Korean J. Math. **21** (2013), No. 4, pp. 401–419
http://dx.doi.org/10.11568/kjm.2013.21.4.401

ON ALMOST n-SIMPLY PRESENTED ABELIAN p-GROUPS

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Abstract. Let $n \geq 0$ be an arbitrary integer. We define the class of *almost n-simply presented* abelian p-groups. It naturally strengthens all the notions of almost simply presented groups introduced by Hill and Ullery in Czechoslovak Math. J. (1996), n-simply presented p-groups defined by the present author and Keef in Houston J. Math. (2012), and almost ω_1-$p^{\omega+n}$-projective groups developed by the same author in an upcoming publication [3]. Some comprehensive characterizations of the new concept are established such as Nunke-esque results as well as results on direct summands and ω_1-bijections.

1. Introduction and Backgrounds

Throughout the current paper, we assume that p is a fixed prime integer and all groups into consideration are additive p-torsion abelian. All crucial notions and notations will follow those from [8] and [9]; the new ones will be explained in the sequel. For instance,

$$G[p^n] = \{g \in G \ : \ p^n g = 0, n \in \mathbb{N}\},$$

is said to be the *p^n-socle* of G that plays a significant role in our further investigation.

Received August 10, 2013. Revised November 28, 2013. Accepted November 28, 2013.

2010 Mathematics Subject Classification: 20K10.

Key words and phrases: abelian p-primary groups, almost Σ-cyclic groups, almost $p^{\omega+n}$-projective groups, almost n-simply presented groups.

Likewise, for any $i \in \mathbb{N}$, we have

$$p^iG = \{p^ig \ : \ g \in G\}$$

which is called the *p^i-power* subgroup of G. Set $p^\omega G = \cap_{i<\omega} p^iG$, called *first Ulm subgroup*, and we will say that G is *separable* if $p^\omega G = \{0\}$. By analogy, for every ordinal α, one can define $p^\alpha G$ as follows: $p^\alpha G = p(p^{\alpha-1}G)$ if α is non-limit, or $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ otherwise. A subgroup N of a group G is said to be *nice* if, for any ordinal α, the equality $p^\alpha(G/N) = (p^\alpha G + N)/N$ is fulfilled.

In their seminal works [11] and [12], Hill and Ullery have given the following critical

Definition 1. The reduced group G is called *almost totally projective* if it has a collection $\mathcal{C}$ consisting of nice subgroups of G satisfying the following three conditions:

(1) $\{0\} \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed with respect to ascending unions, i.e., if $H_i \in \mathcal{C}$ with $H_i \subseteq H_j$ whenever $i \leq j$ $(i, j \in I)$ then $\cup_{i\in I} H_i \in \mathcal{C}$;

(3) If K is a countable subgroup of G, then there is $L \in \mathcal{C}$ (that is, a nice subgroup L of G) such that $K \subseteq L$ and L is countable.

This concept generalizes the notion of *almost Σ-cyclic* groups defined in [10]. Actually separable almost totally projective groups are almost Σ-cyclic.

Moreover, the direct sum of a divisible group and an almost totally projective group was called *almost simply presented*.

Paralleling, the current author defines in [3] the following:

Definition 2. The group G is said to be *almost $p^{\omega+n}$-projective* if there is $B \leq G[p^n]$ such that G/B is almost Σ-cyclic.

In addition, if there exists a countable subgroup $C \leq G$ with the property that G/C is almost $p^{\omega+n}$-projective, then we will say that G is *almost ω_1-$p^{\omega+n}$-projective*.

On the other hand, in [14] was formulated the following:

Definition 3. The group G is called *n-simply presented* if there exists $P \leq G[p^n]$ with the property that G/P is simply presented.

This is a natural enlargement of $p^{\omega+n}$-projectives, so that it seems that we can use this idea to refine the statement of Definition 2.

And so, the goal of this article is to combine Definitions 1 and 3 into the following extension of Definition 2.

DEFINITION 4. The group G is said to be *almost n-simply presented* if there is $H \leq G[p^n]$ such that G/H is almost simply presented.

If G/H is almost totally projective, then we will say that G is *almost n-totally projective*.

In case that H is nice in G, we give

DEFINITION 5. The group G is called *nicely almost n-simply presented* (resp. *nicely almost n-totally projective*) if there exists a p^n-bounded nice subgroup $N \leq G$ with G/N almost simply presented (resp. almost totally projective).

In terms of [14] these groups could be named *strongly almost n-simply presented* and *strongly almost n-totally projective*.

Apparently almost $p^{\omega+n}$-projectives are nicely almost n-totally projective. Since almost simply presented groups of cardinality at most $\aleph_1$ are simply presented (see [12]), it is easy to see that almost n-simply presented groups of cardinality not exceeding $\aleph_1$ are n-simply presented, a class of groups which was comprehensively developed in [14]. Thus to avoid any duplication and triviality of the results, our almost n-simply presented groups will be of cardinalities at least $\aleph_2$ since in [12] was showed that there is an almost simply presented group of cardinality $\aleph_2$ that is not simply presented.

Our main achievements will be proved in details in the next section.

2. Basic Results

We start here with three technicalities.

LEMMA 2.1. *Suppose A is a group with a subgroup B such that A/B is bounded, and suppose $A \leq G$ for some group G. The following two conditions hold:*

(a) If N is nice in B, then N is nice in A.

(b) If M is nice in A, then $M \cap B$ is nice in B.

Proof. (a) Let $p^t A \subseteq B$ for some $t \in \mathbb{N}$. Since $p^\omega A = p^\omega B$, and hence $p^\lambda A = p^\lambda B$ for each $\lambda \geq \omega$, we need show only that $\cap_{k<\omega}(N + p^k A) = N + p^\omega A$. In fact, $\cap_{k<\omega}(N+p^k A) = \cap_{t\leq k<\omega}(N+p^k A) \subseteq \cap_{i<\omega}(N+p^i B) = N + p^\omega B = N + p^\omega A$, as required.

(b) For every limit ordinal β we have with the aid of the modular law that $\cap_{\alpha<\beta}(M \cap B + p^{\alpha}B) \subseteq \cap_{\alpha<\beta}(M + p^{\alpha}A) \cap B = (M + p^{\beta}A) \cap B = (M + p^{\beta}B) \cap B = M \cap B + p^{\beta}B$, as required. □

LEMMA 2.2. *Let λ be an ordinal, k a positive integer, and G a group with a subgroup P. If $P \cap p^{\lambda+k}G$ is nice in $p^{\lambda+k}G$, then $P \cap p^{\lambda}G$ is nice in $p^{\lambda}G$.*

Proof. For each limit ordinal β we write

$$\cap_{\alpha<\beta}(P \cap p^{\lambda+k}G + p^{\lambda+k+\alpha}G) = P \cap p^{\lambda+k}G + p^{\lambda+k+\beta}G = P \cap p^{\lambda+k}G + p^{\lambda+\beta}G.$$

We shall consider two cases about β.

Case 1: $\beta \geq \omega \cdot 2$. Then α can be chosen to be $\geq \omega$. Hence we have

$$\cap_{\alpha<\beta}(P \cap p^{\lambda+k}G + p^{\lambda+\alpha}G) = P \cap p^{\lambda+k}G + p^{\lambda+\beta}G.$$

Summarizing in both sides the intersection $P \cap p^{\lambda}G$, we derive that

$$\cap_{\alpha<\beta}(P \cap p^{\lambda+k}G + p^{\lambda+\alpha}G) + P \cap p^{\lambda}G = P \cap p^{\lambda}G + p^{\lambda+\beta}G.$$

But

$$\cap_{\alpha<\beta}(P \cap p^{\lambda+k}G + p^{\lambda+\alpha}G) + P \cap p^{\lambda}G = \cap_{\alpha<\beta}(P \cap p^{\lambda}G + p^{\lambda+\alpha}G).$$

In fact, the left inclusion "$\subseteq$" is self-evident, so that we consider the right one "$\supseteq$". To that aim, given $x \in \cap_{\alpha<\beta}(P \cap p^{\lambda}G + p^{\lambda+\alpha}G)$, whence $x = a_1 + b_{1\alpha} = \cdots = c_1 + b_{1\tau} = \cdots$ where $a_1, c_1 \in P \cap p^{\lambda}G$ and $b_{1\alpha} \in p^{\lambda+\alpha}G$, $b_{1\tau} \in p^{\lambda+\tau}G$ for some arbitrary ordinal τ such that $\alpha < \tau < \beta$. Observe that $b_{1\alpha} - b_{1\tau} = c_1 - a_1 \in p^{\lambda+k}G \cap (P \cap p^{\lambda}G) = P \cap p^{\lambda+k}G$ because $\alpha > k$. Thus $b_{1\alpha} \in P \cap p^{\lambda+k}G + p^{\lambda+\tau}G$ for every τ, which means that $b_{1\alpha} \in \cap_{\alpha<\beta}(P \cap p^{\lambda+k}G + p^{\lambda+\alpha}G)$. Finally, $x \in P \cap p^{\lambda}G + \cap_{\alpha<\beta}(P \cap p^{\lambda+k}G + p^{\lambda+\alpha}G)$, as wanted. The obtained equality gives that $\cap_{\alpha<\beta}(P \cap p^{\lambda}G + p^{\lambda+\alpha}G) = P \cap p^{\lambda}G + p^{\lambda+\beta}G$, as required.

Case 2: $\beta = \omega$. Therefore, α is natural, and we write

$$\cap_{\alpha<\beta}(P \cap p^{\lambda+k}G + p^{\lambda+k+\alpha}G) = P \cap p^{\lambda+k}G + p^{\lambda+\beta}G.$$

However, we can choose these α such that $\alpha = i - k$ with $i \geq k + 1$ and hence

$$\cap_{\imath<\omega}(P \cap p^{\lambda+k}G + p^{\lambda+i}G) = P \cap p^{\lambda+k}G + p^{\lambda+\omega}G.$$

Again as above adding $P \cap p^{\lambda}G$ in both sides of the last identity, we deduce as before that

$$\cap_{\imath<\omega}(P \cap p^{\lambda}G + p^{\lambda+i}G) = P \cap p^{\lambda}G + p^{\lambda+\omega}G,$$

as desired. □

Lemma 2.3. *Let A be a group with a subgroup B such that A/B is bounded. Then*

(i) A is almost Σ-cyclic if and only if B is almost Σ-cyclic.

(ii) A is almost simply presented if and only if B is almost simply presented.

(iii) A is (nicely) almost n-simply presented if and only if B is (nicely) almost n-simply presented.

Proof. (i) The necessity follows immediately from [1].

Concerning the sufficiency, since $p^m A \subseteq B$ for some $m \in \mathbb{N}$, again [1] allows us to deduce that $p^m A$ is almost Σ-cyclic. We henceforth appeal to [12] to obtain that the same is A, as asserted.

(ii) Since $p^m A \subseteq B$ for some $m \in \mathbb{N}$, it readily follows that $p^\omega A = p^\omega B$. Moreover, $B/p^\omega B \subseteq A/p^\omega A$.

To prove the necessity, utilizing [12], $A/p^\omega A$ is almost Σ-cyclic, and thus [1] can be applied to get that so is $B/p^\omega B$. Besides, again [12] tells us that $p^\omega B = p^\omega A$ remains almost simply presented, so that a new third application of [12] guarantees that B must be almost simply presented.

Conversely, to show the sufficiency, as above $p^\omega A = p^\omega B$ and $B/p^\omega B$ are almost simply presented. Moreover, since $A/p^\omega A/B/p^\omega B \cong A/B$ is bounded, point (i) implies that $A/p^\omega A$ is almost simply presented. Finally, [12] again insures that A is almost simply presented, as formulated.

(iii) To prove the necessity, let A/P is almost simply presented for some $P \leq A[p^n]$. But $A/P/(B+P)/P \cong A/(B+P) \cong A/B/(B+P)/B$ is bounded, so that point (ii) forces that $(B+P)/P \cong B/(B \cap P)$ is almost simply presented. And finally, since $P \cap B$ is contained in $B[p^n]$, the claim follows.

To show now the sufficiency, assume that B/P is almost simply presented for some $P \leq A[p^n]$. Since $A/P/B/P \cong A/B$ is bounded, point (ii) gives that A/P is almost simply presented, as required.

The nicely part follows by applying Lemma 2.1. □

Lemma 2.4. *Suppose that A is a group such that $p^\lambda A$ is bounded for some ordinal λ, and suppose $Y \subseteq p^\lambda A$. Then A is almost simply presented if and only if A/Y is almost simply presented.*

Proof. Since

$$A/p^\lambda A \cong A/Y/p^\lambda A/Y = A/Y/p^\lambda(A/Y),$$

we apply a result from [12] which says that A is almost simply presented if and only if $A/p^\lambda A$ is almost simply presented. Again employing the same assertion to A/Y, we are finished. $\square$

We continue with the consideration of the separable case.

Proposition 2.5. *If G is a separable almost n-simply presented group, then G is almost $p^{\omega+n}$-projective.*

Proof. Let G/H be almost simply presented for some $H \leq G$ with $p^n H = \{0\}$. Exploiting [11], $G/H/p^\omega(G/H) \cong G/\cap_{i<\omega}(p^i G + H)$ is almost Σ-cyclic. But $p^n(\cap_{i<\omega}(p^i G + H)) = p^\omega G = \{0\}$, as required. $\square$

For $p^{\omega+n}$-bounded groups, we can obtain even more:

Proposition 2.6. *Suppose G is a group such that $p^{\omega+n}G = \{0\}$. If G is nicely almost n-simply presented, then G is almost $p^{\omega+n}$-projective.*

Proof. Assume that G/N is almost simply presented for some nice subgroup $N \leq G$ such that $p^n N = \{0\}$. Again in virtue of [12], the quotient $G/N/p^\omega(G/N) \cong G/(p^\omega G + N)$ is almost Σ-cyclic. However, $p^n(p^\omega G + N) = \{0\}$, and even $p^\omega G + N$ remains nice in G, as needed. $\square$

We will now explore whether or not Ulm subgroups and Ulm factors reserve the property of being (nicely) almost n-simply presented provided that the full group possesses it, as well as having such a property they imply it on the whole group.

Theorem 2.7. *(a) Suppose that G is almost n-simply presented. Then, for any ordinal λ, both $p^\lambda G$ and $G/p^\lambda G$ are almost n-simply presented.*

(b) Suppose that G is nicely almost n-simply presented. Then, for any ordinal λ, both $p^\lambda G$ and $G/p^\lambda G$ are nicely almost n-simply presented.

Proof. (a) Assume that G/H is almost simply presented for some $H \leq G[p^n]$. By [12] $p^{\lambda+n}(G/H)$ remains almost simply presented, and moreover

$$p^{\lambda+n}(G/H)/(p^{\lambda+n}G + H)/H \cong X/(p^{\lambda+n}G + H)$$

where $X \leq G$ with $X/H = p^{\lambda+n}(G/H)$. But $p^n X \subseteq p^{\lambda+n}G + H$, so that the right hand-side is bounded by p^n. Thus Lemma 2.3 applies to get that $(p^{\lambda+n}G + H)/H$ is almost simply presented. However, the quotient $(p^\lambda G + H)/H/(p^{\lambda+n}G + H)/H \cong (p^\lambda G + H)/(p^{\lambda+n}G + H)$ is also p^n-bounded, so that again Lemma 2.3 works to infer that $(p^\lambda G + H)/H \cong$

$p^\lambda G/(p^\lambda G \cap H)$ is almost simply presented. Since $p^\lambda G \cap H \leq (p^\lambda G)[p^n]$, we are done with the first part.

Now dealing with the second half, we first observe that

$$G/p^\lambda G/(p^\lambda G + H)/p^\lambda G \cong G/(p^\lambda G + H) \cong G/H/(p^\lambda G + H)/H,$$

where $(p^\lambda G + H)/p^\lambda G \cong H/(H \cap p^\lambda G$ is p^n-bounded. But

$$\begin{aligned} &G/H/p^{\lambda+n}(G/H)/(p^\lambda G + H)/H/p^{\lambda+n}(G/H) \\ &\cong G/H/(p^\lambda G + H)/H \cong G/(p^\lambda G + H) \end{aligned}$$

and $(p^\lambda G + H)/H/p^{\lambda+n}(G/H) \subseteq p^\lambda(G/H/p^{\lambda+n}(G/H))$ is p^n-bounded, hence Lemma 2.4 is applicable to conclude that $G/p^\lambda G$ is almost n-simply presented, as stated.

(b) Let us assume that G/N is almost simply presented for some nice $N \leq G$ with $p^n N = \{0\}$. Since $p^\lambda(G/N) = (p^\lambda G + N)/N \cong p^\lambda G/(p^\lambda G \cap N)$ is also almost simply presented and $p^\lambda G \cap N$ is nice in $p^\lambda G$ such that $p^n(p^\lambda G \cap N) = \{0\}$, we conclude that $p^\lambda G$ is nicely almost n-simply presented too.

Moreover,

$$\begin{aligned} &G/p^\lambda G/(N + p^\lambda G)/p^\lambda G \cong G/(N + p^\lambda G) \\ &\cong G/N/(N + p^\lambda G)/N = G/N/p^\lambda(G/N) \end{aligned}$$

must be almost simply presented. However, $N + p^\lambda G$ remains nice in G, so that $(N + p^\lambda G)/p^\lambda G$ is so in $G/p^\lambda G$. Since $(N + p^\lambda G)/p^\lambda G \cong N/(N \cap p^\lambda G)$ is obviously p^n-bounded, the result follows. □

As a useful consequence, we yield:

Corollary 2.8. *If G is an almost n-simply presented group, then $G/p^\omega G$ is almost $p^{\omega+n}$-projective.*

Proof. Follows by a combination of Theorem 2.7 (a) applied to $\lambda = \omega$, along with Proposition 2.5. □

Under some extra circumstances the reverse also holds:

Proposition 2.9. *Suppose G is a group whose $G/p^\lambda G$ is n-simply presented for some ordinal λ. Then G is almost n-simply presented if and only if $p^\lambda G$ is almost n-simply presented.*

Proof. The "and only if" part follows from Theorem 2.7(a).

As for the "if" part, since

$$G/p^\lambda G \cong G/p^{\lambda+n}G/p^\lambda G/p^{\lambda+n}G = G/p^{\lambda+n}G/p^\lambda(G/p^{\lambda+n}G),$$

is n-simply presented, where $p^{\lambda}(G/p^{\lambda+n}G)$ is clearly p^n-bounded, we take into account Theorem 4.5 from [14] to establish that $G/p^{\lambda+n}G$ is n-simply presented.

On the other hand, Lemma 2.3 (iii) ensures that $p^{\lambda+n}G$ is almost n-simply presented because the factor-group $p^{\lambda}G/p^{\lambda+n}G$ is obviously bounded (by p^n). So, Theorem 2.10 presented below yields the wanted claim. □

For ordinals α of the special type $\lambda + n$ for some arbitrary ordinal λ, the last achievement can be somewhat strengthened to the following one:

THEOREM 2.10. *Let G be a group, λ an ordinal and n a positive integer. Then G is (nicely) almost n-simply presented if and only if $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are both (nicely) almost n-simply presented.*

Proof. The implication "$\Rightarrow$" follows by a direct application of Theorem 2.7.

For the implication "$\Leftarrow$", let $P_1 \leq G$ contain $p^{\lambda+n}G$ such that $G/P_1 \cong G/p^{\lambda+n}G/P_1/p^{\lambda+n}G$ is almost simply presented and $p^nP_1 \subseteq p^{\lambda+n}G$.

Suppose Y is a maximal p^n-bounded summand of $p^{\lambda}G$, so that the decomposition $p^{\lambda}G = X \oplus Y$ holds. Observe that $p^{\lambda+n}G = p^nX$, so that $Y \cap p^{\lambda+n}G = \{0\}$. Let H be a $p^{\lambda+n}$-high subgroup of G containing Y. One easily sees that $(p^{\lambda+n}G)[p] = X[p]$, so that $H \cap X = \{0\}$. Moreover, since $G[p] = (p^{\lambda+n}G)[p] \oplus H[p] = X[p] \oplus H[p]$ and since H is pure in G (see, e.g., [8]), it is readily checked that $G[p^n] = X[p^n] \oplus H[p^n]$.

We next claim that

$$(G/p^{\lambda+n}G)[p^n] = (X \oplus H[p^n])/p^{\lambda+n}G.$$

To verify this, since the right hand-side is obviously contained in the left one, choose $x \in G$ with $p^nx \in p^{\lambda+n}G$. Thus $p^nx \in p^nX$ and hence $x \in X + G[p^n] = X \oplus H[p^n]$, as required.

By what we have shown above, $P_1 \subseteq X \oplus H[p^n]$. Setting $P_2 = (P_1 + X) \cap H[p^n]$, we derive with the modular law at hand that

$$X + P_1 = (X + P_1) \cap (X \oplus H[p^n]) = X + (X + P_1) \cap H[p^n] = X + P_2.$$

Consequently, by adding on both sides $p^{\lambda}G$, we have that $P_1 + p^{\lambda}G = P_2 + p^{\lambda}G$.

On the other hand, let $p^{\lambda+n}G/P_3 = p^{\lambda+n}(G/P_3)$ be almost simply presented such that $p^nP_3 = \{0\}$. Letting $P = P_2 + P_3$, we observe that $P \leq G[p^n]$, that $p^{\lambda+n}G \cap P = P_3$ and that $p^{\lambda}G + P = p^{\lambda}G + P_2 = p^{\lambda}G + P_1$.

Hence $p^{\lambda+n}G/(p^{\lambda+n}G\cap P) \cong (p^{\lambda+n}G+P)/P$ is almost simply presented. Since $(p^{\lambda}G+P)/(p^{\lambda+n}G+P)$ is clearly bounded by p^n, it therefore follows from Lemma 2.3 (ii) that $(p^{\lambda}G+P)/P \cong p^{\lambda}G/(p^{\lambda}G\cap P)$ is almost simply presented. And since $p^{\lambda}(G/P)/(p^{\lambda}G + P)/P$ is bounded by p^n because $p^{\lambda+n}(G/P) \subseteq (p^{\lambda}G + P)/P$, we once again refer to Lemma 2.3 to get that $p^{\lambda}(G/P)$ is almost simply presented.

Furthermore, Lemma 2.4 is in use to show that

$$G/(p^{\lambda}G + P) = G/(p^{\lambda}G + P_1) \cong G/P_1/(p^{\lambda}G + P_1)/P_1$$

is almost simply presented, because $(p^{\lambda}G + P_1)/P_1 \subseteq p^{\lambda}(G/P_1)$ and the latter is bounded by p^{2n} since $p^n(P_1/p^{\lambda+n}G) = \{0\}$ and $G/P_1 \cong G/p^{\lambda+n}G/P_1/p^{\lambda+n}G$. So, we further have that

$$G/(p^{\lambda}G + P) \cong G/P/(p^{\lambda}G + P)/P$$

is almost simply presented, and so [12] enables us that

$$\begin{aligned} G/P/p^{\lambda}(G/P) &\cong G/P/(p^{\lambda}G + P)/P/p^{\lambda}(G/P)/(p^{\lambda}G + P)/P \\ &= G/P/(p^{\lambda}G + P)/P/p^{\lambda}(G/P/(p^{\lambda}G + P)/P) \end{aligned}$$

is also almost simply presented, which again by [12] means that G/P is almost simply presented, as wanted.

The "nicely" part follows like this: By definition we assume that $P_1/p^{\lambda+n}G$ is nice in $G/p^{\lambda+n}G$ whence, with the help of [8], the subgroup P_1 should be nice in G. Likewise, by definition P_3 is nice in $p^{\lambda+n}G$ and hence in G (see [8] too). We claim that P is nice in G as well. In fact, P_1 being nice in G plainly implies that $P_1 + p^{\lambda}G = P + p^{\lambda}G$ is nice in G. On the other hand, by what we have shown above, $p^{\lambda+n}G \cap P = P_3$ is nice in G. Now Lemma 2.2 works to infer that $p^{\lambda}G \cap P$ is nice in G. We finally appeal once again to [8] to conclude that P has to be nice in G, indeed, as asserted. □

As an interesting consequence, we deduce:

Corollary 2.11. *Let G be a group and λ an ordinal such that $G/p^{\lambda}G$ is almost simply presented. Then G is (nicely) almost n-simply presented if and only if $p^{\lambda}G$ is (nicely) almost n-simply presented.*

Proof. The necessity follows directly from Theorem 2.7, so that we now concentrate on the sufficiency. To that end, we claim that $p^{\lambda+n}G$ is (nicely) almost n-simply presented and claim that $G/p^{\lambda+n}G$ is almost simply presented, whence Theorem 2.10 will work. In fact, since

$p^\lambda G/p^{\lambda+n}G$ is bounded by p^n, Lemma 2.3 (iii) tells us that our first claim is true. As for the second one,

$$G/p^\lambda G \cong G/p^{\lambda+n}G/p^\lambda G/p^{\lambda+n}G = G/p^{\lambda+n}G/p^\lambda(G/p^{\lambda+n}G),$$

is almost simply presented, where $p^\lambda(G/p^{\lambda+n}G)$ is clearly p^n-bounded, and hence [12] enables us that $G/p^{\lambda+n}G$ is almost simply presented, indeed, as claimed. □

PROPOSITION 2.12. *Suppose G is a group of length λ strictly less than ω^2. Then G is nicely almost n-simply presented if and only if, for each non-negative integer m, $p^{\omega\cdot m+n}G/p^{\omega\cdot(m+1)+n}G$ is almost $p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$". From Theorem 2.7 (b), $G/p^{\omega\cdot(m+1)+n}G$ is also nicely almost n-simply presented. Henceforth, we once again appeal to Theorem 2.7 (b) to get that $p^{\omega\cdot m+n}(G/p^{\omega\cdot(m+1)+n}G) = p^{\omega\cdot m+n}G/p^{\omega\cdot(m+1)+n}G$ is nicely almost n-simply presented too. But $p^{\omega\cdot m+n}G/p^{\omega\cdot(m+1)+n}G$ is obviously $p^{\omega+n}$-bounded, whence Proposition 2.6 assures that this quotient is actually almost $p^{\omega+n}$-projective.

"$\Leftarrow$". For $m = 0$ we have that $p^nG/p^{\omega+n}G = p^n(G/p^{\omega+n}G)$ is almost $p^{\omega+n}$-projective, so that the same holds for $G/p^{\omega+n}G$. When $m = 1$ we obtain that $p^{\omega+n}G/p^{\omega\cdot 2+n}G = p^{\omega+n}(G/p^{\omega\cdot 2+n}G)$ is almost $p^{\omega+n}$-projective. But $G/p^{\omega+n}G \cong G/p^{\omega\cdot 2+n}G/p^{\omega+n}G/p^{\omega\cdot 2+n}G = G/p^{\omega\cdot 2+n}G/p^{\omega+n}(G/p^{\omega\cdot 2+n}G)$ is almost $p^{\omega+n}$-projective. Therefore, Theorem 2.10 works to get that $G/p^{\omega\cdot 2+n}G$ is nicely almost n-simply presented, etc. after final steps to $G/p^\lambda G \cong G$ is nicely almost n-simply presented. □

For groups with countable first Ulm subgroup, the situation is the following:

THEOREM 2.13. *Suppose G is a group whose $p^\omega G$ is countable. Then G is almost n-simply presented if and only if G is almost ω_1-$p^{\omega+n}$-projective.*

Proof. Firstly, note that in [3] was established that G is almost ω_1-$p^{\omega+n}$-projective exactly when $G/p^\omega G$ is almost $p^{\omega+n}$-projective, provided $p^\omega G$ is countable. Thus our claim restricts to that proving that G is almost n-simply presented precisely when $G/p^\omega G$ is almost $p^{\omega+n}$-projective.

The implication "$\Rightarrow$" now follows by a simple combination of Corollary 2.8 and Definition 2.

As for the implication "$\Leftarrow$", $G/p^\omega G$ being almost $p^{\omega+n}$-projective forces that $G/p^\omega G/A/p^\omega G \cong G/A$ is almost Σ-cyclic for some $A \le G$

such that $p^nA \subseteq p^\omega G$. Therefore, $A = K \oplus P$ where K is countable and P is p^n-bounded. Hence $G/A \cong G/P/A/P$ where $A/P \cong K$ is countable. Using [3], G/P must be almost simply presented, as expected. □

Remark 1. The last assertion extends Proposition 2.5. It is also a strengthening of the fact that G is n-simply presented uniquely when G is ω_1-$p^{\omega+n}$-projective whenever $p^\omega G$ is countable.

In that aspect we can strengthen Proposition 2.6 to the following one:

Theorem 2.14. *Let G be a group for which $p^{\omega+n}G$ is countable. If G is nicely almost n-simply presented, then G is almost ω_1-$p^{\omega+n}$-projective.*

Proof. The usage of Theorem 2.7 (b) gives that the factor $G/p^{\omega+n}G$ is nicely almost n-simply presented. This combined with Proposition 2.6 allows us to deduce that $G/p^{\omega+n}G$ is almost $p^{\omega+n}$-projective. And since $p^{\omega+n}G$ is countable, we obtain by Definition 2 that G is almost ω_1-$p^{\omega+n}$-projective, as expected. □

It was shown above in Theorem 2.13 that almost ω_1-$p^{\omega+n}$-projective groups are almost n-simply presented, provided that their first Ulm subgroup is countable. In the next statement this limitation will be dropped off.

Proposition 2.15. *Almost ω_1-$p^{\omega+n}$-projective groups are almost n-simply presented.*

Proof. Assume that G is such a group. From ([3], Theorem 2.21 (2)), it follows that there is $P \leq G[p^n]$ such that G/P is the sum of a countable group and an almost Σ-cyclic group. However, it was shown in the proof of Theorem 2.25 again in [3] that this sum is necessarily almost simply presented, as required. □

Remark 2. Notice that an almost ω_1-$p^{\omega+n}$-projective group need not be nicely almost n-simply presented.

When $n = 0$, i.e., for simply presented groups, Theorem 2.13 can be refined. To achieve this, we first need one more technicality. It is actually a non-trivial generalization of the classical Charles' lemma for Σ-cyclic groups (see, e.g., [7]).

Lemma 2.16. *The group G is the sum of a countable group and an almost Σ-cyclic group if and only if there is a countable subgroup $K \leq G$ such that G/K is almost Σ-cyclic.*

Proof. ”$\Rightarrow$”. Assume $G = L + S$ where L is countable and S is almost Σ-cyclic. Since $L \cap S \subseteq S$ is countable, there is a countable nice subgroup C of S such that $L \cap S \subseteq C$. In accordance with [3], S/C is almost Σ-cyclic. But

$$G/C = (L + S)/C = [(L + C)/C] \oplus [S/C] = (K/C) \oplus (S/C)$$

where we put $K = L + C$. Thus $G/C/K/C \cong G/K \cong S/C$ is almost Σ-cyclic, as stated.

”$\Leftarrow$”. Since $p^\omega G \subseteq K$ is countable, we see that the containing group $p^\omega G \cong (H_G \oplus p^\omega G)/H_G \subseteq G/H_G$ is also countable because $(H_G \oplus p^\omega G)/H_G$ is an essential subgroup of G/H_G due to the fact that H_G is maximal with respect to $H_G \cap p^\omega G = \{0\}$. Furthermore, one may write $G = H_G + C$ for some countable $C \leq G$. However,

$$H_G \cong (H_G \oplus p^\omega G)/p^\omega G \subseteq G/p^\omega G$$

is almost Σ-cyclic because so is $G/p^\omega G$. In fact, $G/K \cong G/p^\omega G/K/p^\omega G$ is almost Σ-cyclic with countable factor-group $K/p^\omega G$, and so we can apply [10] to get that $G/p^\omega G$ is almost Σ-cyclic, as claimed. □

So, we have at our disposal all the information needed to prove the following-compare also with Theorem 2.13 when $n = 0$.

THEOREM 2.17. *The group G is almost simply presented with countable $p^\omega G$ if and only if G is the sum of a countable group and an almost Σ-cyclic group.*

Proof. ”**Necessity**”. It follows that $G/p^\omega G$ is almost Σ-cyclic and $p^\omega G$ is countable, so that Lemma 2.16 works.

”**Sufficiency**”. The preceding Lemma 2.16 tells us that there exists a countable subgroup K such that G/K is almost Σ-cyclic. Thus $p^\omega G \leq K$ must be countable. Moreover, as we have seen above in the proof of the sufficiency of the previous lemma, $G/p^\omega G$ is almost Σ-cyclic. Henceforth, we employ [12] to deduce that G is almost simply presented, as asserted. □

PROPOSITION 2.18. *Let $\phi : G \to A$ be an ω_1-bijective homomorphism. If G is almost n-simply presented, then A is almost n-simply presented.*

Proof. Assume that G/H is almost simply presented for some $H \leq G[p^n]$. Considering the induced homomorphism

$$\Phi : G/H \to A/\phi(H),$$

we deduce that $ker\Phi = (H+ker\phi)/H \cong ker\phi/(ker\phi \cap H)$ which is countable because $ker\phi$ is. In addition, $coker\phi = A/\phi(G) \cong A/\phi(H)/\phi(G)/\phi(H) = A/\phi(H)/\Phi(G/H) = coker\Phi$ are both countable as well. Thus Φ is also an ω_1-bijection. We furthermore apply [6] to conclude that $A/\phi(H)$ is almost simply presented. Since $\phi(H) \subseteq A[p^n]$, the argument is completed. □

As a consequence necessary for further applications, we have:

COROLLARY 2.19. *Suppose that G is an almost n-simply presented group and C is its countable subgroup. Then G/C is almost n-simply presented.*

Proof. Since the natural map $G \to G/C$, being an epimorphism, is an ω_1-bijection, Proposition 2.18 allows us to conclude that the quotient-group G/C is almost n-simply presented as well. □

Almost n-simply presented groups are also closed under taking countable extensions. Specifically, the following is true:

PROPOSITION 2.20. *Let $A \leq G$ with countable factor G/A. If A is almost n-simply presented, then G is almost n-simply presented.*

Proof. Write $G = A + K$ for some countable $K \leq G$, and assume that A/P is almost simply presented for some $P \leq A[p^n]$. Therefore, $G/P = [A/P]+[(K+P)/P]$ where $(K+P)/P \cong K/(K \cap P)$ is countable, and an appeal to [11] or [2] gives that G/P is almost simply presented, as required. □

Before proving a slight refinement of the preceding result concerning ω_1-bijections, two preliminary technical claims are necessary.

LEMMA 2.21. *Suppose that α is an ordinal, and that G and F are groups where F is finite. Then the following formula is fulfilled:*

$$p^\alpha(G + F) = p^\alpha G + F \cap p^\alpha(G + F).$$

Proof. We will use a transfinite induction on α. First, if $\alpha-1$ exists, we have

$$p^\alpha(G+F)=p(p^{\alpha-1}(G+F))=p(p^{\alpha-1}G+F\cap p^{\alpha-1}(G+F))=$$

$$p(p^{\alpha-1}G)+p(F\cap p^{\alpha-1}(G+F))\subseteq p^\alpha G+F\cap p(p^{\alpha-1}(G+F)) \\ =p^\alpha G+F\cap p^\alpha(G+F).$$

Since the reverse inclusion "$\supseteq$" is obvious, we obtain the desired equality.

If now $\alpha-1$ does not exist, we have that $p^\alpha(G+F)=\cap_{\beta<\alpha}(p^\beta(G+F))\subseteq\cap_{\beta<\alpha}(p^\beta G+F)=\cap_{\beta<\alpha}p^\beta G+F=p^\alpha G+F$. In fact, the second sign "$=$" follows like this: Given $x\in\cap_{\beta<\alpha}(p^\beta G+F)$, we write that $x=g_{\beta_1}+f_1=\cdots=g_{\beta_s}+f_s=\cdots$ where $f_1,\cdots,f_s\in F$ are the all elements of F; $g_{\beta_1}\in p^{\beta_1}G,\cdots,g_{\beta_s}\in p^{\beta_s}G$ with $\beta_1<\cdots<\beta_s<\cdots$.

Since F is finite, while the number of equalities is infinite due to the infinite cardinality of α, we infer that $g_{\beta_s}\in p^\beta G$ for any ordinal $\beta<\alpha$ which means that $g_{\beta_s}\in\cap_{\beta<\alpha}p^\beta G=p^\alpha G$. Thus $x\in\cap_{\beta<\alpha}p^\beta G+F=p^\alpha G+F$, as claimed. Furthermore, $p^\alpha(G+F)\subseteq(p^\alpha G+F)\cap p^\alpha(G+F)=p^\alpha G+F\cap p^\alpha(G+F)$ which is obviously equivalent to an equality. □

Lemma 2.22. *Let N be a nice subgroup of a group G. Then*
(i) $N+R$ is nice in G for every finite subgroup $R\leq G$;
(ii) N is nice in $G+F$ for each finite group F.

Proof. (i) For any limit ordinal γ, we deduce that $\cap_{\delta<\gamma}(N+R+p^\delta G)\subseteq R+\cap_{\delta<\gamma}(N+p^\delta G)=R+N+p^\gamma G$, as required. Indeed, the relation "$\subseteq$" follows like this: Given $x\in\cap_{\delta<\gamma}(N+R+p^\delta G)$, we write $x=a_1+r_1+g_1=\cdots=a_s+r_s+g_s=\cdots=a_k+r_1+g_k=\cdots$, where $a_1,\cdots,a_k\in N$; $r_1,\cdots,r_k\in R$; $g_1\in p^{\delta_1}G,\cdots,g_k\in p^{\delta_k}G$ with $\delta_1<\cdots<\delta_k$. So $a_1+g_1=\cdots=a_k+g_k=\cdots\in\cap_{\delta<\gamma}(N+p^\delta G)$ and hence $x\in R+\cap_{\delta<\gamma}(N+p^\delta G)$, as requested.

(ii) Since N is nice in G, we may write $\cap_{\delta<\gamma}[N+p^\delta G]=N+p^\gamma G$ for every limit ordinal γ. Furthermore, with Lemma 2.21 at hand, we subsequently deduce that

$$\cap_{\delta<\gamma}[N+p^\delta(G+F)]=\cap_{\delta<\gamma}[N+p^\delta G+F\cap p^\delta(G+F)]\subseteq$$

$$\cap_{\delta<\gamma}(N+p^\delta G)+F\cap p^\gamma(G+F)=N+p^\gamma G+F\cap p^\gamma(G+F)=N+p^\gamma(G+F).$$

In fact, the inclusion "$\subseteq$" follows thus: Given $x\in\cap_{\delta<\gamma}[N+p^\delta G+F\cap p^\delta(G+F)]$, we write $x=a_1+g_1+f_1=\cdots=a_s+g_s+f_s=\cdots=a_k+g_k+f_1=\cdots$, where $a_1,\cdots,a_k\in N$; $g_1\in p^{\delta_1}G,\cdots,g_k\in p^{\delta_k}G$;

$f_1 \in F \cap p^{\delta_1}(G+F), \cdots, f_k \in F \cap p^{\delta_k}(G+F)$ with $\delta_1 < \cdots < \delta_k$. Hence $a_1 + g_1 = \cdots = a_k + g_k = \cdots \in \cap_{\delta<\gamma}(N + p^\delta G)$ and because the number of the f_i's $(1 \leq i \leq k)$ is finite whereas the number of equalities is not, we can deduce that $f_1 \in \cap_{\delta<\gamma}(F \cap p^\delta(G+F)) = F \cap p^\gamma(G+F)$, as needed. □

A helpful statement which we need to prove the next major assertion is the following:

Proposition 2.23. *Suppose G is a group with a finite subgroup F. The following two points hold:*

(a) Then G is almost n-simply presented if and only if G/F is almost n-simply presented.

In particular, if G is nicely almost n-simply presented, then G/F is nicely almost n-simply presented.

(b) Suppose $A \leq G$ such that G/A is finite. Then G is almost n-simply presented if and only if A is almost n-simply presented.

Proof. (a) Assume first that G is almost n-simply presented, so G/P is almost simply presented for some $P \leq G[p^n]$. But

$$G/P/(F+P)/P \cong G/(F+P) \cong G/F/(F+P)/F$$

where $(F+P)/P \cong F/(F \cap P)$ is finite while $(F+P)/F \cong P/(P \cap F)$ is p^n-bounded, whence Theorem 1.8 from [3] assures that both $G/P/(F+P)/P$ and $G/F/(F+P)/F$ should be almost simply presented. Thus G/F is almost n-simply presented, as stated.

As for the "nicely" case, since P must be by definition nice in G, we observe with the help of Lemma 2.22 that $P+F$ remains nice in G, so $(P+F)/F$ is nice in G/F, as required.

Conversely, given G/F is almost n-simply presented group, we have that $G/F/L/F \cong G/L$ is almost simply presented for some $L \leq G$ with $p^n L \subseteq F$. Therefore, one can write $L = M \oplus Q$ where M is bounded by p^n whereas Q is finite. Now $G/M/L/M \cong G/L$ is almost simply presented with finite $L/M \cong Q$. Again Theorem 1.8 from [3] is utilized to conclude that G/M is almost simply presented, as needed to get that G is almost n-simply presented.

(b) Write $G = A + E$ where E is a finite subgroup. Thus $G/E = (A+E)/E \cong A/(A \cap E)$ where $A \cap E$ remains finite. Hence we can apply the previous point (a) to infer the equivalence. □

We now have all the ingredients needed to establish the following.

THEOREM 2.24. *Let $\varphi : G \to A$ be an ω-bijective homomorphism. Then G is almost n-simply presented if and only if A is almost n-simply presented.*

In addition, if G is nicely almost n-simply presented, then so is A.

Proof. By considering the natural composition $G \to Im\varphi \to A$, we may break the arguments into the two cases (a) and (b) of Proposition 2.23. □

However, when the subgroup is taken a priori to be nice in the whole group, some improvements can be established, thus enlarging Proposition 5.3 from [14] to almost n-simply presented groups (for results of that type the reader can see [6]).

THEOREM 2.25. *Let N be a countable nice subgroup of a group G such that $p^\omega G$ is countable. Then G is almost n-simply presented if and only if G/N is almost n-simply presented.*

Proof. The necessity follows from Corollary 2.19, so that we concentrate on the sufficiency. To that aim, observe with the aid of Proposition 2.8 that

$$G/N/p^\omega(G/N) \cong G/(p^\omega G + N) \cong G/p^\omega G/(p^\omega G + K)/p^\omega G$$

is almost $p^{\omega+n}$-projective. Since $(p^\omega G + K)/p^\omega G \cong K/(K \cap p^\omega G)$ is countable (and nice in $G/p^\omega G$), it follows from Proposition 2.10 of [3] that $G/p^\omega G$ is almost $p^{\omega+n}$-projective too. Thus G is (nicely) almost ω_1-$p^{\omega+n}$-projective, and hence Theorem 2.13 works. □

As a valuable consequence, we derive:

COROLLARY 2.26. *Suppose K is a countable subgroup of G. If G/K is separable almost n-simply presented, then G is almost n-simply presented.*

Proof. Since $p^\omega G \subseteq K$ is obviously countable and K is nice in G, Proposition 2.25 is applicable to infer the statement. □

REMARK 3. The last statement improves Corollary 5.4 from [14] to almost simply presented groups.

It is obvious that the arbitrary direct sum of (nicely) almost n-simply presented groups is again (nicely) almost n-simply presented. So, we finish with the exploration of the direct summand problem for (nicely) almost n-simply presented groups (see [15] as well).

Proposition 2.27. *Let $A = B \oplus C$ be a group for some subgroups B and C. If*

(1) A is almost n-simply presented such that C is countable, then B is almost n-simply presented.

(2) A is nicely almost n-simply presented such that $p^{\omega+n}C = \{0\}$, then B is nicely almost n-simply presented.

Proof. (1) Clearly, using Corollary 2.19, we deduce that $B \cong A/C$ is almost n-simply presented.

(2) Referring to Theorem 2.7 (b), $p^{\omega+n}B = p^{\omega+n}A$ is almost n-simply presented. Moreover, $A/p^{\omega+n}A \cong (B/p^{\omega+n}B) \oplus C$, where in virtue of Theorem 2.7 accomplished with Proposition 2.6 we have that $A/p^{\omega+n}A$ is almost $p^{\omega+n}$-projective. Now, appealing to [3], so is $B/p^{\omega+n}B$ as being a subgroup. Finally, Theorem 2.10 is a guarantor that B is nicely almost n-simply presented. $\square$

3. Concluding Discussion and Open Problems

We were unable to prove that if A is a group such that $p^{\omega+n}A = \{0\}$ and $A/p^{\omega}A$ is almost $p^{\omega+n}$-projective, then A is almost n-simply presented, and vice versa. The crucial moment is to show (if possible) that if A is a group with a subgroup H such that A/H is almost Σ-cyclic and $p^nH \subseteq p^{\omega}A \subseteq A[p^n]$, then A/p^nH is almost simply presented, and as a consequence A will be almost n-simply presented.

If these are true, then

$$A/p^{\omega}A \cong A/p^{\omega+n}A/p^{\omega}A/p^{\omega+n}A = A/p^{\omega+n}A/p^{\omega}(A/p^{\omega+n}A)$$

being almost n-simply presented will imply that $A/p^{\omega+n}A$ is also almost n-simply presented. Moreover, if $p^{\omega}A$ is bounded, then so will be $p^{\omega+n}A$ according to Lemma 2.3, and hence owing to Theorem 2.10 the next statement will be fulfilled.

Conjecture 1. *Suppose A is a group such that both $p^{\omega}A$ and $A/p^{\omega}A$ are almost $p^{\omega+n}$-projective. Then A is almost n-simply presented.*

The last can slightly be improved to the following.

Conjecture 2. *Suppose G is a group of length λ strictly less than ω^2. Then G is almost n-simply presented if and only if, for every non-negative integer m, $p^{\omega\cdot m}G/p^{\omega\cdot(m+1)}G$ is almost $p^{\omega+n}$-projective.*

Idea for proof. "$\Rightarrow$". Referring to Theorem 2.7 (a), $G/p^{\omega\cdot(m+1)}G$ is also almost n-simply presented. Hence again applying Theorem 2.7 (a), we deduce that $p^{\omega\cdot m}(G/p^{\omega\cdot(m+1)}G) = p^{\omega\cdot m}G/p^{\omega\cdot(m+1)}G$ is almost n-simply presented too. Since $p^{\omega\cdot m}G/p^{\omega\cdot(m+1)}G$ is obviously p^{ω}-bounded, Proposition 2.5 applies to infer that this quotient is actually almost $p^{\omega+n}$-projective.

"$\Leftarrow$". For $m = 0$ we have that $G/p^{\omega}G$ is almost $p^{\omega+n}$-projective. When $m = 1$ we obtain that $p^{\omega}G/p^{\omega\cdot 2}G = p^{\omega}(G/p^{\omega\cdot 2}G)$ is almost $p^{\omega+n}$-projective. However, $G/p^{\omega}G \cong G/p^{\omega\cdot 2}G/p^{\omega}G/p^{\omega\cdot 2}G = G/p^{\omega\cdot 2}G/p^{\omega}(G/p^{\omega\cdot 2}G)$ is almost $p^{\omega+n}$-projective. Consequently, Conjecture 1 is applicable to get that $G/p^{\omega\cdot 2}G$ is almost n-simply presented. Hence, Theorem 2.7 (a) gives that $p^{\omega}(G/p^{\omega\cdot 2}G) = p^{\omega}G/p^{\omega\cdot 2}G \cong p^{\omega}G/p^{\omega\cdot 3}G/p^{\omega\cdot 2}G/p^{\omega\cdot 3}G = p^{\omega}G/p^{\omega\cdot 3}G/p^{\omega}(p^{\omega}G/p^{\omega\cdot 3}G)$. But by hypothesis $p^{\omega\cdot 2}G/p^{\omega\cdot 3}G = p^{\omega}(p^{\omega}G/p^{\omega\cdot 3}G)$ is almost $p^{\omega+n}$-projective. Now Conjecture 1 implies that $p^{\omega}G/p^{\omega\cdot 3}G$ is almost n-simply presented. Moreover, $G/p^{\omega}G \cong G/p^{\omega\cdot 3}G/p^{\omega}(G/p^{\omega\cdot 3}G)$ is almost $p^{\omega+n}$-projective. Therefore, again Conjecture 1 forces that $G/p^{\omega\cdot 3}G$ is almost n-simply presented, etc. after final steps to $G/p^{\lambda}G \cong G$ is almost n-simply presented, as wanted. $\square$

In closing, we shall state some left-open problems that still elude us.

Problem 1. Let G be a group such that $P \leq G$ is almost simply presented and G/P is almost Σ-cyclic. What we can say about the structure of G?

Problem 2. Let G be a group such that $P \leq G$ is almost Σ-cyclic and G/P is almost simply presented. What we can say about the structure of G?

Problem 3. Describe the structure of *(nicely) almost ω_1-n-simply presented* groups that are groups G for which there exist countable (nice) subgroups C such that G/C are almost n-simply presented.

For papers related to the last question the reader may see [3] and [4].

Problem 4. For groups A and B decide when $\mathrm{Tor}(A, B)$ is almost n-simply presented (in particular, almost $p^{\omega+n}$-projective) whenever $n \geq 0$.

Acknowledgement. The author owes his sincere thanks to the referee for the remarks made which helped to improve the presentation of the article.

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