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FUZZY SEMIGROUPS IN REDUCTIVE SEMIGROUPS

Inheung Chon

Abstract. We consider a fuzzy semigroup S in a right (or left) reductive semigroup X such that $S(k) = 1$ for some $k \in X$ and find a faithful representation (or anti-representation) of S by transformations of S. Also we show that a fuzzy semigroup S in a weakly reductive semigroup X such that $S(k) = 1$ for some $k \in X$ is isomorphic to the semigroup consisting of all pairs of inner right and left translations of S and that S can be embedded into the semigroup consisting of all pairs of linked right and left translations of S with the property that S is an ideal of the semigroup.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh ([11]). Rosenfeld ([9]) used this concept to formulate the notion of fuzzy groups. Kuroki ([5], [6], [7], [8]) introduced fuzzy semigroups, fuzzy ideals, fuzzy bi-ideals, and fuzzy semiprime ideals in semigroups, and developed some properties of those semigroups and ideals. Subsequently Dos ([4]) studied fuzzy regular subsemigroups in regular semigroups, fuzzy inverse subsemigroups in inverse semigroups, and fuzzy multiplication semigroups in commutative semigroups. As a continuation of these studies, we consider fuzzy semigroups in a reductive semigroup and find some properties of those fuzzy semigroups in this note.

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In section 2 we review some basic definitions and properties of fuzzy sets, fuzzy points, and fuzzy semigroups which will be used in next sections. In section 3 we consider a fuzzy semigroup S in a right (or left) reductive semigroup X such that $S(k) = 1$ for some $k \in X$, find a faithful representation (or anti-representation) of S, and show that S is isomorphic (or anti-isomorphic) to the semigroup of all inner left (or right) translations of S. In section 4 we show that a fuzzy semigroup S in a weakly reductive semigroup X such that $S(k) = 1$ for some $k \in X$ is isomorphic to a semigroup $H(S_0)$ which consists of all pairs of inner right and left translations of S and show that S can be embedded into a semigroup $H(S)$ which consists of all pairs of linked right and left translations of S with the properties that S is an ideal of $H(S)$ and every left (or right) translation of S is induced by some inner left (or right) translation of $H(S)$ iff each left (or right) translation of S is linked with some right (or left) translation of S.

2. Preliminaries

In this section we review some basic definitions and properties of fuzzy sets, fuzzy points, and fuzzy semigroups which will be used in section 3 and section 4.

Definition 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B. The set $\{x \in X : B(x) > 0\}$ is called the *support* of B.

Definition 2.2. A *t-norm* is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying, for each p, q, r, s in [0,1],

(1) $T(p, 0) = 0,\ T(p, 1) = p = T(1, p)$,
(2) $T(p, q) \leq T(r, s)$ if $p \leq r$ and $q \leq s$,
(3) $T(p, q) = T(q, p)$,
(4) $T(p, T(q, r)) = T(T(p, q), r)$.

Definition 2.3. A t-norm $T : [0,1] \times [0,1] \to [0,1]$ is continuous if T is continuous with respect to the usual topologies.

It is well known ([1]) that the function $T_m : [0,1] \times [0,1] \to [0,1]$ defined by $T_m(a, b) = \min(a, b)$, the function $T_p : [0,1] \times [0,1] \to [0,1]$

defined by $T_p(a,b) = ab$, and the function $T_M : [0,1] \times [0,1] \to [0,1]$ defined by $T_M(a,b) = \max(a+b-1, 0)$ are continuous t-norms.

The following definition is due to Sessa ([10]).

DEFINITION 2.4. Let X be a set and let U, V be two fuzzy sets in X. Then $U \circ V$ is defined by

$$(U \circ V)(x) = \begin{cases} \sup\limits_{ab=x} T(U(a), V(b)) & \text{if } \ ab = x \\ 0 & \text{if } \ ab \neq x. \end{cases}$$

DEFINITION 2.5. A fuzzy set in a set X is called a *fuzzy point* iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$, we denote this fuzzy point by x_α, where the point x is called its *support*. The fuzzy point x_α is said to be contained in a fuzzy set A, denoted by $x_\alpha \in A$, iff $\alpha \leq A(x)$.

PROPOSITION 2.6. *Let x_p, y_q be fuzzy points in a groupoid X. Then $x_p \circ y_q = (xy)_{T(p,q)}$.*

Proof. If $z = xy$, then

$$\begin{aligned} (x_p \circ y_q)(z) &= (x_p \circ y_q)(xy) = \sup_{ab=xy} T(x_p(a), y_q(b)) \\ &= T(x_p(x), y_q(y)) = T(p,q) \end{aligned}$$

If $z \neq xy$,

$$(x_p \circ y_q)(z) = \sup_{b=z} \ T(x_p(a), y_q(b)) = 0.$$

Thus, $x_p \circ y_q = (xy)_{T(p,q)}$.

□

PROPOSITION 2.7. *Let A, B, C be fuzzy sets in a set X and let T be a continuous t-norm. If X is associative, then $(A \circ B) \circ C = A \circ (B \circ C)$.*

Proof. See Proposition 2.8 of [2]. □

From now on, we assume that every t-norm in this paper is continuous.

The following definition is due to Anthony and Sherwood ([1]). That is, they replaced the minimum condition proposed by Rosenfeld ([9]) with a t-norm.

Definition 2.8. Let X be a groupoid and T be a t-norm. A function $S : X \to [0,1]$ is a *fuzzy groupoid* in X iff for every x, y in X, $S(xy) \geq T(S(x), S(y))$. If X is a group, a fuzzy groupoid G is a *fuzzy group* in X iff for each $x \in X$, $G(x^{-1}) = G(x)$.

See [1] for examples of fuzzy groups.

Proposition 2.9. *Let A be a non-empty fuzzy set of a groupoid X. Then the followings are equivalent.*

(1) *A is a fuzzy groupoid.*
(2) *For any $x_p, y_q \in A$, $x_p \circ y_q \in A$.*
(3) *$A \circ A \subseteq A$.*

Proof. See Proposition 2.7 of [2]. $\square$

If B is a fuzzy groupoid in a semigroup X, $(x_p \circ y_q) \circ z_r = x_p \circ (y_q \circ z_r)$ for every $x_p, y_q, z_r \in B$ from Proposition 2.7. We call B a fuzzy semigroup in X.

Example of a fuzzy semigroup. Let X be a set of all natural numbers which are greater than or equal to 2, that is, $X = \{2, 3, 4, \dots\}$, and let $\cdot$ be a multiplication. Then $(X, \cdot)$ is a semigroup. Let $S : X \to [0,1]$ be a function defined by $S(a) = \frac{a}{a+1}$. Then $S(p \cdot q) = \frac{p \cdot q}{p \cdot q + 1}$, and hence $S(p \cdot q) \geq S(q)$. Thus

$$S(p \cdot q) = T(1, S(p \cdot q)) \geq T(S(p), S(p \cdot q)) \geq T(S(p), S(q)).$$

That is, S is a fuzzy semigroup in S.

We write $x_p y_q$ for $x_p \circ y_q$ in the next section 3 and section 4.

3. Representations of fuzzy semigroups in a reductive semigroup

In this section we discuss the representations of fuzzy semigroups in a semigroup and a reductive semigroup. First of all, we define a left and a right translation of a fuzzy semigroup which play important roles for the representations of fuzzy semigroups in semigroups.

Definition 3.1. Let S be a fuzzy semigroup in a semigroup X. A transformation $l : S \to S$ is called a *left translation* of S if $l(x_p)y_q = l(x_p y_q)$ for all $x_p, y_q \in S$. A transformation $r : S \to S$ is called a *right translation* of S if $x_p r(y_q) = r(x_p y_q)$ for all $x_p, y_q \in S$.

It is easily checked that a transformation $l_{a_p} : S \to S$ defined by $l_{a_p}(b_q) = a_p b_q$ for $a_p \in S$ is a left translation and a transformation $r_{a_p} : S \to S$ defind by $r_{a_p}(b_q) = b_q a_p$ for $a_p \in S$ is a right translation. We call l_{a_p} a *inner left translation* of S and call r_{a_p} a *inner right translation* of S.

Definition 3.2. Let f be a mapping from a set X to a set Y. Let A be a fuzzy set in X. Then the *image* of A, written $f(A)$, is the fuzzy set in Y with membership function defined by

$$f(A)(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \text{ is nonempty,} \\ 0, & \text{otherwise,} \end{cases}$$

for all $y \in Y$.

Proposition 3.3. *Let S be a fuzzy semigroup in a set X. Then the set $\mathcal{W}_S$ of all transformations of S forms a semigroup under the operation of composition $\circ$.*

Proof. Let $f, g, h \in \mathcal{W}_S$ and let $x_p \in S$. Then it is easy to see $f(x_p) = [f(x)]_p$ from Definition 3.2. Since $x_p \in S$ and $f \in \mathcal{W}_S$, $f(x_p) \in S$, and hence $[f(x)]_p \in S$. Since $g \in \mathcal{W}_S$, $g([f(x)]_p) \in S$. Since $(g \circ f)(x_p) = g([f(x)]_p)$, $(g \circ f)(x_p) \in S$. That is, $g \circ f \in \mathcal{W}_S$. Clearly $[(f \circ g) \circ h](x_p) = [f \circ (g \circ h)](x_p)$. □

We define a representation of a fuzzy semigroup in a semigroup and find a representation of the fuzzy semigroup.

Definition 3.4. Let S be a fuzzy semigroup in a semigroup X, let L be a fuzzy set, and let $\mathcal{W}_L$ be the semigroup of all transformations of L. A homomorphism $\psi : S \to \mathcal{W}_L$ is called a *representation* of S by transformations of L and a representation ψ of S is called *faithful* if it is one-to-one. An anti-homomorphism $\phi : S \to \mathcal{W}_L$ is called an *anti-representation* of S by transformations of L and an anti-representation ϕ of S is called *faithful* if it is one-to-one.

Proposition 3.5. *Let S be a fuzzy semigroup in a semigroup X and let $\mathcal{W}_S$ be the semigroup of all transformations of S. Then there is a representation $\psi : S \to \mathcal{W}_S$ of S and there is an anti-representation $\phi : S \to \mathcal{W}_S$ of S.*

Proof. Let $r_{a_p} : S \to S$ be an inner right translation and let $l_{a_p} : S \to S$ be an inner left translation. Let $\psi : S \to \mathcal{W}_S$ be a map defined by $\psi(a_p) = l_{a_p}$ and let $\phi : S \to \mathcal{W}_S$ be a map defined by $\phi(a_p) = r_{a_p}$. Then $[\psi(a_p b_q)](c_r) = l_{a_p b_q}(c_r) = (a_p b_q)c_r$. $[\psi(a_p) \circ \psi(b_q)](c_r) = (l_{a_p} \circ l_{b_q})(c_r) = l_{a_p}(b_q c_r) = a_p(b_q c_r)$. Since X is associative, $(a_p b_q)c_r = a_p(b_q c_r)$ by Proposition 2.7. Thus, $\psi(a_p b_q) = \psi(a_p) \circ \psi(b_q)$. Similarly we may show $\phi(a_p b_q) = \phi(b_q) \circ \phi(a_p)$. □

We now turn to a faithful representation of a fuzzy semigroup in a reductive semigroup.

Definition 3.6. A semigroup X is called *right reductive* if $ax = bx$ for all $x \in X$ implies $a = b$. A semigroup X is called *left reductive* if $xa = xb$ for all $x \in X$ implies $a = b$. A semigroup X is called *reductive* if X is right reductive and left reductive. A semigroup X is called *weakly reductive* if $ax = bx$ and $xa = xb$ for all $x \in X$ imply $a = b$.

Theorem 3.7. Let S be a fuzzy semigroup in a right (or left) reductive semigroup X such that $S(k) = 1$ for some $k \in X$ and let $\mathcal{W}_S$ be the semigroup of all transformations of S. Then there is a faithful representation $\psi : S \to \mathcal{W}_S$ of S (or a faithful anti-representation $\phi : S \to \mathcal{W}_S$ of S).

Proof. Let $l_{a_p} : S \to S$ be an inner left translation and let $\psi : S \to \mathcal{W}_S$ be a map defined by $\psi(a_p) = l_{a_p}$. Then ψ is a representation of S by Proposition 3.5. Suppose $\psi(a_p) = \psi(b_q)$. Then $l_{a_p} = l_{b_q}$. For all $c_r \in S$, $l_{a_p}(c_r) = l_{b_q}(c_r)$, that is, $a_p c_r = b_q c_r$. By Proposition 2.6, $(ac)_{T(p,r)} = (bc)_{T(q,r)}$ for all $c_r \in S$. Since a, b, and c are in a right reductive semigroup X, $a = b$. Since $S(k) = 1$, $k_1 \in S$, and hence $a_p k_1 = b_q k_1$. That is, $(ak)_{T(p,1)} = (bk)_{T(q,1)}$. Since $a = b$, $T(p,1) = T(q,1)$, and hence $p = q$. Thus, $a_p = b_q$, that is, ψ is injective. Similarly we may show that for an inner right translation r_{a_p} of S, a map $\phi : S \to \mathcal{W}_S$ defined by $\phi(a_p) = r_{a_p}$ is a faithful anti-representation. □

Corollary 3.8. Let S be a fuzzy semigroup in a reductive semigroup X such that $S(k) = 1$ for some $k \in X$ and let $\mathcal{W}_S$ be the semigroup of all transformations of S. Then there are a faithful representation $\psi : S \to \mathcal{W}_S$ of S and a faithful anti-representation $\phi : S \to \mathcal{W}_S$ of S.

Proof. Immediate from Theorem 3.7. □

Theorem 3.9. Let S be a fuzzy semigroup in a right (or left) reductive semigroup X such that $S(k) = 1$ for some $k \in X$. Then S is isomorphic (or anti-isomorphic) to the semigroup of all inner left (or right) translations of S.

Proof. Let $\mathcal{W}_S^*$ be the set of all inner left translations of S. Let $l_{a_p}, l_{b_q} \in \mathcal{W}_S^*$. Then $(l_{a_p} \circ l_{b_q})(c_r) = a_p(b_q c_r)$ and $l_{a_p b_q}(c_r) = (a_p b_q)c_r$. Since S is a fuzzy semigroup, $a_p b_q \in S$ and $(a_p b_q)c_r \in S$ from Proposition 2.9. Thus, $l_{a_p b_q} \in W_S^*$. Since X is associative, $a_p(b_q c_r) = (a_p b_q)c_r$ from Proposition 2.7, and hence, $l_{a_p b_q} = l_{a_p} \circ l_{b_q}$. Thus $l_{a_p} \circ l_{b_q} \in W_S^*$. Clearly $[(l_{a_p} \circ l_{b_q}) \circ l_{c_r}](x_t) = [l_{a_p} \circ (l_{b_q} \circ l_{c_r})](x_t)$. Hence, W_S^* is a semigroup. Let $\psi : S \to \mathcal{W}_S^*$ be a map defined by $\psi(a_p) = l_{a_p}$. Then ψ is an isomorphism by Theorem 3.7. Similarly we may prove that S is anti-isomorphic to the semigroup of all inner right translations of S. □

Corollary 3.10. Let S be a fuzzy semigroup in a reductive semigroup X such that $S(k) = 1$ for some $k \in X$. Then S is isomorphic to the semigroup of all inner left translations of S and S is anti-isomorphic to the semigroup of of all inner right translations of S.

Proof. Immediate from Theorem 3.9. □

4. Embedding of fuzzy semigroups into semigroups

In this section we discuss the embedding problem of a fuzzy semigroup in a reductive semigroup into a semigroup. First we define a translational hull of a fuzzy semigroup into which the fuzzy semigroup is embedded.

Definition 4.1. A right translation r and a left translation l of a fuzzy semigroup S in a semigroup X are said to be *linked* if $x_p l(y_q) = r(x_p)y_q$. The set of all pairs (r, l) of linked right and left translations r and l of S is called the *translational hull* of S and is denoted by $H(S)$.

We define an operation in $H(S)$ by $(r, l)(r', l') = (r' \circ r, l \circ l')$ for $(r, l), (r', l') \in H(S)$. The following proposition shows that $H(S)$ is a semigroup under this operation.

Proposition 4.2. *The translational hull $H(S)$ of a fuzzy semigroup S in a semigroup X is a semigroup.*

Proof. Let $(r, l), (r', l') \in H(S)$. It is easy to check that $r' \circ r$ is a right translation of S and $l \circ l'$ is a left translation of S. Since (r, l) and (r', l') are linked pairs, $a_p l(b_q) = r(a_p)b_q$ and $a_p l'(b_q) = r'(a_p)b_q$ for all $a_p, b_q \in S$. $a_p[(l \circ l')(b_q)] = a_p[l(l'(b_q))] = r(a_p)l'(b_q) = r'(r(a_p))b_q = [(r' \circ r)(a_p)]b_q$. Thus, $a_p[(l \circ l')(b_q)] = [(r' \circ r)(a_p)]b_q$, that is, $(r, l)(r', l') = (r' \circ$

$r, l \circ l') \in H(S)$. Let $(r_1, l_1), (r_2, l_2), (r_3, l_3) \in H(S)$. $[(r_1, l_1)(r_2, l_2)](r_3, l_3) = (r_2 \circ r_1, l_1 \circ l_2)(r_3, l_3) = (r_3 \circ r_2 \circ r_1, l_1 \circ l_2 \circ l_3)$. $(r_1, l_1)[(r_2, l_2)(r_3, l_3)] = (r_1, l_1)(r_3 \circ r_2, l_2 \circ l_3) = (r_3 \circ r_2 \circ r_1, l_1 \circ l_2 \circ l_3)$. Thus, $H(S)$ is associative. $\square$

For a fuzzy semigroup S in a semigroup X, let $H(S_0) = \{(r_{a_p}, l_{a_p}) : a_p \in S\}$, where r_{a_p} is an inner right translation of S and l_{a_p} is an inner left translation of S. It is esay to see that r_{a_p} and l_{a_p} are linked, that is, $H(S_0) \subset H(S)$. We characterize $H(S_0)$ in Theorem 4.3 and Lemma 4.5.

Theorem 4.3. Let S be a fuzzy semigroup in a weakly reductive semigroup X such that $S(k) = 1$ for some $k \in X$. Then $H(S_0)$ is a subsemigroup of $H(S)$ and S is isomorphic to $H(S_0)$.

Proof. Let $(r_{a_p}, l_{a_p}), (r_{b_q}, l_{b_q}) \in H(S_0)$. Then $(r_{b_q} \circ r_{a_p})(c_r) = (c_r a_p) b_q = r_{a_p b_q}(c_r)$ and $(l_{a_p} \circ l_{b_q})(c_r) = a_p(b_q c_r) = l_{a_p b_q}(c_r)$. Thus, $(r_{a_p}, l_{a_p})(r_{b_q}, l_{b_q}) = (r_{b_q} \circ r_{a_p}, l_{a_p} \circ l_{b_q}) = (r_{a_p b_q}, l_{a_p b_q})$. Since $a_p, b_q \in S$, $a_p b_q \in S$ from Proposition 2.9. Hence, $(r_{a_p}, l_{a_p})(r_{b_q}, l_{b_q}) \in H(S_0)$. Clearly $H(S_0)$ is associative. Thus, $H(S_0)$ is a subsemigroup of $H(S)$. Let $\psi : S \to H(S_0)$ be a map defined by $\psi(a_p) = (r_{a_p}, l_{a_p})$. Then $\psi(a_p b_q) = (r_{a_p b_q}, l_{a_p b_q}) = (r_{b_q} \circ r_{a_p}, l_{a_p} \circ l_{b_q}) = (r_{a_p}, l_{a_p})(r_{b_q}, l_{b_q}) = \psi(a_p)\psi(b_q)$. Suppose $\psi(a_p) = \psi(b_q)$. Then $(r_{a_p}, l_{a_p}) = (r_{b_q}, l_{b_q})$, that is, $r_{a_p} = r_{b_q}$ and $l_{a_p} = l_{b_q}$. Since $r_{a_p}(c_r) = r_{b_q}(c_r)$ for all $c_r \in S$, $c_r a_p = c_r b_q$. By Proposition 2.6, $(ca)_{T(r,p)} = (cb)_{T(r,q)}$ for all $c_r \in S$. Thus, $ca = cb$ for all $c \in X$. Since $l_{a_p}(c_r) = l_{b_q}(c_r)$ for all $c_r \in S$, $a_p c_r = b_q c_r$. By Proposition 2.6, $(ac)_{T(p,r)} = (bc)_{T(q,r)}$ for all $c_r \in S$. Thus $ac = bc$ for all $c \in X$. Since a, b, c are in a weakly reductive semigroup X, $a = b$. Since $S(k) = 1$, $k_1 \in S$, and hence, $(ka)_{T(1,p)} = (kb)_{T(1,q)}$. Since $a = b$, $T(1,p) = T(1,q)$, and hence, $p = q$. Thus, $a_p = b_q$, that is, ψ is injective. Clearly ψ is surjective. Hence, S is isomorphic to $H(S_0)$. $\square$

Definition 4.4. A non-empty set L (or R) of a semigroup S is a *left ideal* (or *right ideal*) of S if $SL \subseteq L$ (or $RS \subseteq R$). I is an ideal of S if $IS \cup SI \subseteq I$.

Lemma 4.5. *For a fuzzy semigroup S in a semigroup X, $H(S_0)$ is an ideal of $H(S)$.*

Proof. Let $g \in H(S_0)H(S)$. Then $g = (r_{a_p}, l_{a_p})(r, l) = (r \circ r_{a_p}, l_{a_p} \circ l)$, where r and l are linked. Since $r_{r(a_p)}(b_q) = b_q r(a_p) = r(b_q a_p) = r(r_{a_p}(b_q)) = (r \circ r_{a_p})(b_q)$, $r_{r(a_p)} = r \circ r_{a_p}$. Since $a_p l(b_q) = r(a_p) b_q$, $(l_{a_p} \circ l)(b_q) = l_{a_p}(l(b_q)) = a_p l(b_q) = r(a_p) b_q = l_{r(a_p)} b_q$, that is, $(l_{a_p} \circ l) =$

$l_{r(a_p)}$. Thus, $g = (r \circ r_{a_p}, l_{a_p} \circ l) = (r_{r(a_p)}, l_{r(a_p)}) \in H(S_0)$. That is, $H(S_0)H(S) \subseteq H(S_0)$. Let $g \in H(S)H(S_0)$. Then $g = (r,l)(r_{a_p}, l_{a_p}) = (r_{a_p} \circ r, l \circ l_{a_p})$, where r and l are linked. Since $l_{l(a_p)}(b_q) = l(a_p)b_q = l(a_p b_q) = l(l_{a_p}(b_q)) = (l \circ l_{a_p})(b_q)$, $l_{l(a_p)} = l \circ l_{a_p}$. Since $b_q l(a_p) = r(b_q)a_p$, $(r_{a_p} \circ r)(b_q) = r_{a_p}(r(b_q)) = r(b_q)a_p = b_q l(a_p) = r_{l(a_p)}(b_q)$, that is, $r_{a_p} \circ r = r_{l(a_p)}$. Thus, $g = (r_{a_p} \circ r, l \circ l_{a_p}) = (r_{l(a_p)}, l_{l(a_p)}) \in H(S_0)$. That is, $H(S)H(S_0) \subseteq H(S_0)$. Hence, $H(S_0)$ is an ideal of $H(S)$. □

It is well known ([3]) that a weakly reductive semigroup S can be embedded in a semigroup T with the properties that S is an ideal of T and every left (or right) translation of S is induced by some inner left (or right) translation of T iff each left (or right) translation of S is linked with some right (or left) translation of S. The following theorem may be considered as the corresponding one in fuzzy semigroups.

Theorem 4.6. Let S be a fuzzy semigroup on a weakly reductive semigroup X such that $S(k) = 1$ for some $k \in X$. Then S can be embedded into a semigroup $H(S)$ with the properties that

(1) S is an ideal of $H(S)$.
(2) every left (or right) translation of S is induced by some inner left (or right) translation of $H(S)$ iff each left (or right) translation of S is linked with some right (or left) translation of S.

Proof. We may identify S with $H(S_0)$ by Theorem 4.3. By Lemma 4.5, S is an ideal of $H(S)$.

Suppose l is a left translation of S such that $(r,l) \in H(S)$ for some right translation r of S. Since $r_{a_p} \circ r = r_{l(a_p)}$ and $l \circ l_{a_p} = l_{l(a_p)}$, $(r,l)(r_{a_p}, l_{a_p}) = (r_{a_p} \circ r, l \circ l_{a_p}) = (r_{l(a_p)}, l_{l(a_p)})$. We may identify a_p with (r_{a_p}, l_{a_p}) and identify $l(a_p)$ with $(r_{l(a_p)}, l_{l(a_p)})$ from Theorem 4.3. By the identification, $(r,l)(a_p) = (r,l)(r_{a_p}, l_{a_p}) = (r_{l(a_p)}, l_{l(a_p)}) = l(a_p)$. Thus, $l_{(r,l)}(a_p) = (r,l)a_p = l(a_p)$. Hence, $l = l_{(r,l)}|_S$. Conversely, suppose that the left translation l is induced by some inner left translation of $H(S)$, that is, $l = l_{(r_1,l_1)}|_S$ for some right translation r_1 of S and some left translation l_1 of S. Then $x_p l(y_q) = x_p l_{(r_1,l_1)}(y_q) = x_p(r_1,l_1)(y_q)$. Let $r = r_{(r_1,l_1)}|_S$. Then r is a right translation of S and $r(x_p)y_q = r_{(r_1,l_1)}(x_p)y_q = x_p(r_1,l_1)(y_q)$. Thus, $x_p l(y_q) = r(x_p)y_q$, that is, r and l are linked. Similarly we may prove the dual case. □

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Department of Mathematics
Seoul Women's University
Seoul 139-774, Korea
E-mail: ihchon@swu.ac.kr