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σ-COMPLETE BOOLEAN ALGEBRAS AND BASICALLY DISCONNECTED COVERS

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Abstract. In this paper, we show that for any σ-complete Boolean subalgebra $\mathcal{M}$ of $\mathcal{R}(X)$ containing $Z(X)^{\#}$, the Stone-space $S(\mathcal{M})$ of $\mathcal{M}$ is a basically diconnected cover of βX and that the subspace $\{\alpha \mid \alpha \text{ is a fixed } \mathcal{M}\text{-ultrafilter}\}$ of the Stone-space $S(\mathcal{M})$ is the the minimal basically disconnected cover of X if and only if it is a basically disconnected space and $\mathcal{M} \subseteq \{\Lambda_X(A) \mid A \in Z(\Lambda X)^{\#}\}$.

1. Introduction

All spaces in this paper are Tychonoff spaces and $(\beta X, \beta_X)$ or simply βX denotes the Stone-Čech compactification of a space X.

Iliadis constructed the absolute of a Hausdorff space X, which is the minimal extremally disconnected cover (EX, π_X) of X and they turn out to be the perfect onto projective covers([6]). To generalize extremally disconnected spaces, basically disconnected spaces, quasi-F spaces and cloz-spaces have been introduced and their minimal covers have been studied by various aurthors([3], [5], [7], [9]). In these ramifications, minimal covers of compact spaces can be nisely characterized.

Vermeer([9]) showed that every space X has the minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and if X is a compact space, then ΛX is

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given by the Stone-space $S(\sigma Z(X)^{\#})$ of a σ-complete Boolean subalgebra $\sigma Z(X)^{\#}$ of $\mathcal{R}(X)$. Henriksen, Vermeer and Woods([5])(Kim [7], resp.) showed that the minmal basically disconnected cover of a weakly Lindelöf space (a locally weakly Lindelöf space, resp.) X is given by the subspace $\{\alpha \mid \alpha$ is a fixed $\sigma Z(X)^{\#}$-ultrafilter$\}$ of the Stone-space $S(\sigma Z(X)^{\#})$.

In this paper, we show that for any σ-complete Booeal subalgebra $\mathcal{M}$ of $\mathcal{R}(X)$ containing $Z(X)^{\#}$, the Stone-space $S(\mathcal{M})$ of $\mathcal{M}$ is a basically diconnected cover of βX and that the subspace $\{\alpha \mid \alpha$ is a fixed $\mathcal{M}$-ultrafilter$\}$ of the Stone-space $S(\mathcal{M})$ is the the minimal basically disconnected cover of X if and only if it is a basically disconnected space and $\mathcal{M} \subseteq \{\Lambda_X(A) \mid A \in Z(\Lambda X)^{\#}\}$.

For the terminology, we refer to [1], [2] and [8].

2. Minimal basically disconnected covers of spaces

The set $\mathcal{R}(X)$ of all regular closed sets in a space X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows : for any $A \in \mathcal{R}(X)$ and any $\{A_i : i \in I\} \subseteq \mathcal{R}(X)$,

$\vee\{A_i : i \in I\} = cl_X(\cup\{A_i : i \in I\})$,

$\wedge\{A_i : i \in I\} = cl_X(int_X(\cap\{A_i : i \in I\}))$, and

$A' = cl_X(X - A)$

and a sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, X and is closed under finite joins and finite meets.

Recall that a map $f : Y \longrightarrow X$ is called *a covering map* if it is a continuous, onto, perfect, and irreducible map.

Lemma 2.1. *([5])*

(1) *Let $f : Y \longrightarrow X$ be a covering map. Then the map $\psi : \mathcal{R}(Y) \longrightarrow \mathcal{R}(X)$, defined by $\psi(A) = f(A)$, is a Boolean isomorphism and the inverse map ψ^{-1} of ψ is given by $\psi^{-1}(B) = cl_Y(f^{-1}(int_X(B))) = cl_Y(int_Y(f^{-1}(B)))$.*

(2) *Let X be a dense subspace of a space K. Then the map $\phi : \mathcal{R}(K) \longrightarrow \mathcal{R}(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ϕ^{-1} of ϕ is given by $\phi^{-1}(B) = cl_K(B)$.*

A lattice L is called *σ-complete* if every countable subset of L has join and meet. For any subset $\mathcal{M}$ of a Boolean algebra L, there is the smallest σ-complete Boolean subalgebra $\sigma\mathcal{M}$ of L containing $\mathcal{M}$.

Let X be a space and $Z(X)$ the set of all zero-sets in X. Then $Z(X)^{\#} = \{cl_X(int_X(Z)) \mid Z \in Z(X)\}$ is a sublattice of $\mathcal{R}(X)$. Since X is C^*-embedded in βX, by Lemma 2.1., $\sigma Z(X)^{\#}$, $\sigma Z(\upsilon X)^{\#}$ and $\sigma Z(\beta X)^{\#}$ are Boolean isomorphic.

Let X be a space and $\mathcal{B}$ a Boolean subalgebra of $\mathcal{R}(X)$. Let $S(\mathcal{B}) = \{\alpha \mid \alpha \text{ is a } \mathcal{B}\text{-ultrafilter}\}$ and for any $B \in \mathcal{B}$, $\Sigma_B^{\mathcal{B}} = \{\alpha \in S(\mathcal{B}) \mid B \in \alpha\}$. Then the space $S(\mathcal{B})$, equipped with the topology for which $\{\Sigma_B^{\mathcal{B}} \mid B \in \mathcal{B}\}$ is a base, called *the Stone-space of* $\mathcal{B}$. Then $S(\mathcal{B})$ is a compact, zero-dimensional space and the map $s_{\mathcal{B}} : S(\mathcal{B}) \longrightarrow \beta X$, defined by $s_{\mathcal{B}}(\alpha) = \cap\{A \mid A \in \mathcal{B}\}$, is a covering map ([8]).

Definition 2.2. A space X is called *basically disconnected* if for any zero-set Z in X, $int_X(Z)$ is closed in X, equivalently, $Z(X)^{\#} = B(X)$, where $B(X)$ is the set of clopen sets in X.

A space X is a basically disconnected space if and only if βX is a basically disconnected space. Since X is a basically disconnected space, $Z(X)^{\#} = B(X)$ and so X is a basically disconnected space if and only if $Z(X)^{\#}$ is a σ-complete Boolean algebra.

Definition 2.3. Let X be a space. Then a pair (Y, f) is called

(1) *a cover of* X if $f : Y \longrightarrow X$ is a covering map,

(2) *a basically disconnected cover of* X if (Y, f) is a cover of X and Y is a basically disconnected space, and

(3) *a minimal basically disconnected cover of* X if (Y, f) is a basically disconnected cover of X and for any basically disconnected cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ such that $f \circ h = g$.

Vermeer([9]) showed that every space X has a minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and that if X is a compact space, then ΛX is the Stone-space $S(\sigma Z(X)^{\#})$ of $\sigma Z(X)^{\#}$ and $\Lambda_X(\alpha) = \cap\{A \mid A \in \alpha\}$ $(\alpha \in \Lambda X)$.

Let X be a space. Since $\sigma Z(X)^{\#}$ and $\sigma Z(\beta X)^{\#}$ are Boolean isomorphic, $S(\sigma Z(X)^{\#})$ and $S(\sigma Z(\beta X)^{\#})$ are homeomorphic.

Let X, Y be spaces and $f : Y \longrightarrow X$ a map. For any $U \subseteq X$, let $f_U : f^{-1}(U) \longrightarrow U$ denote the restriction and co-restriction of f with respect to $f^{-1}(U)$ and U, respectively.

For any space X, let $(\Lambda\beta X, \Lambda_\beta)$ denote the minimal basically disconnected cover of βX.

LEMMA 2.4. *([7]) Let X be a space. If $\Lambda_\beta^{-1}(X)$ is a basically disconnected space, then $(\Lambda_\beta^{-1}(X), \Lambda_{\beta_X})$ is the minimal basically disconnected cover of X.*

PROPOSITION 2.5. *Let X be a space and $\mathcal{M}$ a σ-complete Boolean subalgebra of $\mathcal{R}(X)$ containing $Z(X)^{\#}$. Then $S(\mathcal{M})$ is a basically disconnected space.*

Proof. Let D be a cozero-set in $S(\mathcal{M})$. Since $S(\mathcal{M})$ is a compact space([8]), D is a Lindelöf space and hence there is a sequense (A_n) in $\mathcal{M}$ such that $D = \cup\{\Sigma_{A_n}^{\mathcal{M}} \mid n \in \mathbb{N}\}$. Clearly, $cl_{S(\mathcal{M})}(D) \subseteq \Sigma_{\vee\{A_n|n\in\mathbb{N}\}}^{\mathcal{M}}$ because $\mathcal{M}$ is $\sigma-$complete.

Let $\alpha \in S(\mathcal{M}) - cl_{S(\mathcal{M})}(\cup\{\Sigma_{A_n}^{\mathcal{M}} \mid n \in \mathbb{N}\})$. Then there is a $B \in \mathcal{M}$ such that $\alpha \in \Sigma_B^{\mathcal{M}}$ and $(\cup\{\Sigma_{A_n}^{\mathcal{M}} \mid n \in \mathbb{N}\}) \cap \Sigma_B^{\mathcal{M}} = \emptyset$ and hence for any $n \in \mathbb{N}$, $\Sigma_{A_n}^{\mathcal{M}} \cap \Sigma_B^{\mathcal{M}} = \Sigma_{A_n\wedge B} = \emptyset$. Thus $A_n \wedge B = \emptyset$ for all $n \in \mathbb{N}$ and so $\vee\{A_n \wedge B \mid n \in \mathbb{N}\} = (\vee\{A_n \mid n \in \mathbb{N}\}) \wedge B = \emptyset$. Since $B \in \alpha$, $\vee\{A_n \mid n \in \mathbb{N}\} \notin \alpha$ and so $\alpha \notin \Sigma_{\vee\{A_n|n\in\mathbb{N}\}}^{\mathcal{M}}$. That is $\Sigma_{\vee\{A_n|n\in\mathbb{N}\}} \subseteq cl_{S(\mathcal{M})}(\cup\{\Sigma_{A_n}^{\mathcal{M}} \mid n \in \mathbb{N}\})$. Hence $cl_{S(\mathcal{M})}(D)$ is open in $S(\mathcal{M})$ and thus $S(\mathcal{M})$ is a basically disconnected space. □

Let X be a space and $\mathcal{M}$ a σ-complete Boolean subalgebra of $\mathcal{R}(X)$ containg $Z(X)^{\#}$. Since $s_{\mathcal{M}} : S(\mathcal{M}) \longrightarrow \beta X$ is a covering map ([8]), we have the following corollary :

COROLLARY 2.6. *Let X be a space and $\mathcal{M}$ a σ-complete Boolean subalgebra of $\mathcal{R}(X)$ containg $Z(X)^{\#}$. Then $(S(\mathcal{M}), s_{\mathcal{M}})$ is a basically disconnected cover of βX.*

Let X be a space. Then $Z(\Lambda X)^{\#}$ is a σ-complete Boolean subalgebra of $\mathcal{R}(\Lambda X)$, because ΛX is a basically disconnected space. By Lemma 2.1, $\{\Lambda_X(A) \mid A \in Z(\Lambda X)^{\#}\}$ is a σ-complete Boolean subalgebra of $\mathcal{R}(X)$.

Let X be a space and $\mathcal{M}$ a σ-complete Boolean subalgebra of $\mathcal{R}(X)$ containg $Z(X)^{\#}$. By Corollary 2.6, there is a covering map $t : S(\mathcal{M}) \longrightarrow \Lambda\beta X$ such that $\Lambda_\beta \circ t = s_{\mathcal{M}}$.

Note that for any $D \in S(\mathcal{M}), S_{\mathcal{M}}(\Sigma_D^{\mathcal{M}}) = cl_{\beta X}(D)$([8]).

THEOREM 2.7. *Let X be a space and $\mathcal{M}$ a σ-complete Boolean subalgebra of $\mathcal{R}(X)$ containg $Z(X)^{\#}$. Then $(s_{\mathcal{M}}^{-1}(X), s_{\mathcal{M}_X})$ is the minimal* *basically disconnected cover of X if and only if $s_{\mathcal{M}}^{-1}(X)$ is a basically disconnected space and $\mathcal{M} \subseteq \{\Lambda_X(A) \mid A \in Z(\Lambda X)^{\#}\}$.*

Proof. ($\Rightarrow$) Suppose that $s_{\mathcal{M}}^{-1}(X)$ is the minimal basically disconnected cover of X. Then there is a homeomorhpism $l : s_{\mathcal{M}}^{-1}(X) \longrightarrow \Lambda X$ such that $\Lambda_X \circ l = s_{\mathcal{M}_X}$ and there is a covering map $f : \beta\Lambda X \longrightarrow S(\mathcal{M})$ such that $f \circ \beta_{\Lambda X} \circ l = j$, where $j : s_{\mathcal{M}}^{-1}(X) \longrightarrow S(\mathcal{M})$ is the inclusion map. Note that there is a continuous map $g : \beta\Lambda X \longrightarrow \beta X$ such that $g \circ \beta_{\Lambda X} = \beta_X \circ \Lambda_X$. Since $\beta\Lambda X$ is a basically disconnected space, $(\beta\Lambda X, g)$ is a basically disconnected cover of βX and so there is a covering map $h : \beta\Lambda X \longrightarrow \Lambda\beta X$ such that $g = \Lambda_\beta \circ h$.

Take any $D \in \mathcal{M}$. Then $f^{-1}(\Sigma_D^{\mathcal{M}})$ is a clopen set in $\beta\Lambda X$ and hence $f^{-1}(\Sigma_D^{\mathcal{M}}) \cap \Lambda X \in Z(\Lambda X)^{\#}$. Since $\Lambda_\beta \circ t = s_{\mathcal{M}}$,

$$\begin{aligned}
\Lambda_X(f^{-1}(\Sigma_D^{\mathcal{M}}) \cap \Lambda X) &= g(f^{-1}(\Sigma_D^{\mathcal{M}})) \cap X \\
&= \Lambda_\beta(h(f^{-1}(\Sigma_D^{\mathcal{M}}))) \cap X \\
&= \Lambda_\beta(t(f(f^{-1}(\Sigma_D^{\mathcal{M}})))) \cap X \\
&= \Lambda_\beta(t(\Sigma_D^{\mathcal{M}})) \cap X \\
&= S_{\mathcal{M}}(\Sigma_D^{\mathcal{M}}) \cap X \\
&= cl_{\beta X}(D) \cap X = D.
\end{aligned}$$

Since $f^{-1}(\Sigma_D^{\mathcal{M}}) \cap \Lambda X \in Z(\Lambda X)^{\#}$, $\mathcal{M} \subseteq \{\Lambda_X(A) \mid A \in Z(\Lambda X)^{\#}\}$.

($\Leftarrow$) Since $s_{\mathcal{M}}^{-1}(X)$ is a basically disconnected space, there is a covering map $l : s_{\mathcal{M}}^{-1}(X) \longrightarrow \Lambda X$ such that $\Lambda_X \circ l = s_{\mathcal{M}_X}$.

We claim that $l : s_{\mathcal{M}}^{-1}(X) \longrightarrow \Lambda X$ is one-to-one. Let $\alpha \neq \delta$ in $s_{\mathcal{M}}^{-1}(X)$. Then there are $A \in \alpha$ and $B \in \delta$ such that $A \wedge B = \emptyset$ and since $A, B \in \mathcal{M}$, $\Sigma_A^{\mathcal{M}} \cap \Sigma_B^{\mathcal{M}} = \emptyset$. Since $\mathcal{M} \subseteq \{\Lambda_X(E) \mid E \in Z(\Lambda X)^{\#}\}$, there are $C, D \in Z(\Lambda X)^{\#}$ such that $\Lambda_X(C) = A$ and $\Lambda_X(D) = B$. Since ΛX is a basically disconnected space,

$$\Lambda_X(C \cap D) = \Lambda_X(C \wedge D) = \Lambda_X(C) \wedge \Lambda_X(D) = \emptyset.$$

By Lemma 2.1, $C \cap D = \emptyset$.

Note that

$$s_{\mathcal{M}_X}(\Sigma_A^{\mathcal{M}} \cap S_{\mathcal{M}}^{-1}(X)) = \Lambda_X(l(\Sigma_A^{\mathcal{M}} \cap S_{\mathcal{M}}^{-1}(X)) = S_{\mathcal{M}}(\Sigma_A^{\mathcal{M}}) \cap X = A = \Lambda_X(C),$$

$$s_{\mathcal{M}_X}(\Sigma_B^{\mathcal{M}} \cap S_{\mathcal{M}}^{-1}(X)) = \Lambda_X(l(\Sigma_B^{\mathcal{M}} \cap S_{\mathcal{M}}^{-1}(X)) = S_{\mathcal{M}}(\Sigma_B^{\mathcal{M}}) \cap X = B = \Lambda_X(D)$$

Hence by Lemma 2.1,

$$l(\Sigma_A^{\mathcal{M}} \cap S_{\mathcal{M}}^{-1}(X)) = C,\ l(\Sigma_B^{\mathcal{M}} \cap S_{\mathcal{M}}^{-1}(X)) = D.$$

Since $\alpha \in \Sigma_A^{\mathcal{M}}$ and $\delta \in \Sigma_B^{\mathcal{M}}$, $l(\alpha) \in C$ and $l(\delta) \in D$. Since $C \cap D = \emptyset$, $l(\alpha) \neq l(\delta)$ and so $l : s_{\mathcal{M}}^{-1}(X) \longrightarrow \Lambda X$ is a one-to-one. Hence $(s_{\mathcal{M}}^{-1}(X), s_{\mathcal{M}_X})$ is the minimal basically disconnected cover of X. □

Let X be a space and $\mathcal{M}$ a σ-complete Boolean subalgebra of $\mathcal{R}(X)$ containg $Z(X)^{\#}$. Then a $\mathcal{M}$-ultrafilter α is called *fixed* if $\cap\{A \mid A \in \alpha\} \neq \emptyset$. For any $\alpha \in S(\mathcal{M})$, $s_{\mathcal{M}}(\alpha) = \cap\{A \mid A \in \alpha\}$ and hence $s_{\mathcal{M}}^{-1}(X) = \{\alpha \in S(\mathcal{M}) \mid \alpha \text{ is fixed}\}$. Thus we have the following corollary

Corollary 2.8. *Let X be a space and $\mathcal{M}$ a σ-complete Boolean subalgebra of $\mathcal{R}(X)$ containg $Z(X)^{\#}$. Then $s_{\mathcal{M}}^{-1}(X)$ is a basically disconnected space and $\mathcal{M} \subseteq \{\Lambda_X(A) \mid A \in Z(\Lambda X)^{\#}\}$ if and only if the subspace $\{\alpha \in S(\mathcal{M}) \mid \alpha \text{ is fixed}\}$ of $S(\mathcal{M})$ is homeomorphic to ΛX.*

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