}\}$ has $\mathcal{L}(\frac{1}{n}) = (-\infty, 0]$.

DEFINITION 2.7. Let $\{x_n\}$ be a sequence in a partial metric space (X, p), then $a \in X$ is a proper limit of $\{x_n\}$, written $x_n \to a(properly)$, if $x_n \to a$ in (X, p^*). If a sequence has a proper limit then we say that the sequence is properly convergent.

In $\mathbb{R}$ with the usual partial metric the proper limit of the sequence $\{\frac{1}{n}\}$ is 0. It is known that [13] if $\{x_n\}$ is a sequence in a partial metric space (X, p), and $x_n \to a(properly)$, then $a = \sup \mathcal{L}(x_n)$.

A partial metric space (X, p) is complete if every Cauchy sequence converges.

3. Statistical Convergence

Let us start with the following definition.

DEFINITION 3.1. Let X be a partial metric space and $\{x_n\}$ be a sequence in X.

(a) We say that the sequence $\{x_n\}$ is statistically convergent to $x \in X$ if for every $\varepsilon > 0$,
$$\delta(\{n \in \mathbb{N} : |p(x, x_n) - p(x, x)| \geq \varepsilon\}) = 0.$$
(b) The sequence $\{x_n\}$ is called statistically Cauchy if for each $\varepsilon > 0$ there is a positive integer N and $l \geq 0$ such that
$$\delta(\{n \in \mathbb{N} : |p(x_k, x_N) - l| \geq \varepsilon\}) = 0.$$
(c) We say that $a \in X$ is a proper statistical limit of $\{x_n\}$, written $st - \lim x_n = a(properly)$, if $st - \lim x_n = a$ in (X, p^*). If a sequence has a proper statistical limit then we say that the sequence is properly statistical convergent.

Theorem 3.2. *Let $\{x_n\}$ be a sequence in a partial metric space (X, p) and $a \in X$, then $st - \lim x_n = a(properly)$ if and only if*
$$st - \lim p(x_n, a) = st - \lim p(x_n, x_n) = p(a, a).$$

Proof.
$$\begin{aligned} & st - \lim x_n = a(properly) \\ \Leftrightarrow\ & p^*(x_n, a) < \varepsilon \text{ a.a. } n \\ \Leftrightarrow\ & |2p(x_n, a) - p(x_n, x_n) - p(a, a)| < \varepsilon \text{ a.a. } n \\ \Leftrightarrow\ & |p(x_n, a) - p(a, a)| < \frac{\varepsilon}{2} \text{ and } |p(x_n, x_n) - p(a, a)| < \frac{\varepsilon}{2} \text{ a.a. } n \\ \Leftrightarrow\ & st - \lim p(x_n, a) = st - \lim p(x_n, x_n) = p(a, a). \end{aligned}$$
□

4. Strongly Cesàro Summability and Inclusion Relations

Definition 4.1. Let X be a partial metric space and $\{x_n\}$ be a sequence in X, and q be a positive real number. We say that $\{x_n\}$ is strongly q-Cesàro summable to $x \in X$ if
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |p(x_k, x) - p(x, x)|^q = 0.$$
In this case we write $[C, q] - \lim x_n = x$

Definition 4.2. Let X be a partial metric space and $\{x_n\}$ be a sequence in X, and q be a positive real number. We say that $\{x_n\}$ is strongly q-Cesàro summable to $a \in X$ (properly) if
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |p^*(x_k, a)|^q = 0.$$
In this case, we write $[C, q] - \lim x_n = a(properly)$.

Theorem 4.3. *Let $\{x_n\}$ be a sequence in a partial metric space (X, p) and $a \in X$, then $st - \lim x_n = a(properly)$ if and only if*
$$[C, q] - \lim p(x_n, a) = [C, q] - \lim p(x_n, x_n) = p(a, a).$$

Proof.

$$\begin{aligned} & [C,q]-\lim x_n = a(properly) \\ \Leftrightarrow\ & [C,q]-\lim p^*(x_n,a)=0 \\ \Leftrightarrow\ & [C,q]-\lim(2p(x_n,a)-p(x_n,x_n)-p(a,a))=0 \\ \Leftrightarrow\ & [C,q]-\lim p(x_n,a)=[C,q]-\lim p(x_n,x_n)=p(a,a). \end{aligned}$$

□

In the following theorem, we will state the relation between strongly q-Cesàro convergence and statistical convergence in partial metric spaces.

THEOREM 4.4. *Let X be a partial metric space, $\{x_n\}$ be a sequence in X, and $q\in\mathbb{R}$, $0<q<\infty$. If a sequence is strongly q-Cesàro summable to $x\in X$, then it is statistically convergent to $x\in X$. If a bounded sequence is statistically convergent to $x\in X$, then it is strongly q-Cesàro summable to $x\in X$.*

Proof. For any sequence $\{x_n\}$ of elements of X and $\varepsilon>0$, we have that

$$\begin{aligned} & \sum_{k=1}^{n}|p(x_k,x)-p(x,x)|^q \\ = & \sum_{\substack{k=1\\|p(x_k,x)-p(x,x)|\geq\varepsilon}}^{n}|p(x_k,x)-p(x,x)|^q + \sum_{\substack{k=1\\|p(x_k,x)-p(x,x)|<\varepsilon}}^{n}|p(x_k,x)-p(x,x)|^q \\ \geq & \sum_{\substack{k=1\\|p(x_k,x)-p(x,x)|\geq\varepsilon}}^{n}|p(x_k,x)-p(x,x)|^q \\ \geq & \ card\{k\leq n:\quad |p(x_k,x)-p(x,x)|\geq\varepsilon\}\varepsilon^q. \end{aligned}$$

It follows that if $\{x_n\}$ is strongly q-Cesàro summable to x then $\{x_n\}$ is statistically convergent to x.

Now suppose that $\{x_n\}$ is bounded and statistically convergent to x and set $|p(x_k,x)-p(x,x)|)<M$. Let $\varepsilon>0$ be given and select n_ε such that

$$\frac{1}{n}card\left\{k\leq n:\quad |p(x_k,x)-p(x,x)|>\left(\frac{\varepsilon}{2}\right)^{\frac{1}{q}}\right\}<\frac{\varepsilon}{2M^q}$$

for all $n>n_\varepsilon$ and set $P_n=\left\{k\leq n:\quad |p(x_k,x)-p(x,x)|>(\frac{\varepsilon}{2})^{\frac{1}{q}}\right\}$. Now, for $n>n_\varepsilon$ we have that

$$\begin{aligned} & \frac{1}{n}\sum_{k=1}^{n}|p(x_k,x)-p(x,x)|^q \\ = & \frac{1}{n}\left(\sum_{k\in P_n}|p(x_k,x)-p(x,x)|^q+\sum_{\substack{k\in P_n\\k\leq n}}|p(x_k,x)-p(x,x)|^q\right) \\ < & \frac{1}{n}\left(\frac{n\varepsilon}{2M^q}\right)M^q+\frac{1}{n}n\frac{\varepsilon}{2}=\varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is strongly q-Cesàro summable to x. □

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