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HIGHER CYCLOTOMIC UNITS FOR MOTIVIC COHOMOLOGY

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Abstract. In the present article, we describe specific elements in a motivic cohomology group $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_l),\, \mathbb{Z}(2)\big)$ of cyclotomic fields, which generate a subgroup of finite index for an odd prime l. As $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_l),\, \mathbb{Z}(1)\big)$ is identified with the group of units in the ring of integers in $\mathbb{Q}(\zeta_l)$ and cyclotomic units generate a subgroup of finite index, these elements play similar roles in the motivic cohomology group.

1. Introduction

When $K = \mathbb{Q}(\zeta_m)$ is a cyclotomic field, Dirichlet's Unit Theorem implies that the group $\mathcal{O}_K^\times$ of units in the ring of integers in K is a finitely generated abelian group of rank $\phi(m)/2 - 1$. This is proved by using Dirichlet's regulator map $\mathcal{O}_K^\times$ onto a full lattice in a hyperplane in the vector space $\mathbb{R}^{\phi(m)}$ ([7]).

In [5], a chain complex for motivic cohomology of a regular local ring R, by Goodwillie and Lichtenbaum, is defined to be the chain complex associated to the simplicial abelian group $d \mapsto K_0(R\Delta^d,\, \mathbb{G}_m^{\wedge t})$, together with a shift of degree by $-t$. Here, $K_0(R\Delta^d,\, \mathbb{G}_m^{\wedge t})$ is the Grothendieck group of the exact category of projective R-modules with t commuting

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automorphisms factored by the subgroup generated by classes of the objects one of whose t automorphisms is the identity map. Walker showed, in Theorem 6.5 of [10], that it agrees with motivic cohomology given by Voevodsky and thus various other definitions of motivic cohomology for smooth schemes over an algebraically closed field.

A higher regulator map is originally invented by A. Borel in [3]. Bloch ([1]) introduced a single-valued analogue D_2 of the dilogarithm function to describe the regulator map on $K_3(\mathbb{C})$ explicitly. The author ([9]) introduced another way to formulate a single-valued dilogarithm function and use it to explicitly define a motivic regulator map for $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{C},\, \mathbb{Z}(2)\big)$ defined via Goowillie-Lichtenbaum complex.

The purpose of this paper is to find a set of rational generators of the motivic cohomology $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_),\, \mathbb{Z}(2)\big)$ for an odd prime l. As cyclotomic units play such roles in $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_l),\, \mathbb{Z}(2)\big) \simeq \mathcal{O}^{\times}_{\mathbb{Q}(\zeta_l)}$, we term these generators as higher cyclotomic units.

2. The group $H^1_{\mathcal{M}}\big(\mathrm{Spec}K,\, \mathbb{Z}(1)\big)$

For a field K, let $K_0(K\Delta^d,\, \mathbb{G}_m^{\wedge t})$ be the abelian group generated by symbols $(A_1, \dots, A_t)$ where $A_1, \dots, A_t$ are commuting matrices in $GL_n(K[T_1, \dots, T_d]$ for some $n \geq 1$ subject to the relations:
$(A_1, \dots, A_t) = (C^{-1}A_1C, \dots, C^{-1}A_tC)$ for any $C \in GL_n(K[T_1, \dots, T_d]$,
$(A_1, \dots, A_t) + (B_1, \dots, B_t) = \left(\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \dots, \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} \right)$
and $(A_1, \dots, A_t) = 0$ if some A_i is the identity matrix. In particular, by the second relation, any element in $K_0(K\Delta^d,\, \mathbb{G}_m^{\wedge t})$ may be represented by a single symbol $(A_1, \dots, A_t)$.

Then, $H^1_{\mathcal{M}}\big(\mathrm{Spec}K,\, \mathbb{Z}(1)\big)$ is the cokernel of the homomorphism

$$\partial : K_0(K\Delta^1,\, \mathbb{G}_m^{\wedge 1}) \to K_0(K\Delta^0,\, \mathbb{G}_m^{\wedge 1}).$$

More explicitly the symbol represented by an invertible $n \times n$ matrix $A(T)$ is mapped to $(A(1)) - (A(0))$. But, the units in the ring $K[T]$ is the same as the units in the field K. Therefore, $\det A(0) = \det A(1)$ in $K^{\times}$. Hence determinant induces a map $H^1_{\mathcal{M}}\big(\mathrm{Spec}K,\, \mathbb{Z}(1)\big)$ onto $K^{\times}$.

On the other hand, the Whitehead group $K_1(K)$ is defined as the quotient group $GL(K)/E(K)$ where $E(K)$ is a subgroup of $GL(K)$ generated by elementary matrices $e_{ij}(r)$ whose diagonal entries are all 1 and whose (i, j) component is r and 0 everywhere else. Let $A(T)$ be the matrix of the same size as $e_{ij}(r)$ and whose diagonal entries are all 1 and whose (i,j) component is rT and 0 everywhere else. Then $A(0)$ is the identity matrix while $A(1)$ is the elementary matrix $e_{ij}(r)$. So, any symbol represented by an elementary matrix is in the image of $\partial : K_0(K\Delta^1, \mathbb{G}_m^{\wedge 1}) \to K_0(K\Delta^0, \mathbb{G}_m^{\wedge 1})$. Therefore, we have a map from $K_1(K)$ onto $H^1_{\mathcal{M}}\big(\mathrm{Spec}K, \mathbb{Z}(1)\big)$ which fits into a commutative diagram

$$\begin{tikzcd} K_1(K) \arrow[r, two heads] \arrow[d, "\simeq"] & H^1_{\mathcal{M}}\big(\mathrm{Spec}K, \mathbb{Z}(1)\big) \arrow[dl] \\ K^\times & \end{tikzcd}$$

Therefore, $H^1_{\mathcal{M}}\big(\mathrm{Spec}K, \mathbb{Z}(1)\big) \simeq K_1(K) \simeq K^\times$. Now define a homomorphism $RLog : H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{C}, \mathbb{Z}(1)\big) \to (\mathbb{R}, +)$ by sending the symbol A to $\log|\det A|$.

If K is a number field, any embedding σ of K into $\mathbb{C}$ induces a map $H^1_{\mathcal{M}}\big(\mathrm{Spec}K, \mathbb{Z}(1)\big) \to H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{C}, \mathbb{Z}(1)\big)$. Let $\sigma_1, \ldots, \sigma_r$ be real embeddings of K and $\sigma_{r+1}, \ldots, \sigma_{r+s}$ be complex embeddings of K so that $r_1 + 2r_2 = [K : \mathbb{Q}]$. Then

$$R = (RLog \circ \sigma_1, \ldots, RLog \circ \sigma_{r_1}, 2RLog \circ \sigma_{r+1}, \ldots, 2RLog \circ \sigma_{r+s})$$

is the usual Dirichlet reulgator map. $R : \mathcal{O}_K^\times \to \mathbb{R}^{r_1+r_2}$ is a map onto a full lattice in a hyperplane in $\mathbb{R}^{r_1+r_2}$ with a finite kernel. In fact, the kernel is the set of roots of unity in $\mathcal{O}_K^\times$.

3. Generators and Relations in $H^1_{\mathcal{M}}\big(\mathrm{Spec}K, \mathbb{Z}(2)\big)$

$K_0(\mathbb{C}\Delta^1, \mathbb{G}_m^{\wedge 2})$ can be recognized as the abelian group generated by pairs (A, B) $\big(= \big(A(T), B(T)\big)\big)$ and certain explicit relations, where A, B are commuting matrices in $GL_n(\mathbb{C}[T])$ for $n \geq 0$. On the other hand, $K_0(\mathbb{C}\Delta^2, \mathbb{G}_m^{\wedge 2})$ is recognized as the abelian group generated by the symbols $\big(A(X,Y), B(X,Y)\big)$ with commuting $A(X,Y)$, $B(X,Y) \in GL_n(\mathbb{C}[X,Y])$ and certain relations, and the boundary map ∂ on the Goodwillie-Lichtenbaum motivic complex sends the symbol $\big(A(X,Y), B(X,Y)\big)$ to $\big(A(1-T, T), B(1-T, T)\big) - \big(A(0,T), B(0,T)\big) + \big(A(T,0), B(T,0)\big)$ in $K_0(\mathbb{C}\Delta^1, \mathbb{G}_m^{\wedge 2})$. The same symbol (A, B) will denote the element in $K_0(\mathbb{C}\Delta^1, \mathbb{G}_m^{\wedge 2})/\partial K_0(\mathbb{C}\Delta^2, \mathbb{G}_m^{\wedge 2})$ represented by (A, B), by abuse of notation. The motivic cohomology group $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{C}, \mathbb{Z}(2)\big)$ is a subgroup of this quotient group.

In $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{C},\, \mathbb{Z}(2)\big)$, note that we have the following two simple relations for any two commuting matrices A, B in $GL_n(\mathbb{C}[T])$:

$$-\,(A(T),\, B(T)) = (A(1-T),\, B(1-T)) \tag{1}$$

$$\big(A_1(T),\, B_1(T)\big) + \big(A_2(T),\, B_2(T)\big) = \big(A_1(T) \oplus A_2(T),\, B_1(T) \oplus B_2(T)\big).$$

The first relation can be shown by applying the boundary map ∂ to the symbol $\big(A(X),\, B(X)\big)$ in $K_0(\mathbb{C}\Delta^2,\, \mathbb{G}_m^{\wedge 2})$ and by noting that $(A, B) = 0$ in $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{C},\, \mathbb{Z}(2)\big)$ when A and B are constant matrices. The fact that $(A, B) = 0$ for constant matrices A and B is obtained simply by applying the boundary map ∂ to the symbol $\big(A,\, B\big)$ in $K_0(\mathbb{C}\Delta^2,\, \mathbb{G}_m^{\wedge 2})$. Hence, an element of $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{C},\, \mathbb{Z}(2)\big)$ can be represented by a single expression (A, B), where A, B are commuting matrices in $GL_n(\mathbb{C}[T])$ for some positive integer n.

4. Motivic Regulator Map for $H^1_{\mathcal{M}}\big(\mathrm{Spec}K,\, \mathbb{Z}(2)\big)$

For $A \in GL_n(\mathbb{C}[T])$, let $P_A(\lambda)$ be the characteristic polynomial associated with A. It is a polynomial in λ of degree n with coefficients in $\mathbb{C}[T]$. Let x be a point in $\mathbb{C}$ and $\mathcal{O}_x$ be the local ring of germs of analytic functions at x. Identifying T with the identity function $\mathbb{C} \to \mathbb{C}$ embeds $\mathbb{C}[T]$ into $\mathcal{O}_x$. Then for commuting matrices $A, B \in GL_n(\mathbb{C}[T])$, let $x \in \mathbb{C}$ be such that $P_A(\lambda) = (\lambda - a_1(T))(\lambda - a_2(T)) \cdots (\lambda - a_n(T))$ and $P_B(\lambda) = (\lambda - b_1(T))(\lambda - b_2(T)) \cdots (\lambda - b_n(T))$ for some $a_1(T), \ldots, a_n(T)$ and $b_1(T), \ldots, b_n(T) \in \mathcal{O}_x$. Then there exists $S \in GL_n(\mathcal{O}_x)$ such that $S^{-1}AS$ and $S^{-1}BS$ are upper triangular matrices in $GL_n(\mathcal{O}_x)$,i.e., A, B are simultaneously triangularizable in $GL_n(\mathcal{O}_x)$ ([8] or [9]).

Let $(\lambda_1(T), \lambda_2(T), \ldots, \lambda_n(T))$ and $(\mu_1(T), \mu_2(T), \ldots, \mu_n(T))$ be the ordered n-tuples of diagonal entries of $S^{-1}AS$ and $S^{-1}BS$ Then, the set of pairs $\{(\lambda_1, \mu_1),\, (\lambda_2, \mu_2),\, \ldots,\, (\lambda_n, \mu_n)\}$ of elements of $\mathcal{O}_x$ is determined only by A, B and $x \in \mathbb{C}$ and is independent of the choice of S.

For $A \in GL_n(\mathbb{C}[T])$, let $P_A = P_{A,1}P_{A,2} \cdots P_{A,s}$ be the factorization of the characteristic polynomial P_A of A into irreducible polynomials in $\mathbb{C}[\lambda, T]$. The discriminant $\mathrm{disc}_{A,i}$ of each irreducible polynomial $P_{A,i}$ is a nonzero polynomial in $\mathbb{C}[T]$. Let $S_A = \{z \in \mathbb{C} |\, \mathrm{disc}_{A,i} = 0 \text{ for some } i\}$. Then S_A is a finite set.

Now divide the unit interval $[0, 1]$ into subintervals $[t_0, t_1], [t_0, t_1], \ldots,$ $[t_{r-1}, t_r]$ such that each open interval (t_{i-1}, t_i) is contained in a simply connected open subset U of $\mathbb{C} - (S_A \cup S_B)$. Using the analytic continuation, we have the set $\{(\lambda_{i,1}, \mu_{i,1}), \ldots, (\lambda_{i,n}, \mu_{i,n})\}$ of pairs of analytic functions on U which are locally pairs. At each $x \in U$, there is $S \in GL_n(\mathcal{O}(V))$ for some open neighborhood $V \subseteq U$ of x such that $S^{-1}AS$ and $S^{-1}BS$ are both upper triangular matrices in $GL_n(\mathcal{O}(V))$. Here, $\mathcal{O}(V)$ denotes the ring of analytic functions on V. For each subinterval (t_{i-1}, t_i), let $\{(\lambda_{i,1}, \mu_{i,1}), (\lambda_{i,2}, \mu_{i,2}), \ldots, (\lambda_{i,n}, \mu_{i,n})\}$ be the set of pairs of elements in $\mathcal{O}(U)$ which are locally ordered n-tuples of diagonal entries of $S^{-1}AS$ and $S^{-1}BS$. Then $\lambda_{i,l}$ and $\mu_{i,l}$ are smooth maps from (t_{i-1}, t_i) into $\mathbb{C} - \{0\}$ and may be thought of as paths into $\mathbb{C} - \{0\}$.

For paths γ and σ in $\mathbb{C} - \{0\}$. Let $D(\gamma_1, \gamma_2)$ be the real number defined by

$$D(\gamma, \sigma) = \operatorname{Im}\left(\int_0^1 \log|\gamma(t)| \frac{\sigma'(t)}{\sigma(t)} dt - \int_0^1 \log|\sigma(t)| \frac{\gamma'(t)}{\gamma(t)} dt\right)$$

For two commuting matrices $A, B \in GL_n(\mathbb{C}[T])$, we define

$$D(A, B) = \sum_{i=1}^{r} \sum_{l=1}^{n} D(\lambda_{i,l}, \mu_{i,l})$$

Then the integral which defines each term $D(\lambda_{i,l}, \mu_{i,l})$ is convergent and thus D gives a map from the set of pairs of commuting matrices in $GL_n(\mathbb{C}[T])$ into $\mathbb{R}$.

For notational convenience, we write

$$D(A, B) = \sum_{l=1}^{n} D(\lambda_l, \mu_l)$$

where, for each t,

$$\{\big(\lambda_1(t), \mu_1(t)\big), \ \big(\lambda_2(t), \mu_2(t)\big), \ \ldots, \ \big(\lambda_n(t), \mu_n(t)\big)\}$$

are pairs of eigenvalues of $A(t)$ and $B(t)$, which are piecewise smooth paths.

For any continuous piecewise smooth path σ from $[0,1]$ into $\mathbb{C}$, we may divide the interval $[0,1]$ into subintervals $[t_0, t_1], [t_0, t_1], \ldots, [t_{r-1}, t_r]$ such that, for each $i = 1, \ldots, r$, $\sigma\big((t_{i-1}, t_i)\big)$ is contained in an open subset U of $\mathbb{C}$ such that there is $S \in GL_n\big(\mathcal{O}(U)\big)$ such that $S^{-1}AS$ and $S^{-1}BS$ are upper triangular matrices in $GL_n\big(\mathcal{O}(U)\big)$. Then we may define $D\big(A(\sigma), B(\sigma)\big)$ as the sum

$$D\big(A(\sigma), B(\sigma)\big) = \sum_{i=1}^{r}\sum_{l=1}^{n} D\big(\lambda_{i,l}\circ\sigma, \mu_{i,l}\circ\sigma\big).$$

A proof of the following theorem was given in [8].

Theorem 4.1. *With the same notation as above, for two commuting matrices* $A, B \in GL_n(\mathbb{C}[T])$, *we define* $D(A,B) = \sum_{l=1}^{n} D(\lambda_l, \mu_l)$. *Then* D *gives a homomorphism from* $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{C},\, \mathbb{Z}(2)\big)$ *into* $\mathbb{R}$. *In fact, it is a homomorphism on* $K_0(\mathbb{C}\Delta^1, \mathbb{G}_m^{\wedge 2})/\partial K_0(\mathbb{C}\Delta^2, \mathbb{G}_m^{\wedge 2})$.

We also have the following fundamental properties of our D-map ([8] or [9]):

(i) (Skew-Symmetry) $D(A,B) = -D(B,A)$ for commuting matrices $A,\ B \in GL_n(\mathbb{C}[T])$.

(ii) (Vanishing of Constant Matrix) $D(A,B) = 0$ if $A, B \in GL_n(\mathbb{C}[T])$ are commuting and either A or B is in $GL_n(\mathbb{C})$.

(iii) (Bilinearity) $D(A_1A_2, B) = D(A_1, B) + D(A_2, B)$ whenever A_1, $A_2, B \in GL_n(\mathbb{C}[T])$ are commute with each other.

(iv) (Vanishing of Matrices with Real Coefficients) $D(A,B) = 0$ if $A, B \in GL_n(\mathbb{R}[T])$

5. Technique of constructing elements in $H^1_{\mathcal{M}}\big(\mathrm{Spec}K,\, \mathbb{Z}(2)\big)$

In [9], the following technical lemma was introduced to construct explicit elements in the motivic cohomology group $H^1_{\mathcal{M}}\big(\mathrm{Spec}K,\, \mathbb{Z}(2)\big)$. Let K be a subfield of $\mathbb{C}$.

Lemma 5.1. *Let* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *be elements in* $\mathbb{C}$ *not equal to either 0 or 1. Suppose also that* $a_1a_2\cdots a_n = b_1b_2\cdots b_n$ *and* $(1-a_1)(1-a_2)\cdots(1-a_n) = (1-b_1)(1-b_2)\cdots(1-b_n)$. *If all the elementary symmetric functions evaluated at* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *are in* K, *then there is a matrix* $A(T)$ *in* $GL_n(K[T])$ *such that* $I - A(T)$ *is also invertible and the eigenvalues of* $A(0)$ *and* $A(1)$ *are* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$, *respectively.*

We use this construction to define a map $\theta : \mathcal{B}(K) \to H^1_{\mathcal{M}}\big(\mathrm{Spec}K,\, \mathbb{Z}(2)\big)$, which will be used to compare the Bloch's dilogarithmic map to our motivic regulator map.

The group $\mathcal{B}(K)$ of a field K is defined as the kernel of the homomorphism

$$\mathcal{A}(K) \xrightarrow{\lambda} K^\times \wedge_{\mathbb{Z}} K^\times$$

where $\mathcal{A}(K)$ is a free abelian group generated by the symbols $[a]$ with $a \in K - \{0, 1\}$, $K^\times \wedge_{\mathbb{Z}} K^\times$ is $K^\times \otimes_{\mathbb{Z}} K^\times$ divided by the subgroup generated by $a \otimes (-a)$ with $a \in K^\times$ and where $\lambda([a]) = a \wedge (1-a)$ ([4] or [2])).

Define $\theta_1 : \mathcal{A}(K) \to K_0(K\Delta^1, \mathbb{G}_m^{\wedge 2})$ by $\theta_1([a]) = 2\big(A(a,T), I - A(a,T)\big)$ for every $a \in K - \{0,1\}$, where

$$A(a,T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4a & (4-a)T + a & (a-4)T + 4 \end{pmatrix}.$$

Then θ_1 induces a map $\mathcal{A}(K) \to K_0(K\Delta^1, \mathbb{G}_m^{\wedge 2})/\partial K_0(K\Delta^2, \mathbb{G}_m^{\wedge 2})$, which we denote again by θ_1 by abuse of notation.

In [9], it was shown that there exists a map $\theta : \mathcal{B}(K) \to H^1_{\mathcal{M}}\big(\mathrm{Spec}K, \mathbb{Z}(2)\big)$ as a lifting of θ_1 and we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{B}(K) \arrow[r, hook] \arrow[d, "\theta"] & \mathcal{A}(K) \arrow[d, "\theta_1"] \\ H^1_{\mathcal{M}}\big(\mathrm{Spec}K, \mathbb{Z}(2)\big) \arrow[r, hook] & K_0(K\Delta^1, \mathbb{G}_m^{\wedge 2})/\partial K_0(K\Delta^2, \mathbb{G}_m^{\wedge 2}) \end{tikzcd}.$$

6. Compatibility With Bloch-Wigner Function

Bloch-Wigner function $D_2 : \mathbb{C} \to \mathbb{R}$ may be defined as below ([2] or [6]). When $|z - \frac{1}{2}| < \frac{1}{2}$, it is given by

$$D_2(z) = -\mathrm{Im} \int_0^z \log(1-t)\,\frac{dt}{t} + \arg(1-z)\,\log|z|$$

where the principal branches of log and arg are used. Then it can be shown that D_2 as a real analytic function is invariant under the continuation along small loops around 0 and 1. Thus D_2 is extended to a single-valued, real analytic function on $\mathbb{C} - \{0,1\}$. The function D_2 extends to a continuous function on all of $\mathbb{C}$ by setting $D_2(0) = D_2(1) = 0$. Then we have the following basic properties of the Bloch-Wigner function:

(i) D_2 vanishes on the real line.

(ii) For any $z \in \mathbb{C}$, we have

$$D_2(z) + D_2(1-z) \ = D_2(z) + D_2(1/z) \ = D_2(z) + D_2(\bar{z}) \ = 0.$$

(iii) (Duplication Formula (c.f. [4])) For any $z \in \mathbb{C}$, we have

$$D_2(z) + D_2(-z) = \frac{1}{2} D_2(z^2).$$

Then the most important lemma which shows the connection between our D-map and the Bloch-Wigner function is as follows ([9])

Lemma 6.1. *Let γ_1 be a path from $[0,1]$ into $\mathbb{C} - \{0,1\}$ and $\gamma_2(t) = 1 - \gamma_1(t)$ for every $t \in [0,1]$. Then*

$$D(\gamma_1, \gamma_2) = D_2(\gamma_1(1)) - D_2(\gamma_1(0)),$$

where D is as in Section 4.

Corollary 6.2. *Let $A(T)$ be an invertible matrix in $GL_n(K[T])$ such that $I - A(T)$ is also invertible. Let $A(1)$ and $A(0)$ have eigenvalues $b_1, b_2, \ldots, b_n$ and $a_1, a_2, \ldots, a_n$ in $\mathbb{C}$, respectively. Then*

$$D\big(A(T),\, I - A(T)\big) = \sum_{i=1}^{n} D_2(b_i) - \sum_{i=1}^{n} D_2(a_i).$$

Proposition 6.3. *The Bloch-Wigner function $D_2 : \mathcal{B}(K) \to \mathbb{R}$ is the composite $D \circ \theta$ where The map $\theta : \mathcal{B}(K) \to H^1_{\mathcal{M}}\big(\mathrm{Spec}K,\, \mathbb{Z}(2)\big)$ is given in Section 5.*

Proof. In the construction of θ, the matrix $A(a,T)$ was such that

$$\begin{aligned} D\big(A(T),\, I - A(T)\big) &= D_2(-2) + D_2(2) + D_2(a) \\ &\quad - D_2(4) - D_2(\sqrt{a}) - D_2(-\sqrt{a}) \\ &= D_2(a) - D_2(\sqrt{a}) - D_2(-\sqrt{a}) = \frac{1}{2} D_2(a). \end{aligned}$$

by the Duplication Formula of D_2. Hence, $\theta_1([a]) = 2(A,\, I - A)$ will yield $D_2(a)$ under D. $\square$

7. Higher Cyclotomic Units

Let ζ_m be a primitive m-th root of unity where m is an odd positive integer. and let $K = \mathbb{Q}(\zeta_m)$ be a cyclotomic field.

Let $\mathbf{Z}_D = \mathrm{Ker}\big(D : K_0(K\Delta^1, \mathbb{G}_m^{\wedge 2}) \to \mathbb{R}\big)$. The the image $\partial\mathbf{Z}_D$ of $\mathbf{Z}_D$ under the boundary homomorphism $\partial : K_0(K\Delta^1, \mathbb{G}_m^{\wedge 2}) \to K_0(K\Delta^0, \mathbb{G}_m^{\wedge 2})$. Then we have the following lemma ([9]).

Lemma 7.1. *$\partial\mathbf{Z}_D$ contains elements of the following forms and for any element of these forms, we may find an explicit $z \in \mathbf{Z}_D$ whose image under ∂ is equal to the element.*

(i) $(AB, C) - (A, C) - (B, C)$ and $(C, AB) - (C, A) - (C, B)$, for commuting matrices A, B, $C \in GL_n(K)$;

(ii) $(x,\, 1-x) - (y,\, 1-y)$, for x, $y \in K \cap \mathbb{R}^+ - \{1\}$.

Proof. (i) Let $A(T)$ be the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & I \\ -AB & T(I+AB) + (1-T)(A+B) \end{pmatrix}.$$

Then, $A(T)$ is in $GL_{2n}(K[T])$, $\big(A(T),\, C \oplus C\big)$ is in $\mathbf{Z}_D$ since C is a constant matrix. But, the boundary of $\big(A(T),\, C \oplus C\big)$ is $(I \oplus AB, C \oplus C) - (A \oplus B, C \oplus C) = (AB, C) - (A, C) - (B, C)$. The proof for $(C, AB) - (C, A) - (C, B)$ is similar.

For (ii), note that Bloch-Wigner function vanishes on the real line and that a square root of a positive real number is a real number. Apply Lemma 5.1 to $a_1 = x,\ a_2 = \sqrt{y},\ a_3 = -\sqrt{y},\ b_1 = -\sqrt{x},\ b_2 = \sqrt{x},\ b_3 = y$. to get $A(T) \in GL_3((K \cap \mathbb{R})[T])$. Then $z = 2\big(A(T),\, I - A(T)\big)$ is in

$\mathbf{Z}_D$. But, by the theory of rational canonical form, ∂z is equal to

$$
\begin{aligned}
2&\left((y,1-y)+\left(\begin{pmatrix}0&1\\x&0\end{pmatrix},\begin{pmatrix}1&-1\\-x&1\end{pmatrix}\right)\right)\\
&\qquad-\left((x,1-x)+\left(\begin{pmatrix}0&1\\y&0\end{pmatrix},\begin{pmatrix}1&-1\\-y&1\end{pmatrix}\right)\right)\\
&=\left(\begin{pmatrix}y&0\\0&y\end{pmatrix},\begin{pmatrix}1-y&0\\0&1-y\end{pmatrix}\right)-\left(\begin{pmatrix}y&0\\0&y\end{pmatrix},\begin{pmatrix}1&-1\\-y&1\end{pmatrix}\right)\\
&\qquad-\left(\begin{pmatrix}x&0\\0&x\end{pmatrix},\begin{pmatrix}1-x&0\\0&1-x\end{pmatrix}\right)+\left(\begin{pmatrix}x&0\\0&x\end{pmatrix},\begin{pmatrix}1&-1\\-x&1\end{pmatrix}\right)\\
&=\left(\begin{pmatrix}y&0\\0&y\end{pmatrix},\begin{pmatrix}1&1\\y&1\end{pmatrix}\right)-\left(\begin{pmatrix}x&0\\0&x\end{pmatrix},\begin{pmatrix}1&1\\x&1\end{pmatrix}\right)\\
&=\left(\begin{pmatrix}y&0\\0&y\end{pmatrix},\begin{pmatrix}\frac{-y}{1-y}&\frac{1}{1-y}\\0&1\end{pmatrix}\begin{pmatrix}1&1\\y&1\end{pmatrix}\begin{pmatrix}\frac{-y}{1-y}&\frac{1}{1-y}\\0&1\end{pmatrix}^{-1}\right)\\
&\qquad-\left(\begin{pmatrix}x&0\\0&x\end{pmatrix},\begin{pmatrix}\frac{-x}{1-x}&\frac{1}{1-x}\\0&1\end{pmatrix}\begin{pmatrix}1&1\\x&1\end{pmatrix}\begin{pmatrix}\frac{-x}{1-x}&\frac{1}{1-x}\\0&1\end{pmatrix}^{-1}\right)\\
&=\left(\begin{pmatrix}y&0\\0&y\end{pmatrix},\begin{pmatrix}0&1\\y-1&2\end{pmatrix}\right)-\left(\begin{pmatrix}x&0\\0&x\end{pmatrix},\begin{pmatrix}0&1\\x-1&2\end{pmatrix}\right).
\end{aligned}
$$

By taking the boundary of the element

$$
\begin{aligned}
&\left(\begin{pmatrix}y&0\\0&y\end{pmatrix},\begin{pmatrix}0&1\\y-1&(2-y)T+2(1-T)\end{pmatrix}\right)\\
&\qquad-\left(\begin{pmatrix}x&0\\0&x\end{pmatrix},\begin{pmatrix}0&1\\x-1&(2-x)T+2(1-T)\end{pmatrix}\right),
\end{aligned}
$$

which is in $\mathbf{Z}_D$ by the fundamental property (iv) of the D-map in Section 4, we see that

$$\begin{aligned}
\partial z &= \left(\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ y-1 & 2-y \end{pmatrix}\right) - \left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ x-1 & 2-x \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ y-1 & 2-y \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \\
&\qquad - \left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ x-1 & 2-x \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 1-y & 0 \\ 0 & 1 \end{pmatrix}\right) - \left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 1-x & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= (y,\, 1-y) - (x,\, 1-x)
\end{aligned}$$

in modulo $\partial\mathbf{Z}_D$. So, (ii) is the boundary of $2\big(A(T),\, I - A(T)\big)$. □

PROPOSITION 7.2. *(m-th Roots of Unity) If ζ_m is a primitive m-the root of unity for an odd integer $m > 0$, there exists an explicit element $h(\zeta_m)$ in $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_m),\, \mathbb{Z}(2)\big)$ whose value under the dilogarithm D is equal to $mD_2(\zeta_m)$.*

Proof. Let ζ be a primitive $2m$-th root of unity such that $\zeta^2 = \zeta_m$. Then

$$a_1 = 4,\ a_2 = \zeta,\ a_3 = -\zeta,\ b_1 = -2,\ b_2 = 2,\ b_3 = \zeta^2$$

satisfy the conditions of Lemma 5.1 with $K = \mathbb{Q}(\zeta_m)$. Actually,

$$a_1 = x^2,\ a_2 = y,\ a_3 = -y,\ b_1 = -x,\ b_2 = x,\ b_3 = y^2$$

for any $x,\, y \in K$ would do. Let $A(T) = A(\zeta^2, T)$ where $A(a, T)$ is the matrix used to define θ_1 in Section 5. Then by the calculation in the proof of Proposition 6.3, we have $2D\big(A(T),\, I - A(T)\big) = D_2(\zeta_m)$ and thus $2mD\big(A(T),\, I - A(T)\big) = mD_2(\zeta_m)$

Now the only possible problem is that its image $2m\big(A(1),\, I - A(1)\big) - 2\big(A(0),\, I - A(0)\big)$ under ∂ might not be 0 in $K_0\big(\mathbb{Q}(\zeta_m)\Delta^0,\, \mathbb{G}_m^{\wedge 2}\big)$, so $2m\big(A(T),\, I - A(T)\big)$ might not represent an element in $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_m),\, \mathbb{Z}(2)\big)$. So we need to find an element z in $K_0(\mathbb{Q}(\zeta_m)\Delta^1,\, \mathbb{G}_m^{\wedge 2})$ whose image under the boundary map ∂ is equal to $2m\big(A(0),\, I - A(0)\big) - 2m\big(A(1),\, I - A(1)\big)$ and $D(z) = 0$. Then, $2m(A(T), I - A(T)) - z$ would represent an element of $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_m),\, \mathbb{Z}(2)\big)$ and its value under D would be $mD_2(\zeta_m)$. But,

$$\begin{aligned} 2\big(A(1),\, I - A(1)\big) &- 2\big(A(0),\, I - A(0)\big) \\ &= 2(-2,3) + 2(2,-1) + 2(\zeta^2,\, 1-\zeta^2) - 2(4,-3) \\ &\quad -2\left(\begin{pmatrix} 0 & 1 \\ \zeta^2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -\zeta^2 & 1 \end{pmatrix}\right). \end{aligned}$$

Therefore, it is enough to prove that $2mw$ is in $\partial \mathbf{Z}_D$, where

$$w = (-2,3)+(2,-1)+(\zeta^2,\, 1-\zeta^2)-(4,-3)-\left(\begin{pmatrix} 0 & 1 \\ \zeta^2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -\zeta^2 & 1 \end{pmatrix}\right).$$

But,

$$\begin{aligned} 2mw = m\big((4,3) + (2,1) - (4,9)\big) &+ (1,\, 1-\zeta^2) \\ -\left(\begin{pmatrix} 0 & 1 \\ \zeta^2 & 0 \end{pmatrix}^{2m}, \begin{pmatrix} 1 & -1 \\ -\zeta^2 & 1 \end{pmatrix}\right)& \end{aligned}$$

modulo $\partial \mathbf{Z}_D$ by Lemma 7.1 (i). Here $\begin{pmatrix} 0 & 1 \\ \zeta^2 & 0 \end{pmatrix}^{2m} = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^2 \end{pmatrix}^{m} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So,

$$2mw = m\big((4,3) - (4,9)\big) = -m(4,3)$$

modulo $\partial \mathbf{Z}_D$, again by Lemma 7.1 (i). But if we apply Lemma 7.1 (ii) with $x = 2$ and $y = 3$ and multiply by 2, we get $(4,3) = 0$ modulo $\partial \mathbf{Z}_D$. Therefore, $2mw = 0$ modulo $\partial \mathbf{Z}_D$. Hence, by the proof of Lemma 7.1, there exists an explicit $z\mathbf{Z}_D$ such that $h(\zeta_m) = 2m(A(T), I - A(T)) - z$ has the required property. □

Note that we were able to construct an element $h(\zeta_m)$ in $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_m),\, \mathbb{Z}(2)\big)$ whose image under D is $mD_2(\zeta_m)$, where ζ_m is a primitive m-th root of unity.

Now, let $m = l$ be an odd prime and let $\{\sigma_1, \bar{\sigma}_1, \ldots, \sigma_{r_2}, \bar{\sigma}_{r_2}\}$, where $r_2 = \phi(l)/2$, be the set of the complex embeddings of $\mathbb{Q}(\zeta_l)$. Then, we have a homomorphism $\overline{D}$ from $H^1_{\mathcal{M}}\big(\mathrm{Spec}\mathbb{Q}(\zeta_l),\, \mathbb{Z}(2)\big)$ into $\mathbb{R}^{r_2}$ which is defined by

$$\overline{D}(a) = \big(D\sigma_1(a), \ldots, D\sigma_{r_2}(a)\big).$$

If ζ_l is an l-th primitive root of unity, then the element $l[\zeta_l] \in \mathcal{A}(K)$ is mapped to $l(\zeta_l \wedge (1-\zeta_l)) = \zeta_l^l \wedge (1-\zeta_l) = 0$ under the homomorphism

$\lambda : \mathcal{A}(K)K^\times \wedge_\mathbb{Z} K^\times$ as in Section 5. Therefore, $l[\zeta_l]$ is an element of the Bloch's group $\mathcal{B}(K)$.

Theorem 7.2.4 in [2] states that the images of $l[\sigma_1(\zeta)], l[\sigma_2(\zeta)], \ldots, l[\sigma_{r_2}(\zeta)] \in \mathcal{B}(K)$ under the given map $\mathcal{B}(K) \to K_3(\mathbb{Q}(\zeta))_\mathbb{Q}$ form a basis of the target group and after the Borel's regulator map, their images generate a lattice of maximal rank in $\mathbb{R}_2^r$. Therefore, we obtain the following theorem.

THEOREM 7.3. *(Rational Generators of $H^1_\mathcal{M}\big(\mathrm{Spec}\mathbb{Q}(\zeta_m), \mathbb{Z}(2)\big)$) For an odd prime l, $h(\sigma_1\zeta_m), \ldots, h(\sigma_{r_2}\zeta_l)$ rationally generates $H^1_\mathcal{M}\big(\mathrm{Spec}\mathbb{Q}(\zeta_m), \mathbb{Z}(2)\big)$,i.e., they generate a subgroup of finite index in $H^1_\mathcal{M}\big(\mathrm{Spec}\mathbb{Q}(\zeta_m), \mathbb{Z}(2)\big)$.*

Note that by the construction of our map $\theta : \mathcal{B}(K) \to H^1_\mathcal{M}\big(\mathrm{Spec}\mathbb{Q}(\zeta_l), \mathbb{Z}(2)\big)$ in Section 5, $\theta(l[\zeta_l])$ is equal to $h(\zeta_l)$ modulo an element z whose value under D is 0, i.e., a torsion element.

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