Korean J. Math. **18** (2010), No. 4, pp. 343–356

GENERALIZED FUZZY CONGRUENCES ON SEMIGROUPS

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Abstract. We define a G-fuzzy congruence, which is a generalized fuzzy congruence, discuss some of its basic properties, and characterize the G-fuzzy congruence generated by a fuzzy relation on a semigroup. We also give certain lattice theoretic properties of G-fuzzy congruences on semigroups.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([7]). Subsequently, Goguen ([1]) and Sanchez ([6]) studied fuzzy relations in various contexts. In [4] Nemitz discussed fuzzy equivalence relations, fuzzy functions as fuzzy relations, and fuzzy partitions. Murali ([3]) developed some properties of fuzzy equivalence relations and certain lattice theoretic properties of fuzzy equivalence relations. The standard definition of a reflexive fuzzy relation μ on a set X, which most mathematicians used in their papers, is $\mu(x,x) = 1$ for all $x \in X$. Gupta et al. ([2]) weakened this standard definition to $\mu(x,x) > 0$ for all $x \in X$ and $\inf_{t \in X} \mu(t,t) \geq \mu(y,z)$ for all $y \neq z \in X$, which is called G-reflexive fuzzy relation, and redefined a G-fuzzy equivalence relation on a set and developed some properties of that relation. Samhan ([5]) defined a fuzzy congruence based on the standard definition of a reflexive fuzzy relation, found the fuzzy congruence generated by a fuzzy relation on a semigroup, and developed some lattice theoretic properties of fuzzy congruences. The present work has been started as a continuation of these studies.

Received September 6, 2010. Revised November 13, 2010. Accepted November 19, 2010.

2000 Mathematics Subject Classification: 03E72.

Key words and phrases: G-fuzzy equivalence relation, G-fuzzy congruence .

This work was supported by a special research grant from Seoul Women's University (2010).

In section 2 we define a generalized fuzzy congruence based on the G-reflexive fuzzy relation, which is called a G-fuzzy congruence in this note, and review some basic properties of fuzzy relations which will be used in next sections. In section 3 we discuss some basic properties of G-fuzzy congruences, find the G-fuzzy congruence generated by a fuzzy relation μ on a semigroup S such that $\mu(x, y) > 0$ for some $x \neq y \in S$, and characterize the G-fuzzy congruence generated by a fuzzy relation μ on a semigroup S such that $\mu(x, y) = 0$ for all $x \neq y \in S$. In section 4 we find sufficient conditions for the composition $\mu \circ \nu$ of two G-fuzzy congruences μ and ν on a semigroup to be the G-fuzzy congruence generated by $\mu \cup \nu$, show that for the collection $C(S)$ of all G-fuzzy congruences on a semigroup S and $0 < k \leq 1$, $C_k(S) = \{\mu \in C(S) : \mu(c, c) = k \text{ for all } c \in S\}$ is a complete lattice and any sublattice H of $C_k(S)$ such that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in H$ is modular, and show that if S is a group, $(C_k(S), +, \cdot)$ is modular.

2. Preliminaries

We recall some definitions and properties of fuzzy relations and G-fuzzy congruences which will be used in next sections.

Definition 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B.

The standard definition of a fuzzy reflexive relation μ in a set X demands $\mu(x, x) = 1$. Gupta et al. ([2]) weakened this definition as follows.

Definition 2.2. A *fuzzy relation* μ in a set X is a fuzzy subset of $X \times X$. μ is *G-reflexive* in X if $\mu(x, x) > 0$ and $\inf\limits_{t \in X} \mu(t, t) \geq \mu(x, y)$ for all $x, y \in X$ such that $x \neq y$. μ is *symmetric* in X if $\mu(x, y) = \mu(y, x)$ for all x, y in X. The composition $\lambda \circ \mu$ of two fuzzy relations λ, μ in X is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x, y) = \sup_{z \in X} \min(\lambda(x, z), \mu(z, y)).$$

A fuzzy relation μ in X is *transitive* in X if $\mu \circ \mu \subseteq \mu$. A fuzzy relation μ in X is called *G-fuzzy equivalence relation* if μ is G-reflexive, symmetric, and transitive.

Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set X. Then it is easy to see that the composition $\circ$ is associative, $\mathcal{F}_X$ is a monoid under the operation of composition $\circ$, and a G-fuzzy equivalence relation is an idempotent element of $\mathcal{F}_X$.

DEFINITION 2.3. A fuzzy relation μ in a set X is called *fuzzy left (right) compatible* if $\mu(x,y) \leq \mu(zx,zy)$ ($\mu(x,y) \leq \mu(xz,yz)$) for all $x,y,z \in X$. A G-fuzzy equivalence relation on X is called a *G-fuzzy left congruence (right congruence)* if it is fuzzy left compatible (right compatible). A G-fuzzy equivalence relation on X is a *G-fuzzy congruence* if it is a G-fuzzy left and right congruence.

DEFINITION 2.4. Let μ be a fuzzy relation in a set X. μ^{-1} is defined as a fuzzy relation in X by $\mu^{-1}(x,y) = \mu(y,x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations μ and ν.

PROPOSITION 2.5. *Let μ be a fuzzy relation on a set X. Then $\cup_{n=1}^{\infty} \mu^n$ is the smallest transitive fuzzy relation on X containing μ, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.3 of [5]. □

PROPOSITION 2.6. *Let μ be a fuzzy relation on a set X. If μ is symmetric, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.4 of [5]. □

PROPOSITION 2.7. *If μ is a fuzzy relation on a semigroup S that is fuzzy left and right compatible, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 3.6 of [5]. □

PROPOSITION 2.8. *If μ is a G-reflexive fuzzy relation on a set X, then $\mu^{n+1}(x,y) \geq \mu^n(x,y)$ for all natural numbers n and all $x,y \in X$.*

Proof. Note that

$$\begin{aligned}\mu^2(x,y) = (\mu \circ \mu)(x,y) &= \sup_{z \in X} \min[\mu(x,z), \mu(z,y)] \\ &\geq \min[\mu(x,x), \mu(x,y)] = \mu(x,y).\end{aligned}$$

Suppose $\mu^{k+1}(x,y) \geq \mu^k(x,y)$ for all $x, y \in X$. Then

$$\begin{aligned}\mu^{k+2}(x,y) = (\mu \circ \mu^{k+1})(x,y) &= \sup_{z \in S} \min[\mu(x,z),\ \mu^{k+1}(z,y)] \\ &\geq \sup_{z \in S} \min[\mu(x,z),\ \mu^k(z,y)] \\ &= (\mu \circ \mu^k)(x,y) = \mu^{k+1}(x,y).\end{aligned}$$

By the mathematical induction, $\mu^{n+1}(x,y) \geq \mu^n(x,y)$ for $n = 1, 2, \ldots$. □

PROPOSITION 2.9. *Let μ and each ν_i be fuzzy relations in a set X for all $i \in I$. Then $\mu \circ (\bigcap_{i \in I} \nu_i) \subseteq \bigcap_{i \in I} (\mu \circ \nu_i)$ and $(\bigcap_{i \in I} \nu_i) \circ \mu \subseteq \bigcap_{i \in I} (\nu_i \circ \mu)$.*

Proof. Straightforward. □

3. G-fuzzy congruences on semigroups

In this section we develop some basic properties of G-fuzzy congruences and characterize the G-fuzzy congruence generated by a fuzzy relation on a semigroup.

PROPOSITION 3.1. *Let μ be a fuzzy relation on a set S. If μ is G-reflexive, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. Clearly $\mu^1 = \mu$ is G-reflexive. Suppose μ^k is G-reflexive.

$$\begin{aligned}\mu^{k+1}(x,x) = (\mu^k \circ \mu)(x,x) &= \sup_{z \in S} \min[\mu^k(x,z), \mu(z,x)] \\ &\geq \min[\mu^k(x,x), \mu(x,x)] > 0\end{aligned}$$

for all $x \in S$. Let $x, y \in S$ with $x \neq y$. Then

$$\begin{aligned}&\inf_{t \in S} \mu^{k+1}(t,t) = \inf_{t \in S} (\mu^k \circ \mu)(t,t) \\ &= \inf_{t \in S} \sup_{z \in S} \min[\mu^k(t,z), \mu(z,t)] \geq \inf_{t \in S} \min[\mu^k(t,t), \mu(t,t)] \\ &\geq \min\,[\inf_{t \in S} \mu^k(t,t), \inf_{t \in S} \mu(t,t)] \\ &\geq \min[\mu^k(x,z), \mu(z,y)]\end{aligned}$$

for all $z \in S$ such that $z \neq x$ and $z \neq y$. That is, $\inf_{t\in S} \mu^{k+1}(t,t) \geq \sup_{z\in S-\{x,y\}} \min[\mu^k(x,z), \mu(z,y)]$. Clearly $\inf_{t\in S} \mu(t,t) \geq \min\ [\mu^k(x,x), \mu(x,y)]$ and $\inf_{t\in S} \mu^k(t,t) \geq \min\ [\mu^k(x,y), \mu(y,y)]$. Since $\mu^{k+1}(t,t) \geq \mu^k(t,t) \geq \mu(t,t)$ for $k \geq 1$ by Proposition 2.8,

$$\inf_{t\in S} \mu^{k+1}(t,t) \geq \min\ [\mu^k(x,x), \mu(x,y)]$$

and $\inf_{t\in S} \mu^{k+1}(t,t) \geq \min\ [\mu^k(x,y), \mu(y,y)]$. Thus

$$\begin{aligned}\inf_{t\in S} \mu^{k+1}(t,t) \geq \max\ [&\sup_{z\in S-\{x,y\}} \min(\mu^k(x,z),\ \mu(z,y)),\\ &\min\ (\mu^k(x,x), \mu(x,y)), \min\ (\mu^k(x,y), \mu(y,y))]\\ = \sup_{z\in S} \min[\mu^k(x,z), \mu(z,y)] &= (\mu^k \circ \mu)(x,y) = \mu^{k+1}(x,y).\end{aligned}$$

That is, μ^{k+1} is G-reflexive. By the mathematical induction, μ^n is G-reflexive for $n = 1, 2, \ldots$. Thus $\inf_{t\in S} [\cup_{n=1}^{\infty} \mu^n](t,t) = \inf_{t\in S} \sup[\mu(t,t), (\mu \circ \mu)(t,t), \ldots] \geq \sup\ [\inf_{t\in S} \mu(t,t), \inf_{t\in S}\ (\mu \circ \mu)(t,t), \ldots] \geq \sup[\mu(x,y), (\mu \circ \mu)(x,y), \ldots] = [\cup_{n=1}^{\infty} \mu^n](x,y)$. Clearly $[\cup_{n=1}^{\infty} \mu^n](x,x) > 0$. Hence $\cup_{n=1}^{\infty} \mu^n$ is G-reflexive. □

Proposition 3.2. *Let μ and ν be G-fuzzy congruences in a set X. Then $\mu \cap \nu$ is a G-fuzzy congruence.*

Proof. It is clear that $\mu \cap \nu$ is G-reflexive and symmetric. By Proposition 2.9, $[(\mu \cap \nu) \circ (\mu \cap \nu)] \subseteq [\mu \circ (\mu \cap \nu)] \cap [\nu \circ (\mu \cap \nu)] \subseteq [(\mu \circ \mu) \cap (\mu \circ \nu)] \cap [(\nu \circ \mu) \cap (\nu \circ \nu)] \subseteq [\mu \cap (\mu \circ \nu)] \cap [(\nu \circ \mu) \cap \nu] \subseteq \mu \cap \nu$. That is, $\mu \cap \nu$ is transitive. Clearly $\mu \cap \nu$ is fuzzy left and right compatible. Thus $\mu \cap \nu$ is a G-fuzzy congruence. □

It is easy to see that even though μ and ν are G-fuzzy congruences, $\mu \cup \nu$ is not necessarily a G-fuzzy congruence. We provide an explicit form of the G-fuzzy congruence generated by $\mu \cup \nu$ in the following proposition.

Proposition 3.3. *Let μ and ν be G-fuzzy congruences on a semigroup S. Then the G-fuzzy congruence generated by $\mu \cup \nu$ in S is* $\cup_{n=1}^{\infty}(\mu \cup \nu)^n = (\mu \cup \nu) \cup [(\mu \cup \nu) \circ (\mu \cup \nu)] \cup \ldots$.

Proof. Clearly $(\mu \cup \nu)(x, x) > 0$ and

$$\begin{aligned}
&\inf_{t \in S} (\mu \cup \nu)(t, t) = \inf_{t \in S} \max(\mu(t, t), \nu(t, t)) \\
&\geq \max\,(\inf_{t \in S} \mu(t, t), \inf_{t \in S} \nu(t, t)) \\
&\geq \max\,(\mu(x, y), \nu(x, y)) \\
&= (\mu \cup \nu)(x, y)
\end{aligned}$$

for all $x \neq y$ in S. That is, $\mu \cup \nu$ is G-reflexive. By Proposition 3.1, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is G-reflexive. Clearly $\mu \cup \nu$ is symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is transitive. Hence $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is a G-fuzzy equivalence relation containing $\mu \cup \nu$. It is straightforward to see that $\mu \cup \nu$ is fuzzy left and right compatible. By Proposition 2.7, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is fuzzy left and right compatible. Thus $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is a G-fuzzy congruence containing $\mu \cup \nu$. Let λ be a G-fuzzy congruence in S containing $\mu \cup \nu$. Then $\cup_{n=1}^{\infty}(\mu \cup \nu)^n \subseteq \cup_{n=1}^{\infty}\lambda^n = \lambda \cup (\lambda \circ \lambda) \cup (\lambda \circ \lambda \circ \lambda) \cup \cdots \subseteq \lambda \cup \lambda \cup \cdots = \lambda$. Thus $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is the G-fuzzy congruence generated by $\mu \cup \nu$.□

We now turn to the characterization of the G-fuzzy congruence generated by a fuzzy relation on a semigroup.

Definition 3.4. Let μ be a fuzzy relation on a semigroup S and let $S^1 = S \cup \{e\}$, where e is the identity of S. We define the fuzzy relation μ^* on S as

$$\mu^*(c, d) = \bigcup_{\substack{x, y \in S^1, \\ xay = c, \\ xby = d}} \mu(a, b) \quad \text{for all } c, d \in S.$$

Proposition 3.5. *Let μ and ν be two fuzzy relations on a semigroup S. Then*

(1) $\mu \subseteq \mu^*$
(2) $(\mu^*)^{-1} = (\mu^{-1})^*$

(3) *If* $\mu \subseteq \nu$, *then* $\mu^* \subseteq \nu^*$
(4) $(\mu \cup \nu)^* = \mu^* \cup \nu^*$
(5) $\mu = \mu^*$ *if and only if* μ *is fuzzy left and right compatible*
(6) $(\mu^*)^* = \mu^*$

Proof. See Proposition 3.5 of [5]. □

Samhan ([5]) found the fuzzy congruence generated by a fuzzy relation on a semigroup. Theorem 3.6 may be considered as a generalization of this work in G-fuzzy congruences.

THEOREM 3.6. *Let* μ *be a fuzzy relation on a semigroup* S.

(1) *If* $\mu(x, y) > 0$ *for some* $x \neq y \in S$, *then the G-fuzzy congruence generated by* μ *is* $\cup_{n=1}^{\infty} [\mu^* \cup (\mu^*)^{-1} \cup \theta^*]^n$, *where* θ *is a fuzzy relation on* S *such that* $\theta(z, z) = \sup_{x \neq y \in S} \mu(x, y)$ *for all* $z \in S$ *and* $\theta(x, y) = \theta(y, x) \leq \min\,[\mu(x, y), \mu(y, x)]$ *for all* $x, y \in S$ *with* $x \neq y$, *and* μ^* *and* θ^* *are fuzzy relations on* S *defined in Definition 3.4.*
(2) *If* $\mu(x, y) = 0$ *for all* $x \neq y \in S$ *and* $\mu(z, z) > 0$ *for all* $z \in S$, *then the G-fuzzy congruence generated by* μ *is* $\cup_{n=1}^{\infty} (\mu^*)^n$, *where* μ^* *is a fuzzy relation on* S *defined in Definition 3.4.*
(3) *If* $\mu(x, y) = 0$ *for all* $x \neq y \in S$, $\mu(z, z) = 0$ *for some* $z \in S$, *and* $\mu^*(z, z) > 0$ *for all* $z \in S$, *then the G-fuzzy congruence generated by* μ *is* $\cup_{n=1}^{\infty} (\mu^*)^n$, *where* μ^* *is a fuzzy relation on* S *defined in Definition 3.4.*
(4) *If* $\mu(x, y) = 0$ *for all* $x \neq y \in S$, $\mu(z, z) = 0$ *for some* $z \in S$, *and* $\mu^*(z, z) = 0$ *for some* $z \in S$, *then there does not exist the G-fuzzy congruence generated by* μ.

Proof. (1) Since $\theta(z, z) > 0$, $\theta^*(z, z) > 0$ for all $z \in S$ by Proposition 3.5 (1). Let $x, y \in S$ with $x \neq y$ and let $S^1 = S \cup \{e\}$, where e is the identity of S. From Definition 3.4, $\mu^*(x, y) = \cup_{\substack{c,d \in S^1,\\ cad = x,\\ cbd = y}} \mu(a, b)$ and $\theta^*(x, y) = \cup_{\substack{c,d \in S^1,\\ cad = x,\\ cbd = y}} \theta(a, b)$. Since $cad = x$ and $cbd = y$ for $c, d \in S^1$, $x \neq y$ implies $a \neq b$. Thus $\mu^*(x, y) \leq \sup_{x \neq y \in S} \mu(x, y) = \theta(t, t)$ for all $t \in S$ and $\theta^*(x,y) \leq \mu^*(x,y)$. That is, $\inf_{z\in S}\theta^*(z,z) \geq \theta(t,t) \geq \mu^*(x,y) \geq \theta^*(x,y)$. Let $\mu_1 = \mu^* \cup (\mu^*)^{-1} \cup \theta^*$. Then

$$\mu_1(z,z) = \max[\mu^*(z,z), (\mu^*)^{-1}(z,z), \theta^*(z,z)] > 0$$

and

$$\begin{aligned}&\inf_{t\in S} \mu_1(t,t) \geq \inf_{t\in S} \theta^*(t,t) \geq \max[\mu^*(x,y),\ (\mu^*)^{-1}(x,y), \theta^*(x,y)]\\&= \mu_1(x,y).\end{aligned}$$

Thus μ_1 is G-reflexive. By Proposition 3.1, $\cup_{n=1}^{\infty} \mu_1^n$ is G-reflexive. Since $\theta(x,y) = \theta(y,x)$, $\theta = \theta^{-1}$. By Proposition 3.5 (2), $\theta^* = (\theta^{-1})^* = (\theta^*)^{-1}$. Thus

$$\begin{aligned}\mu_1(x,y) &= \max\ [\mu^*(x,y), (\mu^*)^{-1}(x,y), \theta^*(x,y)]\\&= \max\ [(\mu^*)^{-1}(y,x), \mu^*(y,x), (\theta^*)^{-1}(x,y)]\\&= \max[(\mu^*)^{-1}(y,x), \mu^*(y,x), \theta^*(y,x)] = \mu_1(y,x).\end{aligned}$$

Thus μ_1 is symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty} \mu_1^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty} \mu_1^n$ is transitive. Hence $\cup_{n=1}^{\infty} \mu_1^n$ is a G-fuzzy equivalence relation containing μ. By Proposition 3.5 (2), (4), and (6), $\mu_1^* = (\mu^* \cup (\mu^*)^{-1} \cup \theta^*)^* = (\mu^* \cup (\mu^{-1})^* \cup \theta^*)^* = (\mu^*)^* \cup ((\mu^{-1})^*)^* \cup (\theta^*)^* = \mu^* \cup (\mu^{-1})^* \cup \theta^* = \mu^* \cup (\mu^*)^{-1} \cup \theta^* = \mu_1$. Thus μ_1 is fuzzy left and right compatible by Proposition 3.5 (5). By Proposition 2.7, $\cup_{n=1}^{\infty} \mu_1^n$ is fuzzy left and right compatible. Thus $\cup_{n=1}^{\infty} \mu_1^n$ is a G-fuzzy congruence containing μ. Let ν be a G-fuzzy congruence containing μ. Then $\mu(x,y) \leq \nu(x,y)$, $\mu^{-1}(x,y) = \mu(y,x) \leq \nu(y,x) = \nu(x,y)$, and $\theta(x,y) \leq \mu(x,y) \leq \nu(x,y)$. That is, $(\mu \cup \mu^{-1} \cup \theta)(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$. Since $\nu(a,a) \geq \nu(x,y) \geq \mu(x,y)$ for all $a, x, y \in S$ such that $x \neq y$, $\theta(a,a) = \sup_{x\neq y\in S} \mu(x,y) \leq \nu(a,a)$ for all $a \in S$. Since $\nu(a,a) \geq \mu(a,a) = \mu^{-1}(a,a)$ and $\nu(a,a) \geq \theta(a,a)$ for all $a \in S$, $\max\ [\mu(a,a), \mu^{-1}(a,a), \theta(a,a)] \leq \nu(a,a)$ for all $a \in S$. Thus $\mu \cup \mu^{-1} \cup \theta \subseteq \nu$. By Proposition 3.5 (2), (3), and (4), $\mu_1 = \mu^* \cup (\mu^*)^{-1} \cup \theta^* = \mu^* \cup (\mu^{-1})^* \cup \theta^* = (\mu \cup \mu^{-1} \cup \theta)^* \subseteq \nu^*$. Since ν is fuzzy left and right compatible, $\nu = \nu^*$ by Proposition 3.5 (5).

Thus $\mu_1 \subseteq \nu$. Suppose $\mu_1^k \subseteq \nu$. Then $\mu_1^{k+1}(b,c) = (\mu_1^k \circ \mu_1)(b,c) = \sup\limits_{d\in S} \min[\mu_1^k(b,d), \mu_1(d,c)] \le \sup\limits_{d\in S} \min\,[\nu(b,d), \nu(d,c)] = (\nu \circ \nu)(b,c)$ for all $b, c \in S$. That is, $\mu_1^{k+1} \subseteq (\nu \circ \nu)$. Since ν is transitive, $\mu_1^{k+1} \subseteq \nu$. By the mathematical induction, $\mu_1^n \subseteq \nu$ for every natural number n. Thus $\cup_{n=1}^{\infty}\,[\mu^* \cup (\mu^*)^{-1} \cup \theta^*]^n = \cup_{n=1}^{\infty}\,{\mu_1}^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu$.

(2) Since $\mu(z,z) > 0$, $\mu^*(z,z) > 0$ for all $z \in S$ by Proposition 3.5 (1). Let $x, y \in S$ with $x \neq y$. Since $\mu^*(x,y) \le \sup\limits_{x\neq y\in S} \mu(x,y)$ and $\mu(x,y) = 0$, $\mu^*(x,y) = 0$. Thus $\inf\limits_{t\in S} \mu^*(t,t) \ge \mu^*(x,y)$. Hence μ^* is G-reflexive. Since $\mu = \mu^{-1}$, $\mu^* = (\mu^{-1})^* = (\mu^*)^{-1}$ by Proposition 3.5 (2). Thus μ^* is symmetric. By Proposition 2.5, Proposition 2.6, and Proposition 3.1, $\cup_{n=1}^{\infty}\,(\mu^*)^n$ is a G-fuzzy equivalence relation containing μ. By Proposition 3.5 (5) and (6), μ^* is fuzzy left and right compatible. By Proposition 2.7, $\cup_{n=1}^{\infty}\,(\mu^*)^n$ is a G-fuzzy congruence containing μ. Let ν be a G-fuzzy congruence containing μ. Since $\mu \subseteq \nu$, $\mu^* \subseteq \nu^*$ by Proposition 3.5 (3). Since ν is fuzzy left and right compatible, $\nu^* = \nu$ by Proposition 3.5 (5). Thus $\mu^* \subseteq \nu$. By the mathematical induction as shown in Theorem 3.6 (1), we may show that $(\mu^*)^n \subseteq \nu$ for every natural number n. Hence $\cup_{n=1}^{\infty}\,(\mu^*)^n = \mu^* \cup (\mu^* \circ \mu^*) \cup (\mu^* \circ \mu^* \circ \mu^*) \cdots \subseteq \nu$.

(3) The proof is similar to that of (2).

(4) Suppose ξ is the G-fuzzy congruence generated by μ. Then $\xi(z,z) > 0$ for every $z \in S$. Let θ be a fuzzy relation such that $\theta(a,b) = \frac{\xi(a,b)}{2}$ for all $a, b \in S$. Then $\theta(z,z) > 0$, and hence $\theta^*(z,z) > 0$ for all $z \in S$ by Proposition 3.5 (1). Let $x, y \in S$ with $x \neq y$. Since $\mu^*(x,y) \le \sup\limits_{x\neq y\in S} \mu(x,y)$ and $\mu(x,y) = 0$, $\mu^*(x,y) = 0$. Since ξ is fuzzy left and right compatible, $\xi^* = \xi$ by Proposition 3.5 (5). Since ξ is G-reflexive and $\xi^* = \xi$, $\inf\limits_{t\in S} \xi^*(t,t) \ge \xi^*(x,y)$. Since $\theta^*(a,b) = \frac{\xi^*(a,b)}{2}$ for all $a, b \in S$, $\inf\limits_{t\in S} \theta^*(t,t) \ge \theta^*(x,y)$. Thus $(\mu^* \cup \theta^*)(z,z) > 0$ for all $z \in S$ and $\inf\limits_{t\in S}\,(\mu^* \cup \theta^*)(t,t) \ge (\mu^* \cup \theta^*)(x,y)$. That is, $\mu^* \cup \theta^*$ is G-reflexive. Since ξ is symmetric, θ is symmetric. Since θ is symmetric and $\mu(x,y) = 0$, $\mu \cup \theta = (\mu \cup \theta)^{-1}$. By Proposition 3.5 (2), $(\mu \cup \theta)^* = [(\mu \cup \theta)^{-1}]^* = [(\mu \cup \theta)^*]^{-1}$. Thus $(\mu \cup \theta)^* = \mu^* \cup \theta^*$ is symmetric. By Proposition 2.5, Proposition 2.6, and Proposition 3.1, $\cup_{n=1}^{\infty}\,(\mu^* \cup \theta^*)^n$ is a G-fuzzy equivalence relation containing μ. By Proposition 3.5 (4)

and (6), $(\mu^* \cup \theta^*)^* = (\mu^*)^* \cup (\theta^*)^* = \mu^* \cup \theta^*$. Thus $\mu^* \cup \theta^*$ is fuzzy left and right compatible by Proposition 3.5 (5). By Proposition 2.7, $\cup_{n=1}^{\infty} (\mu^* \cup \theta^*)^n$ is a G-fuzzy congruence containing μ. Since $\theta(a,b) = \frac{\xi(a,b)}{2} \leq \xi(a,b)$ and $\mu(a,b) \leq \xi(a,b)$ for all $a, b \in S$, $\mu \cup \theta \subseteq \xi$. Let $\mu_1 = \mu^* \cup \theta^*$. By Proposition 3.5 (3) and (4), $\mu_1 = \mu^* \cup \theta^* = (\mu \cup \theta)^* \subseteq \xi^*$. Since $\xi^* = \xi$, $\mu_1 \subseteq \xi$. By the mathematical induction as shown in Theorem 3.6 (1), we may show that $\mu_1^n \subseteq \xi$ for every natural number n. Hence $\cup_{n=1}^{\infty} [\mu^* \cup \theta^*]^n = \cup_{n=1}^{\infty} {\mu_1}^n \subseteq \xi$. Let $v \neq w \in S$. Then $\mu_1(v,w) = (\mu^* \cup \theta^*)(v,w) = \theta^*(v,w) \leq \inf_{t \in S} \theta^*(t,t) \leq \mu_1(z,z)$ for every $z \in S$. Suppose $\mu_1^k(v,w) \leq \mu_1(z,z)$ for every $z \in S$. Then

$$\begin{aligned} \mu_1^{k+1}(v,w) &= \sup_{s \in S} \min\ [\mu_1^k(v,s),\ \mu_1(s,w)] \\ &= \max\ [\sup_{s \in S - \{v,w\}} \min(\mu_1^k(v,s),\ \mu_1(s,w)),\ \min\ (\mu_1^k(v,v), \mu_1(v,w)), \\ &\qquad \min\ (\mu_1^k(v,w), \mu_1(w,w))] \\ &\leq \max\ [\mu_1(z,z),\ \mu_1(z,z),\ \mu_1^k(v,w)] = \mu_1(z,z). \end{aligned}$$

By the mathematical induction, $\mu_1^n(v,w) \leq \mu_1(z,z)$ for every natural number n. Clearly $\mu_1^k(z,z) = \mu_1(z,z)$ for $k = 1$. Suppose $\mu_1^k(z,z) = \mu_1(z,z)$. Since $\mu_1^k(z,s) \leq \mu_1(z,z)$ for $s \neq z \in S$, $\mu_1^{k+1}(z,z) = \sup_{s \in S} \min\ [\mu_1^k(z,s),\ \mu_1(s,z)] = \max\ [\sup_{s \in S - \{z\}} \min(\mu_1^k(z,s),\ \mu_1(s,z)),$ $\min\ (\mu_1^k(z,z), \mu_1(z,z))] = \mu_1(z,z)$. By the mathematical induction, $\mu_1^n(z,z) = \mu_1(z,z)$ for every natural number n and every $z \in S$. Let p be in S with $\mu^*(p,p) = 0$. Since $\theta(a,b) = \frac{\xi(a,b)}{2}$ and ξ is fuzzy left and right compatible, θ is fuzzy left and right compatible. That is, $\theta = \theta^*$. Thus $\mu_1(p,p) = \theta^*(p,p) = \theta(p,p) = \frac{\xi(p,p)}{2} < \xi(p,p)$. Since $\mu_1^n(z,z) = \mu_1(z,z)$ for every natural number n and every $z \in S$, $[\cup_{n=1}^{\infty} (\mu^* \cup \theta^*)^n](p,p) = [\cup_{n=1}^{\infty} {\mu_1}^n](p,p) = \mu_1(p,p) < \xi(p,p)$ for some $p \in S$ such that $\mu^*(p,p) = 0$. Hence $\cup_{n=1}^{\infty} (\mu^* \cup \theta^*)^n$, which is a G-fuzzy congruence containing μ, is contained in ξ. This contradicts that ξ is the G-fuzzy congruence generated by μ. □

4. Lattices of G-fuzzy congruences

In this section we discuss some lattice theoretic properties of G-fuzzy congruences. Let $C(S)$ be the collection of all G-fuzzy congruences on a semigroup S. It is easy to see that $C(S)$ is not a lattice.

THEOREM 4.1. *Let* $0 < k \leq 1$ *and let* $C_k(S) = \{\mu \in C(S) : \mu(c,c) = k$ *for all* $c \in S\}$. *Then* $(C_k(S), \leq)$ *is a complete lattice, where* $\leq$ *is a relation on the set of all G-fuzzy congruences on* S *defined by* $\mu \leq \nu$ *iff* $\mu(x,y) \leq \nu(x,y)$ *for all* $x, y \in S$.

Proof. Clearly $\leq$ is a partial order relation. It is easy to check that the relation σ defined by $\sigma(x,y) = k$ for all $x, y \in S$ is in $C_k(S)$ and the relation λ defined by $\lambda(x,y) = k$ for $x = y$ and $\lambda(x,y) = 0$ for $x \neq y$ is in $C_k(S)$. Also σ is the greatest element and λ is the least element of $C_k(S)$ with respect to the ordering $\leq$. Let $\{\mu_j\}_{j\in J}$ be a non-empty collection of G-fuzzy congruences in $C_k(S)$. Let $\mu(x,y) = \inf\limits_{j\in J} \mu_j(x,y)$ for all $x, y \in S$. It is easy to see that $\mu(x,x) > 0$ for all $x \in S$, $\inf\limits_{t\in X} \mu(t,t) \geq \mu(y,z)$ for all $y \neq z \in X$, $\mu = \mu^{-1}$, $\mu(x,y) \leq \mu(zx,zy)$, and $\mu(x,y) \leq \mu(xz,yz)$ for all $x,y,z \in S$. $\mu \circ \mu(x,y) = \sup\limits_{z\in X} \min[\inf\limits_{j\in J} \mu_j(x,z), \inf\limits_{j\in J} \mu_j(z,y)] = \sup\limits_{z\in X} \inf\limits_{j\in J} \inf\limits_{i\in J} \min[\mu_j(x,z), \mu_i(z,y)] \leq \sup\limits_{z\in X} \inf\limits_{j\in J} \min[\mu_j(x,z), \mu_j(z,y)] \leq \inf\limits_{j\in J} \mu_j \circ \mu_j(x,y) \leq \inf\limits_{j\in J} \mu_j(x,y) = \mu(x,y)$. That is, $\mu \in C_k(S)$. Since μ is the greatest lower bound of $\{\mu_j\}_{j\in J}$, $(C_k(S), \leq)$ is a complete lattice. □

We define addition and multiplication on $C_k(S)$ by $\mu + \nu = <\mu \cup \nu>_c$ and $\mu \cdot \nu = \mu \cap \nu$, where $<\mu \cup \nu>_c$ is the G-fuzzy congruence generated by $\mu \cup \nu$.

DEFINITION 4.2. A lattice $(L, +, \cdot)$ is called *modular* if $(x+y)\cdot z \leq x + (y \cdot z)$ for all $x, y, z \in L$ with $x \leq z$.

LEMMA 4.3. *Let* μ *and* ν *be G-fuzzy congruences on a semigroup* S *such that*

$$\mu(c,c) = \nu(c,c) \text{ for all } c \in S.$$

If $\mu \circ \nu = \nu \circ \mu$, *then* $\mu \circ \nu$ *is the G-fuzzy congruence on* S *generated by* $\mu \cup \nu$.

Proof. Clearly $(\mu \circ \nu)(a,\ a) > 0$ for all $a \in S$. Let $x, y \in S$ with $x \neq y$. Since $\mu(c,\ c) = \nu(c,\ c)$ for all $c \in S$, $\inf\limits_{t\in S} \mu(t,t) = \inf\limits_{t\in S} \nu(t,t) \geq$

$\max\,[\mu(x,y),\nu(x,y)]$. Thus

$$\begin{aligned}\inf_{t\in S}\,(\mu\circ\nu)(t,t) &= \inf_{t\in S}\,\sup_{z\in S}\,\min\,[\mu(t,z),\nu(z,t)]\\ &\geq \inf_{t\in S}\,\min\,[\mu(t,t),\nu(t,t)]\\ &\geq \min\,[\inf_{t\in S}\,\mu(t,t),\ \inf_{t\in S}\,\nu(t,t)]\\ &\geq \max[\mu(x,y),\nu(x,y)].\end{aligned}$$

Also

$$\inf_{t\in S}\,(\mu\circ\nu)(t,t) \geq \min\,[\,\inf_{t\in S}\mu(t,t), \inf_{t\in S}\,\nu(t,t)] \geq \min[\mu(x,z),\ \nu(z,y)]$$

for all $z\in S$ such that $z\neq x$ and $z\neq y$. That is,

$$\inf_{t\in S}\,(\mu\circ\nu)(t,t) \geq \sup_{z\in S-\{x,y\}}\,\min\,[\mu(x,z),\nu(z,y)].$$

Thus

$$\begin{aligned}&\inf_{t\in S}\,(\mu\circ\nu)(t,t)\\ &\geq \max\,[\sup_{z\in S-\{x,y\}}\,\min(\mu(x,z),\nu(z,y)),\ \max(\mu(x,y),\ \nu(x,y))]\\ &= \max[\sup_{z\in S-\{x,y\}}\,\min(\mu(x,z),\nu(z,y)),\ \nu(x,y),\ \mu(x,y)]\\ &= \max\,[\sup_{z\in S-\{x,y\}}\,\min\,(\mu(x,z),\nu(z,y)),\min(\mu(x,x),\nu(x,y)),\\ &\qquad\qquad \min(\mu(x,y),\nu(y,y))]\\ &= \sup_{z\in S}\,\min[\mu(x,z),\nu(z,y)] = (\mu\circ\nu)(x,y).\end{aligned}$$

That is, $\mu\circ\nu$ is G-reflexive. Since μ and ν are symmetric, $(\mu\circ\nu)^{-1} = \nu^{-1}\circ\mu^{-1} = \nu\circ\mu = \mu\circ\nu$. Thus $\mu\circ\nu$ is symmetric. Since μ and ν are transitive and the operation $\circ$ is associative, $(\mu\circ\nu)\circ(\mu\circ\nu) = \mu\circ(\nu\circ\mu)\circ\nu = \mu\circ(\mu\circ\nu)\circ\nu = (\mu\circ\mu)\circ(\nu\circ\nu) \subseteq \mu\circ\nu$. Hence $\mu\circ\nu$ is a G-fuzzy equivalence relation. Since S is a semigroup, $(\mu\circ\nu)(x,y) = \sup_{a\in S}\,\min[\mu(x,a),\nu(a,y)] \leq \sup_{za\in S}\,\min[\mu(zx,za),\nu(za,zy)] \leq \sup_{t\in S}\min[\mu(zx,t),\nu(t,zy)] = (\mu\circ\nu)(zx,zy)$. Thus $\mu\circ\nu$ is fuzzy left compatible. Similarly we may show $\mu\circ\nu$ is fuzzy right compatible. Hence

$\mu\circ\nu$ is a G-fuzzy congruence in S. Since $\nu(y,y)=\mu(y,y)\geq\mu(x,y)$, $(\mu\circ\nu)(x,y)=\sup\limits_{z\in S}\min[\mu(x,z),\nu(z,y)]\geq\min(\mu(x,y),\nu(y,y))=\mu(x,y)$. Since $\mu(x,x)=\nu(x,x)\geq\nu(x,y)$, $(\mu\circ\nu)(x,y)=\sup\limits_{z\in S}\min[\mu(x,z),\nu(z,y)]\geq\min(\mu(x,x),\nu(x,y))=\nu(x,y)$.

Thus $(\mu\circ\nu)(x,y)\geq\max(\mu(x,y),\nu(x,y))=(\mu\cup\nu)(x,y)$ for all $x,y\in S$ such that $x\neq y$. Since $\mu(c,c)=\nu(c,c)$ for all $c\in S$, $(\mu\circ\nu)(c,c)=\sup\limits_{p\in S}\min[\mu(c,p),\nu(p,c)]\geq\min(\mu(c,c),\nu(c,c))=(\mu\cup\nu)(c,c)$ for all $c\in S$. Thus $\mu\cup\nu\subseteq\mu\circ\nu$. Let λ be a G-fuzzy congruence in S containing $\mu\cup\nu$. Since λ is transitive, $\mu\circ\nu\subseteq(\mu\cup\nu)\circ(\mu\cup\nu)\subseteq\lambda\circ\lambda\subseteq\lambda$. Thus $\mu\circ\nu$ is the G-fuzzy congruence generated by $\mu\cup\nu$. □

It is well known that if μ and ν are congruences on a semigroup S and $\mu\circ\nu=\nu\circ\mu$, then $\mu\circ\nu$ is the congruence on S generated by $\mu\cup\nu$. Lemma 4.3 may be considered as a generalization of this in G-fuzzy congruences.

THEOREM 4.4. *Let $0<k\leq 1$ and let S be a semigroup and H be a sublattice of $(C_k(S),+,\cdot)$ such that $\mu\circ\nu=\nu\circ\mu$ for all $\mu,\nu\in H$. Then H is a modular lattice.*

Proof. Let $\mu,\nu,\rho\in H$ with $\mu\leq\rho$. Let $x,y\in S$.

$$\begin{aligned}
&\min[(\mu\circ\nu)(x,y),\rho(x,y)]=\sup_{z\in S}\min\ [\mu(x,z),\nu(z,y),\rho(x,y)]\\
&\leq\sup_{z\in S}\min[\mu(x,z),\rho(x,z),\nu(z,y),\rho(x,y)]\\
&\leq\sup_{z\in S}\min[\mu(x,z),\ \nu(z,y),\rho(z,y)]\\
&=[\mu\circ\min(\nu,\rho)](x,y).
\end{aligned}$$

Thus $(\mu\circ\nu)\cdot\rho\leq\mu\circ(\nu\cdot\rho)$. Since $\mu,\nu\in C_k(S)$, $\mu(c,c)=\nu(c,c)=k$ for all $c\in S$. By Lemma 4.3, $\mu\circ\nu$ is the G-fuzzy congruence generated by $\mu\cup\nu$. That is, $\mu+\nu=\mu\circ\nu$. Similarly we may show $\mu+(\nu\cdot\rho)=\mu\circ(\nu\cdot\rho)$. Thus $(\mu+\nu)\cdot\rho\leq\mu+(\nu\cdot\rho)$. Hence H is modular. □

PROPOSITION 4.5. *If S is a group, then $\mu\circ\nu=\nu\circ\mu$ for all $\mu,\nu\in C_k(S)$.*

Proof. Straightforward. □

Corollary 4.6. *If S is a group and $0 < k \leq 1$, then $(C_k(S), +, \cdot)$ is modular.*

Proof. By Theorem 4.4 and Proposition 4.5, $(C_k(S), +, \cdot)$ is modular. □

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