HOMOMORPHISMS IN PROPER LIE $CQ^*$-ALGEBRAS

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Abstract. Using the Hyers-Ulam-Rassias stability method of functional equations, we investigate homomorphisms in proper $CQ^*$-algebras and proper Lie $CQ^*$-algebras, and derivations on proper $CQ^*$-algebras and proper Lie $CQ^*$-algebras associated with the following functional equation

$$\frac{1}{k} f(kx + ky + kz) = f(x) + f(y) + f(z)$$

for a fixed positive integer $k$.

1. Introduction and preliminaries

Ulam [46] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group $G$ and a metric group $G'$ with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$\rho(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G'$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated.

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Hyers [18] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies Hyers inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Th.M. Rassias [34] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded.

**Theorem 1.1.** (Th.M. Rassias). Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where $\theta$ and $p$ are positive real numbers with $p < 1$. Then the limit

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is $\mathbb{R}$-linear.

Th.M. Rassias [35] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [14] following the same approach as in Th.M. Rassias [34], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [14], as well as by Th.M. Rassias and Šemrl [40] that one cannot prove a Th.M. Rassias’ type theorem when $p = 1$. The counterexamples of Gajda [14], as well as of Th.M. Rassias and Šemrl [40] have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings, cf. P. Găvruta [15], who among others studied the Hyers-Ulam stability of functional equations. The inequality (1.1) that was introduced for the first time by
Th.M. Rassias [34] provided a lot of influence in the development of a
generalization of the Hyers-Ulam stability concept. This new concept is
known as Hyers-Ulam-Rassias stability of functional equations (cf. the
books of P. Czerwik [10, 11], D.H. Hyers, G. Isac and Th.M. Rassias
[19]).

Beginning around the year 1980, the topic of approximate homomor-
phisms and their stability theory in the field of functional equations and
inequalities was taken up by several mathematicians (cf. D.H. Hyers and
Th.M. Rassias [20], Th.M. Rassias [38] and the references therein).

J.M. Rassias [32] following the spirit of the innovative approach of
Th.M. Rassias [34] for the unbounded Cauchy difference proved a similar
stability theorem in which he replaced the factor \( \|x\|^p + \|y\|^p \) by \( \|x\|^p \cdot \|y\|^q \)
for \( p, q \in \mathbb{R} \) with \( p + q \neq 1 \) (see also [33] for a number of other new
results).

**Theorem 1.2.** [31, 32, 33] Let \( X \) be a real normed linear space and
\( Y \) a real complete normed linear space. Assume that \( f : X \to Y \) is an
approximately additive mapping for which there exist constants \( \theta \geq 0 \)
and \( p \in \mathbb{R} - \{1\} \) such that \( f \) satisfies inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \theta \cdot \|x\|^\frac{p}{2} \cdot \|y\|^\frac{q}{2}
\]
for all \( x, y \in X \). Then there exists a unique additive mapping \( L : X \to Y \)
satisfying
\[
\|f(x) - L(x)\| \leq \frac{\theta}{|2p - 2|} \|x\|^p
\]
for all \( x \in X \). If, in addition, \( f : X \to Y \) is a mapping such that the
transformation \( t \to f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \),
then \( L \) is an \( \mathbb{R} \)-linear mapping.

Several mathematicians have contributed works on these subjects (see
[22]–[28], [36]–[39], [42]).

In a series of papers [1]–[13] and [43]–[45], many authors have con-
sidered a special class of quasi \(*\)-algebras, called proper \( CQ^*\)-algebras,
which arise as completions of \( C^*\)-algebras. They can be introduced in
the following way:

Let \( A \) be a Banach module over the \( C^*\)-algebra \( A_0 \) with involution \(*\)
and \( C^*\)-norm \( \| \cdot \|_0 \) such that \( A_0 \subset A \). We say that \((A, A_0)\) is a proper
\( CQ^*\)-algebra if

(i) \( A_0 \) is dense in \( A \) with respect to its norm \( \| \cdot \| ; \)
(ii) an involution $\ast$, which extends the involution of $A_0$, is defined in $A$ with the property $(xy)^\ast = y^\ast x^\ast$ for all $x, y \in A$ whenever the multiplication is defined.

(iii) $\|y\|_0 = \sup_{x \in A, \|x\| \leq 1} \|xy\|$ for all $y \in A_0$.

**Definition 1.3.** Let $(A, A_0)$ and $(B, B_0)$ be proper $CQ^*$-algebras.

(i) A $\mathbb{C}$-linear mapping $H: A \to B$ is called a proper $CQ^*$-algebra homomorphism if $H(z) \in B_0$ and $H(zx) = H(z)H(x)$ for all $z \in A_0$ and all $x \in A$.

(ii) A $\mathbb{C}$-linear mapping $\delta: A_0 \to A$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A_0$ (see [3]).

A $C^*$-algebra $C$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ on $C$, is called a Lie $C^*$-algebra. (see [22], [24], [30]).

**Definition 1.4.** A proper $CQ^*$-algebra $(A, A_0)$, endowed with the Lie product $[z, x] := \frac{zx - xz}{2}$ for all $z \in A_0$ and all $x \in A$, is called a proper Lie $CQ^*$-algebra.

**Definition 1.5.** Let $(A, A_0)$ and $(B, B_0)$ be proper Lie $CQ^*$-algebras.

(i) A $\mathbb{C}$-linear mapping $H: A \to B$ is called a proper Lie $CQ^*$-algebra homomorphism if $H(z) \in B_0$ and $H([z, x]) = [H(z), H(x)]$ for all $z \in A_0$ and all $x \in A$.

(ii) A $\mathbb{C}$-linear mapping $\delta: A_0 \to A$ is called a Lie derivation if $\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$ for all $x, y \in A_0$.

In [16], Gilányi showed that if $f$ satisfies the functional inequality
\begin{equation}
\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \tag{1.2}
\end{equation}
then $f$ satisfies the Jordan-von Neumann functional inequality
\begin{equation}
2f(x) + 2f(y) = f(x + y) + f(x - y).
\end{equation}

This paper is organized as follows: In Section 2, we investigate homomorphisms in proper $CQ^*$-algebras associated with the functional equation
\begin{equation}
\frac{1}{k}f(kx + ky + kz) = f(x) + f(y) + f(z). \tag{1.3}
\end{equation}
In Section 3, we investigate derivations on proper $CQ^*$-algebras associated with the functional equation (1.3).

In Section 4, we investigate homomorphisms in proper Lie $CQ^*$-algebras associated with the functional equation (1.3).

In Section 5, we investigate derivations on proper Lie $CQ^*$-algebras associated with the functional equation (1.3).

2. Homomorphisms in proper $CQ^*$-algebras

Throughout this section, assume that $(A,A_0)$ is a proper $CQ^*$-algebra with $C^*$-norm $\| \cdot \|_A$ and norm $\| \cdot \|_A$, and that $(B,B_0)$ is a proper $CQ^*$-algebra with $C^*$-norm $\| \cdot \|_B$ and norm $\| \cdot \|_B$.

**Proposition 2.1.** Let $X$ and $Y$ be normed spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. Let $f : X \to Y$ be a mapping such that

$$\| f(x) + f(y) + f(z) \|_Y \leq \| \frac{1}{k} f(kx + ky + kz) \|_Y$$

for all $x, y, z \in X$. Then $f$ is Cauchy additive, i.e., $f(x+y) = f(x) + f(y)$.

**Proof.** Letting $x = y = z = 0$ in (2.1), we get

$$\|3f(0)\|_Y \leq \| \frac{1}{k} f(0) \|_Y.$$ 

So $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (2.1), we get

$$\| f(x) + f(-x) \|_Y \leq \| \frac{1}{k} f(0) \|_Y = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ in (2.1), we get

$$\| f(x) + f(y) - f(x + y) \|_Y = \| f(x) + f(y) + f(-x - y) \|_Y$$

$$\leq \| \frac{1}{k} f(0) \|_Y = 0$$

for all $x, y \in X$. Thus

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$, as desired. \hfill \Box

We investigate homomorphisms in proper $CQ^*$-algebras associated with the functional equation (1.3).
Theorem 2.2. Let \( r \neq 1 \) and \( \theta \) be nonnegative real numbers, and \( f : A \to B \) a mapping satisfying \( f(w) \in B_0 \) for all \( w \in A_0 \) such that

\[
\|\mu f(x) + f(y) + f(z)\|_B \leq \frac{1}{k} f(k\mu x + ky + kz)\|_B,
\]

(2.2)

\[
\|f(wx) - f(w)f(x)\|_B \leq \theta(\|w\|_A^2 + \|x\|_A^2)
\]

(2.3)

for \( \mu \in T^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \), all \( w \in A_0 \) and all \( x, y, z \in A \). Then the mapping \( f : A \to B \) is a proper \( CQ^* \)-algebra homomorphism.

Proof. Let \( \mu = 1 \) in (2.2). By Proposition 2.1, the mapping \( f : A \to B \) is Cauchy additive.

Letting \( z = 0 \) and \( y = -\mu x \) in (2.2), we get

\[
\mu f(x) - f(\mu x) = \mu f(x) + f(-\mu x) = 0
\]

for all \( x \in A \). So \( f(\mu x) = \mu f(x) \) for all \( x \in A \). By the same reasoning as in the proof of Theorem 2.1 of [23], the mapping \( f : A \to B \) is \( C \)-linear.

(i) Assume that \( r < 1 \). By (2.3),

\[
\|f(wx) - f(w)f(x)\|_B = \lim_{n \to \infty} \frac{1}{4^n} \|f(4^n wx) - f(2^n w)f(2^n x)\|_B
\]

\[
\leq \lim_{n \to \infty} \frac{4^{nr}}{4^n} \theta(\|w\|_A^{2r} + \|x\|_A^{2r}) = 0
\]

for all \( w \in A_0 \) and all \( x \in A \). So

\[
f(wx) = f(w)f(x)
\]

for all \( w \in A_0 \) and all \( x \in A \).

(ii) Assume that \( r > 1 \). By a similar method to the proof of the case (i), one can prove that the mapping \( f : A \to B \) satisfies

\[
f(wx) = f(w)f(x)
\]

for all \( w \in A_0 \) and all \( x \in A \).

Since \( f(w) \in B_0 \) for all \( w \in A_0 \), the mapping \( f : A \to B \) is a proper \( CQ^* \)-algebra homomorphism, as desired.

Theorem 2.3. Let \( r \neq 1 \) and \( \theta \) be nonnegative real numbers, and \( f : A \to B \) a mapping satisfying (2.2) and \( f(w) \in B_0 \) for all \( w \in A_0 \) such that

\[
\|f(wx) - f(w)f(x)\|_B \leq \theta \cdot \|w\|_A^r \cdot \|x\|_A^r
\]

(2.4)

for all \( w \in A_0 \) and all \( x \in A \). Then the mapping \( f : A \to B \) is a proper \( CQ^* \)-algebra homomorphism.
Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $C$-linear.

(i) Assume that $r < 1$. By (2.4),
\[
\|f(wx) - f(w)f(x)\|_B = \lim_{n \to \infty} \frac{1}{4^n} \|f(4^nwx) - f(2^n w)f(2^n x)\|_B \\
\leq \lim_{n \to \infty} \frac{4^{nr}}{4^n} \theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0
\]
for all $w \in A_0$ and all $x \in A$. So
\[
f(wx) = f(w)f(x)
\]
for all $w \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies
\[
f(wx) = f(w)f(x)
\]
for all $w \in A_0$ and all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.2. \qed

3. Derivations on proper $CQ^*$-algebras

Throughout this section, assume that $(A, A_0)$ is a proper $CQ^*$-algebra with $C^*$-norm $\| \cdot \|_{A_0}$ and norm $\| \cdot \|_A$.

We investigate derivations on proper $CQ^*$-algebras associated with the functional equation (1.3).

**Theorem 3.1.** Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A \to A$ a mapping such that
\[
\|f(\mu x) + f(y) + f(z)\|_A \leq \frac{1}{k} f(k \mu x + ky + kz)\|_A,
\]
\[
\|f(w_0w_1) - f(w_0)w_1 - w_0f(w_1)\|_A \leq \theta(\|w_0\|_A^2 + \|w_1\|_A^2)
\]
for $\mu \in T^1$, all $w_0, w_1 \in A_0$ and all $x, y, z \in A$. Then the mapping $f : A \to A$ is a derivation on $A$.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $C$-linear.
(i) Assume that $r < 1$. By (3.2),
\[
\|f(w_0w_1) - f(w_0)w_1 - w_0f(w_1)\|_A
\leq \lim_{n \to \infty} \frac{1}{4^n} \theta \left( \|w_0\|^2r_A + \|w_1\|^2r_A \right) = 0
\]
for all $w_0, w_1 \in A_0$. So
\[
f(w_0w_1) = f(w_0)w_1 + w_0f(w_1)
\]
for all $w_0, w_1 \in A_0$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies
\[
f(w_0w_1) = f(w_0)w_1 + w_0f(w_1)
\]
for all $w_0, w_1 \in A_0$.

Therefore, the mapping $f : A \to A$ is a derivation on $A$, as desired.

\[\square\]

**Theorem 3.2.** Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying (3.1) such that
\[
\|f(w_0w_1) - f(w_0)w_1 - w_0f(w_1)\|_A \leq \theta \cdot \|w_0\|^2r_A \cdot \|w_1\|^2r_A
\]
for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a derivation on $A$.

**Proof.** The proof is similar to the proofs of Theorems 2.2 and 3.1. \[\square\]

4. Homomorphisms in proper Lie $CQ^*$-algebras

Throughout this section, assume that $(A, A_0)$ is a proper Lie $CQ^*$-algebra with $C^*$-norm $\| \cdot \|_{A_0}$ and norm $\| \cdot \|_A$, and that $(B, B_0)$ is a proper Lie $CQ^*$-algebra with $C^*$-norm $\| \cdot \|_{B_0}$ and norm $\| \cdot \|_B$.

We investigate homomorphisms in proper Lie $CQ^*$-algebras associated with the functional equation (1.3).

**Theorem 4.1.** Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying (2.2) and $f(w) \in B_0$ for all $w \in A_0$ such that
\[
\|f([w, x]) - [f(w), f(x)]\|_B \leq \theta (\|w\|^2r_A + \|x\|^2r_A)
\]
for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper Lie $CQ^*$-algebra homomorphism.
Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (4.1),

$$
\|f([w, x]) - [f(w), f(x)]\|_B = \lim_{n \to \infty} \frac{1}{4^n} \|f(4^n[w, x]) - [f(2^n w), f(2^n x)]\|_B \\
\leq \lim_{n \to \infty} \frac{4^n r}{4^n} \theta \|w\|_A^r + \|x\|_A^r = 0
$$

for all $w \in A_0$ and all $x \in A$. So

$$
\ f([w, x]) = [f(w), f(x)]
$$

for all $w \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$
\ f([w, x]) = [f(w), f(x)]
$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper Lie $CQ^*$-algebra homomorphism.

**Theorem 4.2.** Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying (2.2) and $f(w) \in B_0$ for all $w \in A_0$ such that

$$
\|f([w, x]) - [f(w), f(x)]\|_B \leq \theta \cdot \|w\|_A^r \cdot \|x\|_A^r
$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper Lie $CQ^*$-algebra homomorphism.

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (4.2),

$$
\|f([w, x]) - [f(w), f(x)]\|_B = \lim_{n \to \infty} \frac{1}{4^n} \|f(4^n[w, x]) - [f(2^n w), f(2^n x)]\|_B \\
\leq \lim_{n \to \infty} \frac{4^n r}{4^n} \theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0
$$

for all $w \in A_0$ and all $x \in A$. So

$$
\ f([w, x]) = [f(w), f(x)]
$$

for all $w \in A_0$ and all $x \in A$. 

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies
\[ f([w, x]) = [f(w), f(x)] \]
for all $w \in A_0$ and all $x \in A$.
Therefore, the mapping $f : A \to B$ is a proper Lie $CQ^*$-algebra homomorphism.

**Remark 4.3.** If the Lie products $[\cdot, \cdot]$ in the statements of Theorems 4.1 and 4.2 are replaced by the Jordan products $\cdot \circ \cdot$, then one obtains proper Jordan $CQ^*$-algebra homomorphisms instead of proper Lie $CQ^*$-algebra homomorphisms.

### 5. Derivations on proper Lie $CQ^*$-algebras

Throughout this section, assume that $(A, A_0)$ is a proper Lie $CQ^*$-algebra with $C^*$-norm $\| \cdot \|_{A_0}$ and norm $\| \cdot \|_A$.

We investigate derivations on proper Lie $CQ^*$-algebras associated with the functional equation (1.3).

**Theorem 5.1.** Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying (3.1) such that
\[
\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A 
\leq \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r})
\]
for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a Lie derivation on $A$.

**Proof.** By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $C$-linear.

(i) Assume that $r < 1$. By (5.1),
\[
\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A 
\leq \frac{1}{4^n} \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) = 0
\]
for all $w_0, w_1 \in A_0$. So
\[ f([w_0, w_1]) = [f(w_0), w_1] + [w_0, f(w_1)] \]
for all $w_0, w_1 \in A_0$. 


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(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \rightarrow A$ satisfies
\[ f([w_0, w_1]) = [f(w_0), w_1] + [w_0, f(w_1)] \]
for all $w_0, w_1 \in A_0$.

Therefore, the mapping $f : A \rightarrow A$ is a Lie derivation on $A$, as desired.

**Theorem 5.2.** Let $r \neq 1$ and $\theta$ be nonnegative real numbers, and $f : A \rightarrow A$ a mapping satisfying (3.1) such that
\[ \|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A \leq \theta \cdot \|w_0\|_A^r \cdot \|w_1\|_A^r \]
for all $w_0, w_1 \in A_0$. Then the mapping $f : A \rightarrow A$ is a Lie derivation on $A$.

**Proof.** The proof is similar to the proofs of Theorems 2.2 and 5.1.

**Remark 5.3.** If the Lie products $[\cdot, \cdot]$ in the statements of Theorems 5.1 and 5.2 are replaced by the Jordan products $\cdot \circ \cdot$, then one obtains Jordan derivations instead of Lie derivations.

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