NONTRIVIAL SOLUTIONS FOR THE NONLINEAR
BIHARMONIC SYSTEM WITH DIRICHLET
BOUNDARY CONDITION

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Abstract. We investigate the existence of multiple nontrivial solutions \((\xi, \eta)\) for perturbations \(g_1, g_2\) of the harmonic system with Dirichlet boundary condition
\[
\Delta^2 \xi + c \Delta \xi = g_1(2\xi + 3\eta) \quad \text{in } \Omega,
\]
\[
\Delta^2 \eta + c \Delta \eta = g_2(2\xi + 3\eta) \quad \text{in } \Omega,
\]
where we assume that \(\lambda_1 < c < \lambda_2, g_1'(\infty), g_2'(\infty)\) are finite. We prove that the system has at least three solutions under some condition on \(g\).

1. Introduction

Let \(\Omega\) be a smooth bounded region in \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\). Let \(\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots\) be the eigenvalues of \(-\Delta\) with Dirichlet boundary condition in \(\Omega\). In [8] Jung and Choi studied the multiplicity of solutions of the nonlinear biharmonic equation with Dirichlet boundary condition
\[
\Delta^2 u + c \Delta u = g(u) \quad \text{in } \Omega,
\]
\[
u = 0, \quad \Delta u = 0 \quad \text{on } \partial \Omega,
\]
where $g$ is a differentiable function from $R$ to $R$ such that $g(0) = 0$, $c \in R$ and $\Delta^2$ denotes the biharmonic operator. Here we assume that $g'(\infty) = \lim_{|u| \to \infty} \frac{g(u)}{u} \in R$.

In this paper we investigate the existence of multiple nontrivial solutions $(\xi, \eta)$ for perturbations $g_1, g_2$ of the harmonic system with Dirichlet boundary condition

\begin{align*}
\Delta^2 \xi + c \Delta \xi &= g_1(2\xi + 3\eta) \quad \text{in } \Omega, \\
\Delta^2 \eta + c \Delta \eta &= g_2(2\xi + 3\eta) \quad \text{in } \Omega, \\
\xi = 0, \eta = 0, \Delta \xi = 0, \Delta \eta = 0 \quad \text{on } \partial \Omega,
\end{align*}

where we assume that $\lambda_1 < c < \lambda_2$, $g_1'(\infty)$, $g_2'(\infty)$ are finite.

Problem (1.1) was studied by Choi and Jung in [5], [6]. They showed that problem (1.1) has at least three solutions. The authors proved that (1.1) has at least two solutions by a variation of linking Theorem. The authors also proved in [7] that the problem

\begin{align*}
\Delta^2 u + c \Delta u &= b u^+ + s \quad \text{in } \Omega, \\
u = 0, \Delta u = 0 \quad \text{on } \partial \Omega
\end{align*}

has at least two solutions by a variational reduction method when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ or $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$.

This type problem arises in the study of travelling waves in a suspension bridge ([9,10,11]) or the study of the static deflection of an elastic plate in a fluid ([1,2,3,4,12,13]).

In section 2 we define a Banach space $H$ spanned by eigenfunctions of $\Delta^2 + c \Delta$ with Dirichlet boundary condition. We recall a Linking Scale Theorem which will play a crucial role in our argument. In section 3 we prove that problem (1.1) has at least three solutions under some condition on $g$. In section 4 we investigate the existence of multiple nontrivial solutions $(\xi, \eta)$ for perturbations $g_1, g_2$ of harmonic system (1.2).

2. Linking scale theorem

Let $\lambda_k(k = 1, 2, \ldots)$ denote the eigenvalues and $\phi_k(k = 1, 2, \ldots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in $\Omega$, with the Dirichlet boundary condition, where each eigenvalue $\lambda_k$ is repeated as
often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots, \lambda_i \to +\infty$ and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem $\Delta^2 u + c \Delta u = \mu u$ in $\Omega$ with the Dirichlet boundary condition $u = 0$, $\Delta u = 0$ on $\partial \Omega$, has infinitely many eigenvalues $\lambda_k(\lambda_k - c)$, $k = 1, 2, \ldots$, and corresponding eigenfunctions $\phi_k(x)$. The set of functions $\{\phi_k\}$ is an orthogonal base for $W^{1,2}_0(\Omega)$. Let us denote an element $u$ of $W^{1,2}_0(\Omega)$ as $u = \sum h_k \phi_k$, $\sum h_k^2 < \infty$. Let $c$ be not an eigenvalue of $-\Delta$ and define a subspace $E$ of $W^{1,2}_0(\Omega)$ as follows

$$E = \{u \in W^{1,2}_0(\Omega) : \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty\}.$$ 

Then this is a complete normed space with a norm $\|u\| = \left[\sum |\lambda_k(\lambda_k - c)| h_k^2\right]^{1/2}$.

We need the following some properties which are proved in [6, 7]. Since $\lambda_k \to +\infty$ and $c$ is fixed, we have:

(i) $(\Delta^2 u + c \Delta)u \in E$ implies $u \in E$.

(ii) $\|u\| \geq C \|u\|_{L^2(\Omega)}$, for some $C > 0$.

(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$.

**Definition 2.1.** Let $X$ be a Hilbert space, $Y \subset X$, $\rho > 0$ and $e \in X \setminus Y$, $e \neq 0$. Set:

$$B_\rho(Y) = \{x \in Y : \|x\|_X \leq \rho\},$$

$$S_\rho(Y) = \{x \in Y : \|x\|_X = \rho\},$$

$$\Delta_\rho(e, Y) = \{\sigma e + v : \sigma \geq 0, v \in Y, \|\sigma e + v\|_X \leq \rho\},$$

$$\Sigma_\rho(e, Y) = \{\sigma e + v : \sigma \geq 0, v \in Y, \|\sigma e + v\|_X = \rho\} \cup \{v : v \in Y, \|v\|_X \leq \rho\}.$$ 

Now we recall a theorem of existence of three solutions which is linking scale theorem.

**Theorem 2.1.** (Linking Scale Theorem) Let $X$ be an Hilbert space, which is topological direct sum of the four subspaces $X_0$, $X_1$, $X_2$ and $X_3$. Let $F \in C^1(X, R)$. Moreover assume:

(a) $\dim X_i < +\infty$ for $i = 0, 1, 2$;

(b) there exist $\rho > 0$, $R > 0$ and $e \in X_2$, $e \neq 0$ such that:

$$\rho < R \quad \text{and} \quad \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_\rho(e, X_3)} F.$$
there exist $\rho' > 0$, $R' > 0$ and $e' \in X_1$, $e' \neq 0$ such that:

\[ \rho' < R' \quad \text{and} \quad \sup_{S_{\rho'}(X_0 \oplus X_1)} F \leq \inf_{\Sigma_{R'}(e', X_2 \oplus X_3)} F; \]

(d) $R \leq R'(\Rightarrow \Delta_R(e, X_3) \subset \Sigma_{R'}(e', X_2 \oplus X_3))$;

(e) $-\infty < a = \inf_{\Delta_{R'}(e, X_2 \oplus X_3)} F$;

(f) (P.S.) holds for any $c \in [a, b]$ where $b = \sup_{B_{\rho}(X_0 \oplus X_1 \oplus X_2)} F$.

Then there exist three critical levels $c_1$, $c_2$ and $c_3$ for the functional $F$ such that:

\[ a \leq c_3 \leq \sup_{S_{\rho'}(X_0 \oplus X_1)} F < \inf_{\Sigma_{R'}(e', X_2 \oplus X_3)} F \leq \inf_{\Delta_R(e, X_3)} F \leq c_2 \leq \sup_{S_{\rho}(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e, X_3)} F \leq c_1 \leq b. \]

PROPOSITION 2.1. Assume that $g : E \to R$ satisfies the assumptions of Theorem 1.1. Then all solutions in $L^2(\Omega)$ of

\[ \Delta^2 u + c\Delta u = g(u) \quad \text{in} \quad L^2(\Omega) \]

belong to $E$.

With the aid of Proposition 2.1 it is enough that we investigate the existence of solutions of (1.1) in the subspace $E$ of $L^2(\Omega)$. Let $I : E \to R$ be the functional defined by,

\[ I(u) = \int_{\Omega} \frac{1}{2} |\Delta u|^2 - \frac{c}{2} |\nabla u|^2 - G(u), \quad (2.1) \]

where $G(s) = \int_0^s g(\sigma) \, d\sigma$. Under the assumptions of Theorem 1.1, $I(u)$ is well defined. By the following Proposition, $I$ is of class $C^1$ and the weak solutions of (1.1) coincide with the critical points of $I(u)$.

PROPOSITION 2.2. Assume that $g(u)$ satisfies the assumptions of Theorem 1.1. Then $I(u)$ is continuous and Fréchet differentiable in $E$ and

\[ DI(u)(h) = \int_{\Omega} \Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - g(u) h \]

for $h \in X$. Moreover $\int_{\Omega} G(u) \, dx$ is $C^1$ with respect to $u$. Thus $I \in C^1$.

Let $Z_2$ act on $E$ orthogonally. Then $E$ has two invariant orthogonal subspaces $Fix_{Z_2}$ and $Fix_{Z_2}^\perp$. Let us set

\[ H = Fix_{Z_2}^\perp. \]
The \( Z_2 \) action has the representation \( u \mapsto -u, \forall u \in H \). Thus \( Z_2 \) acts freely on the invariant subspace \( H \). We note that \( H \) is a closed invariant linear subspace of \( E \) compactly embedded in \( L^2(\Omega) \). It is easily checked that \( \Delta^2 + c\Delta \) and \( g \) are equivariant on \( H \), so \( I \) is invariant on \( H \). Moreover \( (\Delta^2 + c\Delta)(H) \subseteq H \), \( \Delta^2 + c\Delta : H \to H \) is an isomorphism and \( DI(H) \subseteq H \). Therefore critical points on \( H \) are critical points on \( E \).

### 3. A single biharmonic equation

In this section we prove the existence of multiple solutions of the a nonlinear biharmonic equation.

**Theorem 3.1.** Assume that \( \lambda_1 < c < \lambda_2 \), \( \lambda_k(\lambda_k - c) < g'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c) \), \( \lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c) \) and \( g'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1} - c) \), where \( m \geq 1 \), \( k > 2 \) and \( \gamma \in \mathbb{R} \). Then problem (1.1) has at least three solutions.

Let \( H_k \) be the subspace of \( H \) spanned by \( \phi_1, \ldots, \phi_k \) whose eigenvalues are \( \lambda_1, \ldots, \lambda_k \). Let \( H_k^+ \) be the orthogonal complement of \( H_k \) in \( H \). Let \( r = \frac{1}{2}(\lambda_k(\lambda_k - c) + \lambda_{k+1}(\lambda_{k+1} - c)) \) and let \( L : H \to H \) be the linear continuous operator such that

\[
(Lu, v) = \int_{\Omega} (\Delta^2 u + c\Delta u) \cdot v dx - r \int_{\Omega} uv dx.
\]

Then \( L \) is symmetric, bijective and equivariant. The spaces \( H_k, H_k^+ \) are the negative space of \( L \) and the positive space of \( L \). Moreover, there exists \( \nu > 0 \) such that

- \( \forall u \in H_k : (Lu, u) \leq (\lambda_k(\lambda_k - r)) \int_{\Omega} u^2 dx \leq -\nu ||u||^2 \),
- \( \forall u \in H_k^+ : (Lu, u) \geq (\lambda_{k+1}(\lambda_{k+1} - c)) \int_{\Omega} u^2 dx \geq \nu ||u||^2 \).

We can write

\[
I(u) = \frac{1}{2} (Lu, u) - \psi(u),
\]

where

\[
\psi(u) = \int_{\Omega} [G(u) - \frac{1}{2}ru^2] dx.
\]

Since \( H \) is compactly embedded in \( L^2 \), the map \( D\psi : X \to X \) is compact.
Lemma 3.1. Assume that $g(u)$ satisfies the assumptions of Theorem 3.1. Then $I(u)$ satisfies the $(P.S.)_M$ condition for any $M \in \mathbb{R}$.

For the proof see [8].

Lemma 3.2. Under the same assumptions of Theorem 3.1, the function $I(u)$ is bounded from above on $H_k$;

\[
\sup_{u \in H_k} I(u) < 0, \tag{3.1}
\]

and from below on $H_k^\perp$; there exists $R_k > 0$ such that

\[
\inf_{\| u \| = R_k} I(u) > 0, \tag{3.2}
\]

and

\[
\inf_{\| u \| < R_k} I(u) > -\infty. \tag{3.3}
\]

Proof. For some constant $d \geq 0$, we have $G_r(s) \geq \frac{1}{2} \alpha s^2 + d$, where $G_r(s) = \int_0^s g_r(\sigma) d\sigma$. For $u \in H_k$,

\[
(Lu, u) \leq (\lambda_k (\lambda_k - c) - \rho) \int_{\Omega} u^2 dx = \frac{\lambda_k (\lambda_k - c) - \lambda_{k+1} (\lambda_{k+1} - c)}{2} \int_{\Omega} u^2,
\]

\[
\int_{\Omega} G_r(u) \geq \frac{\alpha}{2} \int_{\Omega} u^2 + d|\Omega|,
\]

so that

\[
I(u) \leq \frac{1}{2} \cdot \frac{\lambda_k (\lambda_k - c) - \lambda_{k+1} (\lambda_{k+1} - c)}{2} \int_{\Omega} u^2 - \frac{\alpha}{2} \int_{\Omega} u^2 - d|\Omega| < 0,
\]

since $\frac{\lambda_k (\lambda_k - c) - \lambda_{k+1} (\lambda_{k+1} - c)}{2} < \alpha$. Thus the functional $I$ is bounded from above on $H_k$. Next we will prove that (3.2) and (3.3) hold. To get our claim (3.2), it is enough to prove that:

\[
\lim_{u \in H_k^\perp, \| u \| \to +\infty} I(u) = +\infty.
\]
We have
\[
\lim_{\|u\| \to +\infty} I(u) \\
\geq \lim_{\|u\| \to +\infty} \frac{1}{2} (1 - \frac{r}{\lambda_{k+1}(\lambda_{k+1} - c)}) \|u\|^2 - \lim_{\|u\| \to +\infty} \frac{1}{2} \beta \int_{\Omega} u^2 - \tilde{b}|\Omega|
\]
\[
\geq \lim_{\|u\| \to +\infty} \frac{1}{2} (1 - \frac{r}{\lambda_{k+1}(\lambda_{k+1} - c)}) \|u\|^2 - \lim_{\|u\| \to +\infty} \frac{1}{2} \beta \int_{\Omega} u^2 - \tilde{b}|\Omega|
\]
\[
\to +\infty,
\]
since there exists \( \tilde{b} \in R \) such that
\[
G_r(u) < \frac{1}{2} \beta u^2 + \tilde{b},
\]
and
\[
\beta < \frac{\lambda_{k+1}(\lambda_{k+1} - c) - \lambda_k(\lambda_k - c)}{2}.
\]
Now we will prove (3.3). Since
\[
\lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c)
\]
and
\[
g'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1} - c),
\]
there exists
\[
\lambda_{k+m}(\lambda_{k+m} - c) < \tilde{\gamma} < \lambda_{k+m+1}(\lambda_{k+m+1} - c)
\]
and \( \tilde{d} \geq 0 \) such that \( G(u) < \frac{\tilde{\gamma}}{2} u^2 + \tilde{d} \). Thus
\[
\inf_{u \in H_k^+, \|u\| \leq R} I(u) = \inf_{u \in H_k^+, \|u\| \leq R} \left\{ \frac{1}{2} \|u\|^2 - \int_{\Omega} G(u) \right\}
\[
> \inf_{u \in H_k^+, \|u\| \leq R} \left\{ \frac{1}{2} (1 - \frac{\tilde{\gamma}}{\lambda_{k+1}(\lambda_{k+1} - c)}) \|u\|^2 - \tilde{d}|\Omega| \right\} > -\infty.
\]
Lemma 3.3. Under the same assumptions of Theorem 1.1, there exists \( \rho_k > 0 \) such that
\[
\sup_{u \in H_k \atop \|u\| = \rho_k} I(u) < 0.
\]

Proof. Let \( L_\infty : H \to H \) be the linear operator defined by
\[
(L_\infty u, v) = (\Delta^2 u + c\Delta u)v - g'(\infty) \int_\Omega uv dx,
\]
where \( \lambda_{i+1}(\lambda_{i+1} - c) < \lambda_k(\lambda_k - c) < g'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c), k > i + 1. \)
Then \( L_\infty \) is an isomorphism. The spaces \( H_k \) and \( H_k^\perp \) are the negative space of \( L_\infty \) and the positive space of \( L_\infty \) respectively, and \( H = H_k \oplus H_k^\perp. \)

Set \( G_\infty(s) = G(s) - \frac{1}{2}g'(\infty)s^2. \) Then
\[
I(u) = \frac{1}{2}(L_\infty u, u) - \int_\Omega G_\infty(s) dx.
\]
Thus, by Lemma 4.2, \( \lim_{u \in H \atop u \to 0} \frac{1}{\|u\|^2} \int_\Omega G_\infty(u) dx \geq 0. \) Then
\[
\lim_{u \in H_k \atop u \to 0} \frac{I(u)}{\|u\|^2} < \lim_{u \in H_k \atop u \to 0} \frac{1}{2\|u\|^2}[\lambda_k(\lambda_k - c) - g'(\infty)] \int_\Omega u^2
\]
\[
- \lim_{u \in H_k^\perp \atop u \to 0} \frac{1}{\|u\|^2} \int_\Omega G_\infty(u) dx < 0.
\]
thus there exists \( \rho_k > 0 \) such that
\[
\sup_{u \in H_k \atop \|u\| = \rho_k} I(u) < 0.
\]

Lemma 3.4. Under the same assumptions of Theorem 1.1,
\[
\inf_{z \in H_k^\perp, \sigma \geq 0 \atop \|z - \sigma e_1\| = R_k} I(z - \sigma e_1) \geq 0.
\]

Proof. By Lemma 3.2, there exists \( R_k > 0 \) such that
\[
\inf_{u \in H_k^\perp \atop \|u\| = R_k} I(u) > 0.
\]
To get our claim, it is enough to prove that

\begin{equation}
\lim_{z \in H_k^\perp, \sigma \geq 0, \|z - \sigma e_1\| \to +\infty} I(z - \sigma e_1) = +\infty.
\end{equation}

To prove (3.4), we need to show that

\begin{equation}
\max_{z \in H_k^\perp, \|z\| = 1} \int z^2 = \max_{z \in H_k^\perp, \|z - \sigma e_1\| = 1} \int (z - \sigma e_1)^2.
\end{equation}

In fact, we have immediately

\begin{equation}
\max_{z \in H_k^\perp, \|z\| = 1} \int z^2 \leq \max_{z \in H_k^\perp, \|z - \sigma e_1\| = 1} \int (z - \sigma e_1)^2.
\end{equation}

Now we prove that

\begin{equation}
\max_{z \in H_k^\perp, \|z\| = 1} \int z^2 \geq \max_{z \in H_k^\perp, \|z - \sigma e_1\| = 1} \int (z - \sigma e_1)^2.
\end{equation}

If \( \sigma > 0 \), then

\[ 2 \int (z - \sigma e_1)v = \nu(z - \sigma e_1, v), \quad \forall v \in H_1 \oplus H_k^\perp. \]

Taking \( v = z - \sigma e_1 \) we get \( \nu = 2 \int (z - \sigma e_1)^2 \) and taking \( v = e_1 \) we also get

\[ 0 \leq 2 \int (z - \sigma e_1)e_1 = 2 \int (z - \sigma e_1)^2(z - \sigma e_1, e_1) \]

\[ = -2\sigma \int (z - \sigma e_1)^2 < 0 \]

which gives a contradiction. Then \( z - \sigma e_1 = z \in H_k^\perp \) and so

\[ \max_{z \in H_k^\perp, \|z - \sigma e_1\| = 1} \int (z - \sigma e_1)^2 = \max_{z \in H_k^\perp, \|z\| = 1} \int z^2. \]

Thus we proved (3.5). Now we prove (3.4). For some constant \( \beta, b \geq 0 \), we have \( G_\infty(s) \geq \frac{1}{2} \beta s^2 + b \), where

\[ G_\infty(s) = \int_0^s g_\infty(\sigma)d\sigma, \]
\[ g(s) - g'(\infty)s. \] For \( z \in H^1_k \) and \( \sigma \geq 0 \), by (4.5) we get
\[
I(z - \sigma e_1) \\
\geq \frac{1}{2} \| z - \sigma e_1 \|^2 - \frac{1}{2} g'(\infty) \int_\Omega (z - \sigma e_1)^2 - \frac{1}{2} \beta \int_\Omega (z - \sigma e_1)^2 - b |\Omega| \\
= \frac{1}{2} \| z - \sigma e_1 \|^2 (1 - g'(\infty)) \int_\|z - \sigma e_1\| \|z - \sigma e_1\|^2 - \beta \int_\|z - \sigma e_1\|^2 - b |\Omega| \\
\geq \frac{1}{2} \| z - \sigma e_1 \|^2 (1 - (g'(\infty) + \beta) \max_{z \in H^1_k, \sigma \geq 0} \int_\|z - \sigma e_1\|^2 - b |\Omega| \\
\geq \frac{1}{2} \| z - \sigma e_1 \|^2 (1 - (g'(\infty) + \beta) \max_{z \in H^1_k, \|z\| = 1} \int z^2) - b |\Omega| \to \infty.
\]

as \( \| z - \sigma e_1 \| \to +\infty \). Thus we proved the lemma. \( \Box \)

From Lemma 3.3 and Lemma 3.4 we have

**Lemma 3.5.** Under the same assumptions of Theorem 1.1, there exists \( \rho_k > 0 \) such that
\[
\sup_{u \in H^1_k} \frac{I(u)}{\|u\|} \leq \inf_{z \in \Sigma(-e_1, H^1_k)} I(z - \sigma e_1),
\]
where
\[
\Sigma(-e_1, H^1_k) = \{ z \in H^1_k, \|z\| \leq R_k \} \cup \{ z - \sigma e_1 | z \in H^1_k, \sigma \geq 0, \|z - \sigma e_1\| = R_k \},
\]
with \( R_k > \rho_k \).

**Lemma 3.6.** Let \( G_0 : R \to R \) be a continuous function such that
\[
\inf_{s \in R} \frac{G_0(s)}{1 + s^2} > -\infty, \quad \lim_{s \to 0} \frac{G_0(s)}{s^2} \geq 0.
\]
Then
\[
\frac{1}{\|u\|^2} \int_\Omega G_0(u) dx \geq 0.
\]

**Proof.** Let
\[
h(s) = \begin{cases} \frac{G_0(s)}{s^2} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}
\]
Then \( h : R \to R \) is bounded, continuous, with \( h(0) = 0 \) and \( G_0(s) \geq -h(s)s^2 \). If \((u_n)\) is a sequence in \( H \) with \( u_n \to 0 \), then up to a subsequence, \( u_n \to 0 \) a.e., and \( v_n = \frac{u_n}{\|u_n\|} \) is strongly convergent in \( L^2(\Omega) \). Since

\[
\frac{1}{\|u_n\|^2} \int_\Omega G_0(u_n)dx \geq -\int_\Omega h(u_n)v_n^2dx,
\]

the claim follows.

\[\square\]

**Lemma 3.7.** Under the same assumptions of Theorem 1.1, there exists \( \rho_{k+m} > 0 \) such that

\[
\sup_{u \in H_{k+m} \atop \|u\| = \rho_{k+m}} I(u) < \inf_{z \in \Sigma(e_{k+m}, H_{k+m}^+)} I(z),
\]

where \( \Sigma(e_{k+m}, H_{k+m}^+) = \{ w \in H_{k+m}^+ \|w\| \leq R_{k+m} \} \cup \{ w + \sigma e_{k+m} \|w\| \leq R_{k+m}, \sigma \geq 0, \|w + \sigma e_{k+m}\| = R_{k+m} \} \) with \( R_{k+m} > \rho_{k+m} \).

**Proof.** First we will prove that

\[
(3.6) \quad \sup_{u \in H_{k+m} \atop \|u\|=\rho_{k+m}, \rho \to 0} I(u) < 0.
\]

From the assumptions of Theorem 1.1, \( \lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c), \) \( m \geq 1 \). Let \( L_0 : H \to H \) be the linear operator defined by

\[
(L_0u, v) = (\Delta^2 u + c\Delta u)v - g'(0) \int_\Omega uvdx.
\]

Then \( L_0 \) is an isomorphism. The space \( H_{k+m}, H_{k+m}^+ \) are the negative space of \( L_0 \) and the positive space of \( L_0 \), respectively, and

\[
H = H_{k+m} \oplus H_{k+m}^+.
\]

Set \( G_0(s) = G(s) - \frac{1}{2}g'(0)s^2 \). Then

\[
I(u) = \frac{1}{2}(L_0u, u) - \int_\Omega G_0(u)dx.
\]
Note that $\inf_{u \in H} \frac{G_0(s)}{1+s^2} > -\infty$, $\lim_{s \to 0} \frac{G_0(s)}{s^2} \geq 0$. Thus by Lemma 3.1, $\lim_{u \to 0} \frac{1}{\|u\|^2} \int_{\Omega} G_0(u) dx \geq 0$. Then
\[
\lim_{u \to 0} \frac{I(u)}{\|u\|^2} < \lim_{u \to 0} \frac{1}{\|u\|^2} \left[ \frac{\lambda_{k+m}(\lambda_{k+m} - c)}{2} - g'(0) \right] \int_{\Omega} u^2 - \lim_{u \to 0} \frac{1}{\|u\|^2} \int_{\Omega} G_0(u) dx < 0.
\]
Thus there exists $\rho_{k+m} > 0$ such that $\sup_{u \in H_{k+m}} I(u) < 0$. By Lemma 4.2, $\inf_{u \in H_{k+m}} I(u) > 0$. Thus we have
\[
\sup_{u \in H_{k+m}} I(u) < \inf_{u \in H_{k+m}} I(u) < \inf_{u \in H_{k+m}} I(u),
\]
with $R_k > \rho_{k+m}$. In other words, there exists $e_{k+m} \in \text{Span}\{\phi_{k+1}, \ldots, \phi_{k+m}\}$ such that
\[
\sup_{\|u\|=\rho_{k+m}} I(u) < \inf_{\|u\|=\rho_{k+m}} I(u).
\]

\[\square\]

**Proof of Theorem 3.1.** By Lemma 3.5, there exists $\rho_{k} > 0$ such that
\[
\sup_{u \in H_{k}} I(u) \leq \inf_{z \in \Sigma(-e_1, H_{k}^\perp)} I(z - \sigma e_1),
\]
where $\Sigma(-e_1, H_{k}^\perp) = \{ z \in H_{k}^\perp \|z\| \leq R_k \} \cup \{ z - \sigma e_1 | z \in H_{k}^\perp, \sigma \geq 0, \|z - \sigma e_1\| = R_k \}$, with $R_k > \rho_{k}$. By Lemma 3.7, there exists $\rho_{k+m} > 0$ such that
\[
\sup_{u \in H_{k+m}} I(u) < \inf_{z \in \Sigma(e_{k+m}, H_{k+m}^\perp)} I(z),
\]
where $\Sigma(e_{k+m}, H_{k+m}^\perp) = \{ w \in H_{k+m} \|w\| \leq R_{k+m} \} \cup \{ w + \sigma e_{k+m} | w \in H_{k+m}^\perp, \sigma \geq 0, \|w + \sigma e_{k+m}\| = R_{k+m} \}$ with $R_{k+m} > \rho_{k+m}$ and $R_k > R_{k+m}$. Thus by Linking Scale Theorem 2.1, (1.1) has at least three solutions. \[\square\]
4. Nontrivial solutions of biharmonic systems

In this section we investigate the existence of multiple nontrivial solutions \((\xi, \eta)\) for perturbations \(g_1, g_2\) of the harmonic system with Dirichlet boundary condition

\[
\begin{align*}
\Delta^2 \xi + c\Delta \xi &= g_1(2\xi + 3\eta) \quad \text{in } \Omega, \\
\Delta^2 \eta + c\Delta \eta &= g_2(2\xi + 3\eta) \quad \text{in } \Omega, \\
\xi = 0, \eta = 0, \Delta \xi = 0, \Delta \eta &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where we assume that \(\lambda_1 < c < \lambda_2, g_1'(\infty), g_2'(\infty)\) are finite.

**Theorem 4.1.** Assume that \(\lambda_1 < c < \lambda_2, \lambda_k(\lambda_k - c) < 2g_1'(\infty) + 3g_2'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c), \lambda_{k+m}(\lambda_{k+m} - c) < 2g_1'(0) + 3g_2'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c).\)

Assume that \(2g_1'(t) + 3g_2'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1} - c), \) where \(m \geq 1, k > 2 \) and \(\gamma \in \mathbb{R}.\) Then system (4.1) has at least three solutions.

**Proof.** Let \(L = \Delta^2 + c\Delta.\) From problem (4.1) we get the equation

\[
L(2\xi + 3\eta) = g(2\xi + 3\eta + 2) \quad \text{in } \Omega,
\]

\[
\xi = 0, \eta = 0, \Delta \xi = 0, \Delta \eta = 0 \quad \text{on } \partial \Omega,
\]

where the nonlinearity \(g(u) = 2g_1(u) + 3g_2(u).\)

Let \(w = 2\xi + 3\eta.\) Then the above equation is equivalent to

\[
L(u) = g(u) \quad \text{in } \Omega,
\]

\[
u = 0, \Delta u = 0 \quad \text{on } \partial \Omega.
\]

With the condition of the theorem, the above equation has at least three solutions, two of which are nontrivial solutions, say \(w_1, w_2.\) Hence we get the solutions \((\xi, \eta)\) of problem (4.1) from the following systems:

\[
L\xi = g_1(w_i) \quad \text{in } \Omega,
\]

\[
L\eta = g_2(w_i) \quad \text{in } \Omega,
\]

\[
\xi = 0, \eta = 0, \Delta \xi = 0, \Delta \eta = 0 \quad \text{on } \partial \Omega,
\]
where \( i = 0, 1, 2 \) and \( w_0 = 0 \). When \( i = 0 \), from the above equation we get the trivial solution \((\xi, \eta) = (0, 0)\). When \( i = 1, 2 \), from the above equation we get the nontrivial solutions \((\xi_1, \eta_1), (\xi_2, \eta_2)\).

Therefore system (4.1) has at least three solutions \((\xi, \eta)\), two of which are nontrivial solutions.

**Theorem 4.2.** Assume that \( \lambda_1 < c < \lambda_2 \),

\[
2g'_1(\infty) + 3g'_2(\infty) < \lambda_1(\lambda_1 - c),
\]

\[
2g'_1(0) + 3g'_2(0) < \lambda_1(\lambda_1 - c).
\]

Assume that \( 2g'_1(t) + 3g'_2(t) \leq \gamma < \lambda_1(\lambda_1 - c), \) where \( \gamma \in \mathbb{R} \). Then system (4.1) has only the trivial solution \((\xi, \eta) = (0, 0)\).

**Proof.** Let \( L = \Delta^2 + c\Delta \). From problem (4.1) we get the equation

\[
(4.5) \quad L(2\xi + 3\eta) = g(2\xi + 3\eta + 2) \quad \text{in } \Omega,
\]

\[
\xi = 0, \eta = 0, \Delta \xi = 0, \Delta \eta = 0 \quad \text{on } \partial \Omega,
\]

where the nonlinearity \( g(u) = 2g_1(u) + 3g_2(u) \).

Let \( w = 2\xi + 3\eta \). Then the above equation is equivalent to

\[
(4.6) \quad L(u) = g(u) \quad \text{in } \Omega,
\]

\[
u = 0, \Delta v = 0 \quad \text{on } \partial \Omega.
\]

With the condition of the theorem, by Theorem 2.1 the above equation has the trivial solution. Hence we have the trivial solution \((\xi, \eta) = (0, 0)\) of problem (4.1) from the following system:

\[
(4.7) \quad L\xi = 0 \quad \text{in } \Omega,
\]

\[
L\eta = 0 \quad \text{in } \Omega,
\]

\[
\xi = 0, \eta = 0, \Delta \xi = 0, \Delta \eta = 0 \quad \text{on } \partial \Omega.
\]

From (4.7) we get the trivial solution \((\xi, \eta) = (0, 0)\). \(\square\)
References


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