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# REMARK ON AVERAGE OF CLASS NUMBERS OF FUNCTION FIELDS

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Abstract. Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$, where $q$ is a power of an odd prime number, and $\mathbb{A} = \mathbb{F}_q[T]$. Let $\gamma$ be a generator of $\mathbb{F}_q^*$. Let $\mathcal{H}_n$ be the subset of $\mathbb{A}$ consisting of monic square-free polynomials of degree $n$. In this paper we obtain an asymptotic formula for the mean value of $L(1, \chi_{\gamma D})$ and calculate the average value of the ideal class number $h_{\gamma D}$ when the average is taken over $D \in \mathcal{H}_{2g+2}$.

## 1. Introduction and statement of result

Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$, where $q$ is a power of an odd prime number, and $\mathbb{A} = \mathbb{F}_q[T]$. Let $\mathbb{A}^+$ be the set of monic polynomials in $\mathbb{A}$ and $\mathcal{H}$ be the subset of $\mathbb{A}^+$ consisting of monic square-free polynomials. Write $\mathbb{A}_n^+ = \{N \in \mathbb{A}^+ : \deg N = n\}$ and $\mathcal{H}_n = \mathcal{H} \cap \mathbb{A}_n^+$. For any nonconstant square free $D \in \mathbb{A}^+$, let $\mathcal{O}_D$ be the integral closure of $\mathbb{A}$ in $k(\sqrt{D})$ and $h_D$ be the ideal class number of $\mathcal{O}_D$. Hoffstein and Rosen [3] calculated the average value of the ideal class number $h_D$ when the average is taken over all monic polynomials

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$D$ of a fixed odd degree. Andrade [1] obtained an asymptotic formula for the mean value of $L(1, \chi_D)$ and calculated the average value of the ideal class number $h_D$ when the average is taken over $D \in \mathcal{H}_{2g+1}$. We remark that Andrade assumed that $q \equiv 1 \bmod 4$ for simplicity, but his results hold true for any odd $q > 3$. In a recent paper, the author [4] obtained an asymptotic formula for the mean value of $L(1, \chi_D)$ and calculated the average value of the ideal class number $h_D$ when the average is taken over $D \in \mathcal{H}_{2g+2}$. Note that if $D \in \mathcal{H}_{2g+1}$, the infinite place $\infty_k = (1/T)$ of $k$ ramifies in $k(\sqrt{D})$, i.e., $k(\sqrt{D})/k$ is a (ramified) imaginary quadratic extension, and if $D \in \mathcal{H}_{2g+2}$, $\infty_k$ splits in $k(\sqrt{D})$, i.e., $k(\sqrt{D})/k$ is a real quadratic extension. Let $\gamma$ be a fixed generator of $\mathbb{F}_q^*$. Any inert imaginary quadratic extension $K$ of $k$ (i.e., $\infty_k$ is inert in $K$) can be written uniquely in the form $K = k(\sqrt{\gamma D})$ for some $D \in \mathcal{H}_{2g+2}$. The aim of this paper is to study the asymptotic formula for the mean value of $L(1, \chi_{\gamma D})$ and calculate the average value of the ideal class number $h_{\gamma D}$ when the average is taken over $D \in \mathcal{H}_{2g+2}$. We state our main results.

THEOREM 1.1. *We have*

$$\sum_{D \in \mathcal{H}_{2g+2}} L(1, \chi_{\gamma D}) = |D| \mathrm{P}(2) + O\left(2^g q^g\right),$$

*where* $|D| = q^{2g+2}$ *and*

$$\mathrm{P}(s) = \prod_{\substack{P \in \mathbb{A}^+ \\ \text{irreducible}}} \left(1 - \frac{1}{(1 + |P|)|P|^s}\right).$$

Since $\sharp \mathcal{H}_{2g+2} = (q-1) q^{2g+1}$ (see (2.1)), as a corollary of the Theorem 1.1, we have the following.

COROLLARY 1.2. *We have*

$$\frac{1}{\sharp \mathcal{H}_{2g+2}} \sum_{D \in \mathcal{H}_{2g+2}} L(1, \chi_{\gamma D}) \sim |D| \mathrm{P}(2)$$

*as* $g \to \infty$.

For any $D \in \mathcal{H}_{2g+2}$, we have the following class number formula (see [3, Theorem 0.6]):

$$L(1, \chi_{\gamma D}) = \frac{q+1}{2\sqrt{|D|}} h_{\gamma D} = \frac{q \zeta_{\mathbb{A}}(2)}{2 \zeta_{\mathbb{A}}(3) \sqrt{|D|}} h_{\gamma D}. \tag{1.1}$$

By Corollary 1.2 and the class number formula (1.1), we have the following asymptotic formula for the average of the class number $h_{\gamma D}$.

THEOREM 1.3. *We have*

$$\frac{1}{\sharp\mathcal{H}_{2g+2}} \sum_{D\in\mathcal{H}_{2g+2}} h_{\gamma D} \sim \frac{2\zeta_{\mathbb{A}}(3)\mathrm{P}(2)}{q\zeta_{\mathbb{A}}(2)} |D|\sqrt{|D|}$$

*as* $g \to \infty$.

## 2. Preliminaries

**2.1. Quadratic Dirichlet $L$-function.** Let $\mathbb{A}^+$ be the set of all monic polynomials in $\mathbb{A}$ and $\mathbb{A}_n^+ = \{N \in \mathbb{A}^+ : \deg N = n\}$ $(n \geq 0)$. The zeta function $\zeta_{\mathbb{A}}(s)$ of $\mathbb{A}$ is defined by the infinite series

$$\zeta_{\mathbb{A}}(s) = \sum_{N\in\mathbb{A}^+} |N|^{-s}.$$

It is straightforward to see that $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$. For any square-free $D \in \mathbb{A}$, the quadratic character $\chi_D$ is defined by the Jacobi symbol $\chi_D(N) = (\frac{D}{N})$ and the quadratic Dirichlet $L$-function $L(s,\chi_D)$ associated to $\chi_D$ is

$$L(s,\chi_D) = \sum_{N\in\mathbb{A}^+} \chi_D(N)|N|^{-s}.$$

We can write $L(s,\chi_D) = \sum_{n=0}^{\infty} \sigma_n(D)q^{-ns}$ with $\sigma_n(D) = \sum_{N\in\mathbb{A}_n^+} \chi_D(N)$. Since $\sigma_n(D) = 0$ for $n \geq \deg D$, $L(s,\chi_D)$ is a polynomial in $q^{-s}$ of degree $\leq \deg D - 1$. Putting $u = q^{-s}$, write

$$\mathcal{L}(u,\chi_D) = \sum_{n=0}^{\deg D-1} \sigma_n(D)u^n = L(s,\chi_D).$$

The cardinality of $\mathcal{H}_n$ is $\#\mathcal{H}_1 = q$ and $\#\mathcal{H}_n = (1-q^{-1})q^d$ $(n \geq 2)$. In particular, we have

$$\#\mathcal{H}_{2g+2} = (q-1)q^{2g+1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}. \tag{2.1}$$

Fix a generator $\gamma$ of $\mathbb{F}_q^*$. Write $\bar{D} = \gamma D$ for any $D \in H_{2g+2}$. Since $(\frac{\gamma}{N}) = (-1)^{\deg N}$, we have $(\frac{\bar{D}}{N}) = (-1)^{\deg N}(\frac{D}{N})$. Hence, $\sigma_n(\bar{D}) = (-1)^n\sigma_n(D)$.

For $D \in \mathcal{H}_{2g+2}$, $\mathcal{L}(u, \chi_{\bar{D}})$ has a trivial zero at $u = -1$. The complete $L$-function $\tilde{\mathcal{L}}(u, \chi_{\bar{D}})$ is defined by

$$\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = (1+u)^{-1}\mathcal{L}(u, \chi_{\bar{D}}).$$

It is a polynomial of even degree $2g$ and satisfies the functional equation

$$\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = (qu^2)^g \tilde{\mathcal{L}}((qu)^{-1}, \chi_{\bar{D}}). \tag{2.2}$$

LEMMA 2.1. *Let $\chi_{\bar{D}}$ be a quadratic character, where $D \in \mathcal{H}_{2g+2}$. Then*

$$\begin{aligned}\mathcal{L}(q^{-1}, \chi_{\bar{D}}) = \sum_{n=0}^{g} (-1)^n q^{-n} \sum_{N \in \mathbb{A}_n^+} \chi_D(N) + (-1)^g q^{-(g+1)} \sum_{n=0}^{g} \sum_{N \in \mathbb{A}_n^+} \chi_D(N) \\ + (1+q^{-1}) q^{-g} \sum_{n=0}^{g-1} \left( \frac{(-1)^n + (-1)^{g+1}}{2} \right) \sum_{N \in \mathbb{A}_n^+} \chi_D(N).\end{aligned}$$

*Proof.* Write $\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = \sum_{n=0}^{2g} \tilde{\sigma}_n(\bar{D}) u^n$. Since $\mathcal{L}(u, \chi_{\bar{D}}) = (1 + u)\tilde{\mathcal{L}}(u, \chi_{\bar{D}})$, we have $\sigma_0(\bar{D}) = \tilde{\sigma}_0(\bar{D})$, $\sigma_n(\bar{D}) = \tilde{\sigma}_{n-1}(\bar{D}) + \tilde{\sigma}_n(\bar{D})$ $(1 \le n \le 2g)$ and $\sigma_{2g+1}(\bar{D}) = \tilde{\sigma}_{2g}(\bar{D})$, or

$$\tilde{\sigma}_n(\bar{D}) = \sum_{i=0}^{n} (-1)^{n-i} \sigma_i(\bar{D}) \ (0 \le n \le 2g). \tag{2.3}$$

By substituting $\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = \sum_{n=0}^{2g} \tilde{\sigma}_n(\bar{D}) u^n$ into (2.2) and equating coefficients, we have $\tilde{\sigma}_n(\bar{D}) = \tilde{\sigma}_{2g-n}(\bar{D}) q^{-g+n}$ or $\tilde{\sigma}_{2g-n}(\bar{D}) = \tilde{\sigma}_n(\bar{D}) q^{g-n}$. Hence,

$$\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = \sum_{n=0}^{g} \tilde{\sigma}_n(\bar{D}) u^n + q^g u^{2g} \sum_{n=0}^{g-1} \tilde{\sigma}_n(\bar{D}) q^{-n} u^{-n}.$$

In particular, we have

$$\tilde{\mathcal{L}}(q^{-1}, \chi_{\bar{D}}) = \sum_{n=0}^{g} \tilde{\sigma}_n(\bar{D}) q^{-n} + q^{-g} \sum_{n=0}^{g} \tilde{\sigma}_n(\bar{D}). \tag{2.4}$$

By substituting (2.3) into (2.4) and using $\sigma_n(\bar{D}) = (-1)^n \sigma_n(D)$, we have

$$\tilde{\mathcal{L}}(q^{-1}, \chi_{\bar{D}}) = \frac{1}{1+q^{-1}} \sum_{n=0}^{g} (-1)^n q^{-n} \sigma_n(D) + \frac{(-1)^g q^{-(g+1)}}{1+q^{-1}} \sum_{n=0}^{g} \sigma_n(D) + q^{-g} \sum_{n=0}^{g-1} \left( \frac{(-1)^n + (-1)^{g+1}}{2} \right) \sigma_n(D).$$

So we get the result since $\mathcal{L}(q^{-1}, \chi_{\bar{D}}) = (1+q^{-1}) \tilde{\mathcal{L}}(q^{-1}, \chi_{\bar{D}})$. □

**2.2. Contribution of square parts.** The square part contributions in the summation of $L(1, \chi_{\bar{D}})$ over $D \in \mathcal{H}_{2g+2}$ are given as follow.

PROPOSITION 2.2. *We have*

$$\sum_{m=0}^{[\frac{g}{2}]} q^{-2m} \sum_{L \in \mathbb{A}_m^+} \sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (L,D)=1}} 1 = |D| \mathrm{P}(2) - q^{-([\frac{g}{2}]+1)} |D| \mathrm{P}(1) + O(q^g), \tag{2.5}$$

(2.6)

$$(-1)^g q^{-(g+1)} \sum_{m=0}^{[\frac{g}{2}]} \sum_{L \in \mathbb{A}_m^+} \sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (L,D)=1}} 1 = (-1)^g q^{-(g+1)+[\frac{g}{2}]} |D| \mathrm{P}(1) + O(g q^g)$$

*and*

$$\begin{aligned}(1+q^{-1}) q^{-g} \left( \frac{1+(-1)^{g+1}}{2} \right) \sum_{m=0}^{[\frac{g-1}{2}]} \sum_{L \in \mathbb{A}_m^+} \sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (L,D)=1}} 1 \\ = (q+1) q^{-g+[\frac{g-1}{2}]-1} \left( \frac{1+(-1)^{g+1}}{2} \right) |D| \mathrm{P}(1) + O(g q^g).\end{aligned} \tag{2.7}$$

*Proof.* The proofs are mild modifications of those of Proposition 3.7 in [4]. We only give the proof of (2.7). By using the fact that (see [2, Proposition 5.2]))

$$\sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (D,L)=1}} 1 = \frac{|D|}{\zeta_{\mathbb{A}}(2)} \prod_{P|L} (1+|P|^{-1})^{-1} + O\left( \sqrt{|D|} \frac{\Phi(L)}{|L|} \right),$$

we have

$$
\begin{aligned}
&(1+q^{-1})q^{-g}\left(\frac{1+(-1)^{g+1}}{2}\right)\sum_{m=0}^{[\frac{g-1}{2}]}\sum_{L\in\mathbb{A}_m^+}\sum_{\substack{D\in\mathcal{H}_{2g+2}\\(L,D)=1}}1\\
&\quad=(1+q^{-1})q^{-g}\left(\frac{1+(-1)^{g+1}}{2}\right)\frac{|D|}{\zeta_{\mathbb{A}}(2)}\sum_{m=0}^{[\frac{g-1}{2}]}\sum_{L\in\mathbb{A}_m^+}\prod_{P|L}(1+|P|^{-1})^{-1}\\
&\qquad+O\left(q^{-g}\sum_{m=0}^{[\frac{g-1}{2}]}\sum_{L\in\mathbb{A}_m^+}\sqrt{|D|}\frac{\Phi(L)}{|L|}\right),
\end{aligned}
$$

where $\Phi(L)$ is the Euler totient function. Using the fact that $\sum_{L\in\mathbb{A}_m^+}\Phi(L)$ $=(1-q^{-1})q^{2m}$ (see [5, Proposition 2.7]), we have

$$
\begin{aligned}
q^{-g}\sum_{m=0}^{[\frac{g-1}{2}]}\sum_{L\in\mathbb{A}_m^+}\sqrt{|D|}\frac{\Phi(L)}{|L|}&=q\sum_{m=0}^{[\frac{g-1}{2}]}q^{-m}\sum_{L\in\mathbb{A}_m^+}\Phi(L)\\
&=(q-1)\sum_{m=0}^{[\frac{g-1}{2}]}q^{m}\ll q^{\frac{g+1}{2}}.
\end{aligned}
$$

By using the fact that ( [2, Lemma 5.7])

$$
\sum_{L\in\mathbb{A}_m^+}\prod_{P|L}(1+|P|^{-1})^{-1}=q^m\sum_{\substack{M\in\mathbb{A}^+\\ \deg M\le m}}\frac{\mu(M)}{|M|}\prod_{P|M}\frac{1}{1+|P|},
$$

we have

(2.8)

$$(1+q^{-1})q^{-g}\left(\frac{1+(-1)^{g+1}}{2}\right)\frac{|D|}{\zeta_{\mathbb{A}}(2)}\sum_{m=0}^{[\frac{g-1}{2}]}\sum_{L\in\mathbb{A}_m^+}\prod_{P|L}(1+|P|^{-1})^{-1}$$

$$= (1+q^{-1})q^{-g}\left(\frac{1+(-1)^{g+1}}{2}\right)\frac{|D|}{\zeta_{\mathbb{A}}(2)}\sum_{m=0}^{[\frac{g-1}{2}]}q^m \sum_{\substack{M\in\mathbb{A}^+\\ \deg M\le m}}\frac{\mu(M)}{|M|}\prod_{P|M}\frac{1}{1+|P|}$$

$$= (q+1)q^{-g+[\frac{g-1}{2}]-1}\left(\frac{1+(-1)^{g+1}}{2}\right)|D|\sum_{\substack{M\in\mathbb{A}^+\\ \deg M\le[\frac{g-1}{2}]}}\frac{\mu(M)}{|M|}\prod_{P|M}\frac{1}{1+|P|}$$

$$-(q+1)q^{-(g+2)}\left(\frac{1+(-1)^{g+1}}{2}\right)|D|\sum_{\substack{M\in\mathbb{A}^+\\ \deg M\le[\frac{g-1}{2}]}}\mu(M)\prod_{P|M}\frac{1}{1+|P|}.$$

Finally, by using ( [4, Lemma 3.3, Lemma 3.5])

$$\sum_{\substack{M\in\mathbb{A}^+\\ \deg M\le[\frac{g-1}{2}]}}\frac{\mu(M)}{|M|}\prod_{P|M}\frac{1}{1+|P|} = \mathrm{P}(1) + O\left(q^{-\frac{g-1}{2}}\right)$$

and

$$\sum_{\substack{M\in\mathbb{A}^+\\ \deg M\le[\frac{g-1}{2}]}}\mu(M)\prod_{P|M}\frac{1}{1+|P|}\le\frac{g+1}{2},$$

we have

$$(1+q^{-1})q^{-g}\left(\frac{1+(-1)^{g+1}}{2}\right)\frac{|D|}{\zeta_{\mathbb{A}}(2)}\sum_{m=0}^{[\frac{g-1}{2}]}\sum_{L\in\mathbb{A}_m^+}\prod_{P|L}(1+|P|^{-1})^{-1}$$

$$= (q+1)q^{-g+[\frac{g-1}{2}]-1}\left(\frac{1+(-1)^{g+1}}{2}\right)|D|\mathrm{P}(1) + O\left(gq^g\right).$$

$\square$

**2.3. Contribution of non square parts.** The non square part contributions in the summation of $L(1,\chi_{\bar{D}})$ over $D \in \mathcal{H}_{2g+2}$ are given as follow.

Proposition 2.3. *We have*

$$\sum_{D\in\mathcal{H}_{2g+2}} \sum_{n=0}^{g} (-1)^n q^{-n} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) = O\left(2^g q^g\right), \tag{2.9}$$

$$(-1)^g q^{-(g+1)} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{n=0}^{g} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) = O\left(2^g q^g\right) \tag{2.10}$$

*and*

$$(1+q^{-1})q^{-g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{n=0}^{g-1} \left(\frac{(-1)^n + (-1)^{g+1}}{2}\right) \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) = O\left(2^g q^g\right). \tag{2.11}$$

*Proof.* As in [2, Lemma 6.4], for any non-square $N \in \mathbb{A}^+$, we have

$$\sum_{D\in\mathcal{H}_{2g+2}} \chi_D(N) \ll q^{g+1} 2^{\deg N - 1}. \tag{2.12}$$

Using (2.12), we have

$$\begin{aligned} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{n=0}^{g} (-1)^n q^{-n} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) &\ll \sum_{n=0}^{g} q^{-n} \sum_{N\in\mathbb{A}_n^+} q^{g+1} 2^{n-1} \\ &\ll q^g \sum_{n=0}^{g} 2^n \ll 2^g q^g, \end{aligned}$$

$$\begin{aligned} (-1)^g q^{-(g+1)} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{n=0}^{g} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) &\ll q^{-(g+1)} \sum_{n=0}^{g} \sum_{N\in\mathbb{A}_n^+} q^{g+1} 2^{n-1} \\ &\ll \sum_{n=0}^{g} 2^n q^n \ll 2^g q^g \end{aligned}$$

and

$$(1+q^{-1})q^{-g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{n=0}^{g-1}\left(\frac{(-1)^n+(-1)^{g+1}}{2}\right)\sum_{\substack{N\in\mathbb{A}_n^+\\ N\neq\square}}\chi_D(N)$$
$$\ll (1+q^{-1})q^{-g}\sum_{n=0}^{g-1}\sum_{N\in\mathbb{A}_n^+}q^{g+1}2^{n-1}$$
$$\ll (q+1)\sum_{n=0}^{g-1}2^nq^n \ll 2^gq^g.$$

□

## 3. Proof of Theorem 1.1

By Lemma 2.1, we have

(3.1)
$$\sum_{D\in\mathcal{H}_{2g+2}}L(1,\chi_{\bar{D}}) = \sum_{D\in\mathcal{H}_{2g+2}}\sum_{n=0}^{g}(-1)^nq^{-n}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)+(-1)^gq^{-(g+1)}$$
$$\sum_{D\in\mathcal{H}_{2g+2}}\sum_{n=0}^{g}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)$$
$$+(1+q^{-1})q^{-g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{n=0}^{g-1}\left(\frac{(-1)^n+(-1)^{g+1}}{2}\right)\sum_{N\in\mathbb{A}_n^+}\chi_D(N).$$

By (2.5) and (2.9), we have

(3.2)
$$\sum_{D\in\mathcal{H}_{2g+2}}\sum_{n=0}^{g}(-1)^nq^{-n}\sum_{N\in\mathbb{A}_n^+}\chi_D(N)$$
$$= |D|\mathrm{P}(2)-q^{-([\frac{g}{2}]+1)}|D|\mathrm{P}(1)+O\left(2^{g+1}q^{g+1}\right)$$

and, by (2.6) and (2.10), we have

$$(3.3)\qquad (-1)^g q^{-(g+1)} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{n=0}^{g} \sum_{N\in\mathbb{A}_n^+} \chi_D(N)$$
$$= (-1)^g q^{-(g+1)+[\frac{g}{2}]}|D|\mathrm{P}(1) + O\left(2^{g+1}q^{g+1}\right).$$

Similarly, by (2.7) and (2.11), we have

$$(3.4)\qquad (1+q^{-1})q^{-g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{n=0}^{g-1} \left(\frac{(-1)^n+(-1)^{g+1}}{2}\right) \sum_{N\in\mathbb{A}_n^+} \chi_D(N)$$
$$= (q+1)q^{-g+[\frac{g-1}{2}]-1}\left(\frac{1+(-1)^{g+1}}{2}\right)|D|\mathrm{P}(1) + O\left(2^g q^g\right).$$

It is easy to see that

$$(-1)^g q^{-(g+1)+[\frac{g}{2}]} + (q+1)q^{-g+[\frac{g-1}{2}]-1}\left(\frac{1+(-1)^{g+1}}{2}\right) - q^{-([\frac{g}{2}]+1)} = 0.$$

Hence, by inserting (3.2), (3.3) and (3.4) into (3.1), we get

$$\sum_{D\in\mathcal{H}_{2g+2}} L(1,\chi_{\bar{D}}) = |D|\mathrm{P}(2) + O\left(2^g q^g\right).$$

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