t-SPLITTING SETS $S$ OF AN INTEGRAL DOMAIN $D$ SUCH THAT $D_S$ IS A FACTORIAL DOMAIN

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Abstract. Let $D$ be an integral domain, $S$ be a saturated multiplicative subset of $D$ such that $D_S$ is a factorial domain, $\{X_\alpha\}$ be a nonempty set of indeterminates, and $D[[X_\alpha]]$ be the polynomial ring over $D$. We show that $S$ is a splitting (resp., almost splitting, $t$-splitting) set in $D$ if and only if every nonzero prime $t$-ideal of $D$ disjoint from $S$ is principal (resp., contains a primary element, is $t$-invertible). We use this result to show that $D \{0\}$ is a splitting (resp., almost splitting, $t$-splitting) set in $D[[X_\alpha]]$ if and only if $D$ is a GCD-domain (resp., UMT-domain with $Cl(D[[X_\alpha]])$ torsion, UMT-domain).

1. Introduction

Let $D$ be an integral domain with quotient field $K$, and let $F(D)$ be the set of nonzero fractional ideals of $D$. For each $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup\{I_v \mid J \in F(D), J \subseteq I, \text{ and } J \text{ is finitely generated}\}$. An ideal $I \in F(D)$ is called a $t$-ideal if $I_t = I$, and a $t$-ideal is a maximal $t$-ideal if it is maximal among proper integral $t$-ideals. It is well known that each nonzero principal
ideal is a $t$-ideal; each proper integral $t$-ideal is contained in a maximal $t$-ideal; a prime ideal minimal over a $t$-ideal is a $t$-ideal; and each maximal $t$-ideal is a prime ideal. We say that an $I \in F(D)$ is $t$-invertible if $(II^{-1})_t = D$; equivalently, if $II^{-1} \not\subseteq P$ for every maximal $t$-ideal $P$ of $D$. Let $T(D)$ be the group of $t$-invertible fractional $t$-ideals of $D$ under the $t$-multiplication $A \ast B = (AB)_t$, and let $Prin(D)$ be its subgroup of principal fractional ideals. The $(t)$-class group of $D$ is an abelian group $\text{Cl}(D) = T(D)/Prin(D)$. The readers can refer to [12] for any undefined notation or terminology.

Let $S$ be a saturated multiplicative subset of an integral domain $D$. As in [3], we say that $S$ is a $t$-splitting set if for each $0 \neq d \in D$, we have $dD = (AB)_t$ for some integral ideals $A, B$ of $D$, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. We say that $S$ is an almost splitting set of $D$ if for each $0 \neq d \in D$, there is an integer $n = n(d) \geq 1$ such that $d^n = sa$ for some $s \in S$ and $a \in N(S)$, where $N(S) = \{0 \neq x \in D | (x, s')_t = D$ for all $s' \in S\}$. A splitting set is an almost splitting set in which $n = n(d) = 1$ for every $0 \neq d \in D$. Let $\overline{S}$ be the saturation of a multiplicative set $S$ of $D$. Note that a splitting set is saturated, while both $t$-splitting sets and almost splitting sets need not be saturated. Also, note that $S$ is $t$-splitting (resp., almost splitting) if and only if $\overline{S}$ is; so we always assume that $S$ is saturated. It is known that an almost splitting set is $t$-splitting [7, Proposition 2.3]; hence

splitting set $\Rightarrow$ almost splitting $\Rightarrow$ $t$-splitting set.

Moreover, if $\text{Cl}(D)$ is torsion, then a $t$-splitting set is almost splitting [7, Corollary 2.4] and if $\text{Cl}(D) = 0$, then splitting set $\iff$ almost splitting $\iff$ $t$-splitting set.

Let $X$ be an indeterminate over $D$ and $D[X]$ be the polynomial ring over $D$. An upper to zero in $D[X]$ is a nonzero prime ideal $Q$ of $D[X]$ with $Q \cap D = (0)$, and $D$ is called a UMT-domain if each upper to zero in $D[X]$ is a maximal $t$-ideal of $D[X]$. We say that $D$ is a Pr"ufer $v$-multiplication domain (PrvMD) if each nonzero finitely generated ideal of $D$ is $t$-invertible. As in [15], we say that $D$ is an almost GCD-domain (AGCD-domain) if for each $0 \neq a, b \in D$, there is an integer $n = n(a, b) \geq 1$ such that $a^nD \cap b^nD$ is principal. Clearly, GCD-domains are AGCD-domains. It is known that AGCD-domains are UMT-domains with torsion class group [5, Lemma 3.1]; $D$ is a PrvMD if and only if $D$ is an integrally closed UMT-domain [13, Proposition 3.2]; and $D$ is a
GCD-domain if and only if \( D \) is a PrMD and \( Cl(D) = 0 \) [6, Corollary 1.5].

In [9, Theorem 2.8], the authors proved that if \( D_S \) is a principal ideal domain (PID), then \( S \) is a \( t \)-splitting set of \( D \) if and only if every nonzero prime ideal of \( D \) disjoint from \( S \) is \( t \)-invertible. They used this result to show that \( D \setminus \{0\} \) is a \( t \)-splitting set of \( D[X] \) if and only if \( D \) is a UMT-domain [9, Corollary 2.9]. Also, in [8, Theorem 2], the author showed that if \( D_S \) is a PID, then \( S \) is an almost splitting set of \( D \) if and only if every nonzero prime ideal of \( D \) disjoint from \( S \) contains a primary element. (A nonzero element \( a \in D \) is said to be primary if \( aD \) is a primary ideal.)

The purpose of this paper is to show that the results of [9, Theorem 2.8] and [8, Theorem 2] are also true when \( D_S \) is a factorial domain (note that a PID is a factorial domain). Precisely, we show that if \( D_S \) is a factorial domain, then \( S \) is a splitting (resp., almost splitting, \( t \)-splitting) set in \( D \) if and only if every nonzero prime \( t \)-ideal of \( D \) disjoint from \( S \) is principal (resp., contains a primary element, is \( t \)-invertible). Let \( \{X_\alpha\} \) be a nonempty set of indeterminates over \( D \), and note that \( D[\{X_\alpha\}]_{D\setminus\{0\}} \) is a factorial domain. Hence, we then use the results we obtained in this paper to show that \( D \setminus \{0\} \) is a splitting (resp., almost splitting, \( t \)-splitting) set in \( D[\{X_\alpha\}] \) if and only if \( D \) is a GCD-domain (resp., UMT-domain and \( Cl(D[\{X_\alpha\}]) \) is torsion, UMT-domain).

2. Main Results

Let \( D \) be an integral domain, \( D^* = D \setminus \{0\} \), \( \{X_\alpha\} \) be a nonempty set of indeterminates over \( D \), and \( D[\{X_\alpha\}] \) be the polynomial ring over \( D \).

We begin this section with nice characterizations of splitting sets, almost splitting sets, and \( t \)-splitting sets which appear in [2, Theorem 2.2], [4, Proposition 2.7], and [3, Corollary 2.3], respectively.

**Lemma 1.** Let \( S \) be a saturated multiplicative subset of \( D \).

1. \( S \) is a splitting (resp., \( t \)-splitting) set of \( D \) if and only if \( dD_S \cap D \) is principal (resp., \( t \)-invertible) for every \( 0 \neq d \in D \).
2. \( S \) is an almost splitting set of \( D \) if and only if for every \( 0 \neq d \in D \), there is a positive integer \( n = n(d) \) such that \( d^nD_S \cap D \) is principal.

Note that if \( D_S \) is a PID, then every nonzero prime ideal \( P \) of \( D \) disjoint from \( S \) has height-one, and thus \( P \) is a \( t \)-ideal. Hence, our first
result is a generalization of [9, Theorem 2.8] that if $D_S$ is a PID, then $S$ is a $t$-splitting set in $D$ if and only if every nonzero prime ideal of $D$ disjoint from $S$ is $t$-invertible. The proof is similar to those of [9, Theorem 2.8] and [8, Theorem 2].

**Theorem 2.** Let $D$ be an integral domain and $S$ be a saturated multiplicative subset of $D$ such that $D_S$ is a factorial domain. Then $S$ is a $t$-splitting set in $D$ if and only if every prime $t$-ideal of $D$ disjoint from $S$ is $t$-invertible.

**Proof.** ($\Rightarrow$) Assume that $S$ is a $t$-splitting set of $D$, and let $P$ be a prime $t$-ideal of $D$ with $P \cap S = \emptyset$. Then $(PD_S)_t = PD_S$ [3, Theorem 4.9], and hence $PD_S = pD_S$ for some $p \in P$ since $D_S$ is a factorial domain. Thus, by Lemma 1, $P = PD_S \cap D = pD_S \cap D$ is $t$-invertible.

($\Leftarrow$) Let $0 \neq d \in D$. Then $dD_S = p_1^{e_1} \cdots p_k^{e_k}D_S$ for some $p_i \in D$ and positive integers $e_i$ such that every $p_i$ is a prime element in $D_S$ and $p_iD_S \neq p_jD_S$ if $i \neq j$. Let $P_i$ be the prime ideal of $D$ such that $P_iD_S = p_iD_S$. Clearly, $P_i$ is minimal over $dD_S$, and hence $P_i$ is a $t$-ideal. Moreover, $P_i \cap S = \emptyset$; so $P_i$ is $t$-invertible by assumption (and hence a maximal $t$-ideal [13, Proposition 1.3]). Note that $(P_i^{e_i})_t$ is $P_t$-primary [1, Lemma 1] because $P_t$ is a maximal $t$-ideal. Also, $(P_i^{e_i})_tD_S = p_i^{e_i}D_S$, and thus $P_i^{e_i}D_S \cap D = (P_i^{e_i})_t$ and $(P_i^{e_i})_t$ is $t$-invertible. Hence

$$dD_S \cap D = p_1^{e_1} \cdots p_k^{e_k}D_S \cap D = (p_1^{e_1}D_S \cap \cdots \cap p_k^{e_k}D_S) \cap D = (P_1^{e_1}D_S \cap \cdots \cap P_k^{e_k}D_S) \cap D = (P_1^{e_1}D_S \cap D) \cap \cdots \cap (P_k^{e_k}D_S \cap D) = (P_1^{e_1})_t \cap \cdots \cap (P_k^{e_k})_t = ((P_1^{e_1})_t \cdots (P_k^{e_k})_t)_t.$$

Thus, $S$ is a $t$-splitting set by Lemma 1. \hfill \square

The next result is a generalization of [9, Corollary 2.9] that $D^*$ is a $t$-splitting set in $D[X]$, where $X$ is an indeterminate over $D$, if and only if $D$ is a UMT-domain.

**Corollary 3.** $D^*$ is a $t$-splitting set in $D[\{X_\alpha\}]$ if and only if $D$ is a UMT-domain.

**Proof.** ($\Rightarrow$) Let $X \in \{X_\alpha\}$, and let $P$ be a nonzero prime ideal of $D[X]$ with $P \cap D = (0)$. Then $P$ is a prime $t$-ideal of $D[X]$, and hence
\(Q := P[Y],\) where \(Y = \{X_n\} \setminus \{X\}\), is a prime \(t\)-ideal of \(D[\{X_n\}]\) [11, Lemma 2.1(1)] such that \(Q \cap D^* = \emptyset\). Hence, \(Q\) is \(t\)-invertible by Theorem 2 because \(D[\{X_n\}] = (QQ^{-1})_t = ((P[Y])(P[Y])^{-1})_t = (((P[Y])(P^{-1}[Y])_t = ((PP^{-1})[Y])_t = (PP^{-1})[Y] [11, Lemma 2.1(1)]. Hence, \(P\) is \(t\)-invertible, and thus \(P\) is a maximal \(t\)-ideal of \(D[X]\).

(\(\Leftarrow\)) Let \(Q\) be a prime \(t\)-ideal of \(D[\{X_n\}]\) such that \(Q \cap D^* = \emptyset\). Since \(Q \neq (0)\), there are \(X_1, \ldots, X_n \in \{X_n\}\) such that \(Q \cap D[X_1, \ldots, X_{n-1}] = (0)\), but \(Q \cap D[X_1, \ldots, X_n] \neq (0)\). Let \(R = D[X_1, \ldots, X_{n-1}]\) and \(P = Q \cap R[X_n]\). Then \(R\) is a UMT-domain [11, Theorem 2.4] and \(P\) is an upper to zero in \(R[X_n]\). Hence, \(P\) is a \(t\)-invertible prime \(t\)-ideal. Let \(Z = \{X_n\} \setminus \{X_1, \ldots, X_n\}\), and note that \(P[Z] \subseteq Q\) and \(P[Z]\) is a \(t\)-invertible prime \(t\)-ideal of \(D[\{X_n\}]\) (see the proof of \((\Rightarrow)\) above). Hence, \(P[Z]\) is a maximal \(t\)-ideal of \(D[\{X_n\}]\), and thus \(Q = P[Z]\) and \(Q\) is \(t\)-invertible. Thus, by Theorem 2, \(D^*\) is a \(t\)-splitting set.

We next give an almost splitting set analog of Theorem 2. Even though the proof is a word for word translation of the proof of [8, Theorem 2], we give it for the completeness of this paper.

**Theorem 4.** Let \(D\) be an integral domain and \(S\) be a saturated multiplicative subset of \(D\) such that \(D_S\) is a factorial domain. Then \(S\) is an almost splitting set in \(D\) if and only if every prime \(t\)-ideal of \(D\) disjoint from \(S\) contains a primary element.

**Proof.** \((\Rightarrow)\) Assume that \(S\) is an almost splitting set of \(D\), and let \(P\) be a prime \(t\)-ideal of \(D\) disjoint from \(S\). Then \(PD_S = pD_S\) for some \(p \in P\) (see the proof of Theorem 2), and since \(S\) is almost splitting, by Lemma 1, there is a positive integer \(n\) such that \(P \supseteq p^nD_S \cap D = qD\) for some \(q \in D\). Clearly, \(q\) is a primary element. Thus, \(P\) contains a primary element \(q\).

\((\Leftarrow)\) Let \(0 \neq d \in D\). Then \(dD_S = p_1^{e_1} \cdots p_k^{e_k} D_S\), where every \(e_i\) is a positive integer and the \(p_i\)'s are non-associate prime elements in \(D_S\) (see the proof of Theorem 2). Let \(P_i\) be the prime ideal of \(D\) such that \(P_iD_S = p_iD_S\). Then \(P_i\) is a prime \(t\)-ideal of \(D\) and \(P_i \cap S = \emptyset\): so \(P_i\) contains a primary element \(q_i\). Clearly, \(q_iD_S = p_i^{n_i} D_S\) for some positive integer \(n_i\). Let \(n = n_1 \cdots n_k\) and \(m_i = \frac{n}{n_i} e_i\). Then \(p_i^{m_i} D_S = q_i^{m_i} D_S\), and
hence
\[
d^n D_S \cap D = ((p_1^{m_1}) D_S \cap \cdots \cap (p_k^{m_k}) D_S) \cap D \\
= ((q_1^{m_1} D_S) \cap \cdots \cap (q_k^{m_k} D_S)) \cap D \\
= (q_1^{m_1} D_S \cap D) \cap \cdots \cap (q_k^{m_k} D_S \cap D) \\
= (q_1^{m_1} D) \cap \cdots \cap (q_k^{m_k} D) \\
= (q_1^{m_1} \cdots q_k^{m_k}) D,
\]
where the fourth and last equalities follow from the fact that each \( q_i^{m_i} \) is a primary element with \( \sqrt{q_i^{m_i} D} \neq \sqrt{q_j^{m_j} D} \) for \( i \neq j \). Therefore, \( S \) is an almost splitting set by Lemma 1.

Let \( N(D^*) = \{ f \in D'[\{X_\alpha\}] \mid (f, d)_v = D[\{X_\alpha\}] \text{ for all } d \in D^* \} \). It is clear that \( (f, d)_v = D[\{X_\alpha\}] \) for all \( d \in D^* \) if and only if \( c(f)_v = D \), where \( c(f) \) is the ideal of \( D \) generated by the coefficients of \( f \). Hence, \( Cl(D[\{X_\alpha\}]_{N(D^*)}) = 0 \) [14, Theorem 2.14]. The next result is a generalization of [5, Theorem 2.4].

**Corollary 5.** \( D^* \) is an almost splitting set in \( D[\{X_\alpha\}] \) if and only if \( D \) is a UMT-domain and \( Cl(D[\{X_\alpha\}]) \) is torsion.

**Proof.** (\( \Rightarrow \)) If \( D^* \) is an almost splitting set in \( D[\{X_\alpha\}] \), then \( Cl(D[\{X_\alpha\}]_{D^*}) = Cl((D[\{X_\alpha\}])_{N(D^*)}) = 0 \). Thus, \( Cl(D[\{X_\alpha\}]) \) is torsion [7, Theorem 2.10(2)]. Also, since almost splitting sets are \( t \)-splitting sets, \( D \) is a UMT-domain by Corollary 3.

(\( \Leftarrow \)) Assume that \( D \) is a UMT-domain and \( Cl(D[\{X_\alpha\}]) \) is torsion. Then \( D^* \) is a \( t \)-splitting set by Corollary 3, and since \( Cl(D[\{X_\alpha\}]) \) is torsion, \( D^* \) is an almost splitting set.

**Corollary 6.** If \( D \) is integrally closed, then \( D^* \) is an almost splitting (resp., a \( t \)-splitting) set in \( D[\{X_\alpha\}] \) if and only if \( D \) is an AGCD-domain (resp., a PeMD).

**Proof.** Note that \( Cl(D[\{X_\alpha\}]) = Cl(D) \) [10, Corollary 2.13]; an integrally closed UMT-domain is a PeMD; and an integrally closed AGCD-domain is a PeMD with torsion class group. Hence, the result follows directly from Corollaries 3 and 5.

**Theorem 7.** Let \( D \) be an integral domain and \( S \) be a saturated multiplicative subset of \( D \) such that \( D_S \) is a factorial domain. Then \( S \) is a splitting set in \( D \) if and only if every prime \( t \)-ideal of \( D \) disjoint from \( S \) is principal.
Proof. (⇒) Let \( P \) be a prime \( t \)-ideal of \( D \) with \( P \cap S = \emptyset \). Then \( PD_S = pD_S \) for some prime element \( p \) of \( D_S \) (see the proof of Theorem 2), and thus \( PD_S \cap D = pD_S \cap D \) is principal by Lemma 1.

(⇐) An argument similar to the proof (⇐) of Theorem 4 shows that \( dD_S \cap D \) is principal for every \( 0 \neq d \in D \). Thus, by Lemma 1, \( S \) is a splitting set.

Let \( X \) be an indeterminate over \( D \). In [9, p. 77] (cf. [2, Example 4.7]), it was noted that \( D^* \) is a splitting set in \( D[X] \) if and only if \( D \) is a GCD-domain.

**Corollary 8.** \( D^* \) is a splitting set in \( D[\{X_\alpha\}] \) if and only if \( D \) is a GCD-domain.

Proof. If \( D^* \) is a splitting set in \( D[\{X_\alpha\}] \), then \( Cl(D) = Cl(D[\{X_\alpha\}]) = 0 \) [2, Corollary 3.8] because \( Cl(D[\{X_\alpha\}]_{D^*}) = Cl(D[\{X_\alpha\}]_{N(D^*)}) = 0 \). Hence, \( D \) is integrally closed [10, Corollary 2.13] and \( D \) is a UMT-domain by Corollary 3. Thus, \( D \) is a GCD domain because \( D \) is an integrally closed UMT-domain with \( Cl(D) = 0 \). Conversely, assume that \( D \) is a GCD-domain. Then \( D^* \) is a \( t \)-splitting set in \( D[\{X_\alpha\}] \) by Corollary 3 and \( Cl(D[\{X_\alpha\}]) = Cl(D) = 0 \). Thus, \( D^* \) is a splitting set.

Let \( S \) be a saturated multiplicative subset of an integral domain \( D \) such that \( D_S \) is a factorial domain. The proofs of Theorems 2, 4, and 7 show that \( S \) is splitting (resp., almost splitting, \( t \)-splitting) if and only if for every nonzero prime element \( p \) of \( D_S \), the ideal \( pD_S \cap D \) is principal (resp., contains a primary element, \( t \)-invertible).

**Acknowledgement.** The author would like to thank the referees for their several helpful comments and suggestions.

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