A REFINED ENUMERATION OF $p$-ARY LABELED TREES

Seunghyun Seo† and Heesung Shin‡

Abstract. Let $T_p^{(n)}$ be the set of $p$-ary labeled trees on \{1, 2, \ldots, n\}. A maximal decreasing subtree of a $p$-ary labeled tree is defined by the maximal $p$-ary subtree from the root with all edges being decreasing. In this paper, we study a new refinement $T_{n,k}^{(p)}$ of $T_n^{(p)}$, which is the set of $p$-ary labeled trees whose maximal decreasing subtree has $k$ vertices.

1. Introduction

Let $p$ be a fixed integer greater than 1. A $p$-ary tree $T$ is a tree such that

(i) either $T$ is empty or has a distinguished vertex $r$ which is called the root of $T$, and

(ii) $T-r$ consists of a weak ordered partition $(T_1, \ldots, T_p)$ of $p$-ary trees.
A 2-ary (resp. 3-ary) tree is called binary (resp. ternary) tree. Figure 1 exhibits all the ternary tree with 3 vertices. A full $p$-ary tree is a $p$-ary tree, where each vertex has either 0 or $p$ children. It is well known (see [6, 6.2.2 Proposition]) that the number of full $p$-ary trees with $n$ internal vertices is given by the $n$th order-$p$ Fuss-Catalan number \[ C_n^{(p)} = \frac{1}{pn+1} \binom{pn+1}{n}. \] Clearly a full $p$-ary tree $T$ with $m$ internal vertices corresponds to a $p$-ary tree with $m$ vertices by deleting all the leaves in $T$, so the number of $p$-ary trees with $n$ vertices is also $C_n^{(p)}$.

**Figure 1.** All 12 ternary trees with 3 vertices

A $p$-ary labeled tree is a $p$-ary tree whose vertices are labeled by distinct positive integers. In most cases, a $p$-ary labeled tree with $n$ vertices is identified with a $p$-ary tree on the vertex set \([n] := \{1, 2, \ldots, n\}\). Let $T_n^{(p)}$ be the set of $p$-ary labeled trees on \([n]\). Clearly the cardinality of $T_n^{(p)}$ is given by

\[ |T_n^{(p)}| = n! C_n^{(p)} = (pn)_{n-1}, \]

where $m_{(k)} := m(m - 1) \cdots (m - k + 1)$ is a falling factorial.

For a given $p$-ary labeled tree $T$, a maximal decreasing subtree of $T$ is defined as the maximal $p$-ary subtree from the root with all edges being decreasing, denoted by MD($T$). Figure 2 illustrates the maximal decreasing subtree of a given ternary tree $T$. Let $T_{n,k}^{(p)}$ be the set of $p$-ary labeled trees on \([n]\) with its maximal decreasing subtree having $k$ vertices.

In this paper we present a formula for $|T_{n,k}^{(p)}|$, which makes a refined enumeration of $T_n^{(p)}$, or a generalization of equation (1). Note that similar refinements for rooted labeled trees and ordered labeled trees were done before (see [4,5]), but the $p$-ary case is much more complicated and has quite different features.
A refined enumeration of $p$-ary labeled trees

8
7
11
3
5
9
1
2
3
4
6
9
1
2
3
4

Figure 2. The maximal decreasing subtree of $T$

2. Main results

From now on we will consider labeled trees only. So we will omit the word “labeled”. Recall that $T_{n,k}^{(p)}$ is the set of $p$-ary trees on $[n]$, whose maximal decreasing subtree has $k$ vertices. Let $Y_{n,k}^{(p)}$ be the set of $p$-ary trees $T$ on $[n]$, where $T$ is given by attaching additional $(n - k)$ increasing leaves to a decreasing tree with $k$ vertices. Let $F_{n,k}^{(p)}$ be the set of (non-ordered) forests on $[n]$ consisting of $k$ $p$-ary trees, where the $k$ roots are not ordered. In Figure 3, the first two forests are the same, but the third one is a different forest in $F_{4,2}^{(2)}$.

Figure 3. Forests in $F_{4,2}^{(3)}$

Define the numbers

\[ t(n, k) = \left| T_{n,k}^{(p)} \right|, \]
\[ y(n, k) = \left| Y_{n,k}^{(p)} \right|, \]
\[ f(n, k) = \left| F_{n,k}^{(p)} \right|. \]

We will show that a $p$-ary tree can be “decomposed” into a $p$-ary tree in $\bigcup_{n,k} Y_{n,k}^{(p)}$ and a forest in $\bigcup_{n,k} F_{n,k}^{(p)}$. Thus it is important to count the numbers $y(n, k)$ and $f(n, k)$. 
Lemma 2.1. For $0 \leq k < n$, the number $y(n, k)$ satisfies the recursion:

$$y(n+1, k+1) = \sum_{m=0}^{p} \binom{n}{m} p(m) \left( kp - n + m + 1 \right) \cdot y(n-m, k)$$

with the following boundary conditions:

$$y(n, n) = \prod_{j=0}^{n-1} (1 + (p-1)j) \quad \text{for } n \geq 1$$

$$y(n, k) = 0 \quad \text{for } k < \max \left( \frac{n-1}{p}, 1 \right)$$

Proof. Consider a tree $Y$ in $\mathcal{Y}^{(p)}_{n+1, k+1}$. The tree $Y$ with $(n+1)$ vertices consists of its maximal decreasing tree with $(k+1)$ vertices and $(n-k)$ increasing leaves. Note that the vertex 1 is always contained in $\text{MD}(Y)$.

If the vertex 1 is a leaf of $Y$, consider the tree $Y'$ obtained by deleting the leaf 1 from $Y$. The number of vertices in $Y'$ and $\text{MD}(Y')$ are $n$ and $k$, respectively. So the number of possible trees $Y'$ is $y(n, k)$. Since we cannot attach the vertex 1 to $(n-k)$ increasing leaves of $Y'$, there are $kp - (n-1)$ ways of recovering $Y$. Thus the number of $Y$ with the leaf 1 is

$$kp - (n-1+1) \cdot y(n, k).$$

If the vertex 1 is not a leaf of $Y$, then the vertex 1 has increasing leaves $\ell_1, \ldots, \ell_m$, where $1 \leq m \leq p$. Consider the tree $Y''$ obtained by deleting $\ell_1, \ldots, \ell_m$ from $Y$. Clearly 1 is a leaf of $Y''$ and the number of vertices in $Y''$ and $\text{MD}(Y'')$ are $n-m+1$ and $k+1$, respectively. Thus by (5), the number of possible trees $Y''$ is $(kp - (n-m) + 1) \cdot y(n-m, k)$. To recover $Y$ is to relabel all the vertices except 1 of $Y''$ with the label set $\{2, 3, \ldots, n+1\} \setminus \{\ell_1, \ldots, \ell_m\}$ and to attach the leaves $\ell_1, \ldots, \ell_m$ to the vertex 1 of $Y''$. Clearly $\ell_1, \ldots, \ell_m$ is a subset of $\{2, 3, \ldots, n+1\}$. It is obvious that a way of attaching $\ell_1, \ldots, \ell_m$ to vertex 1 can be regarded as an injection from $\ell_1, \ldots, \ell_m$ to $[p]$. Thus the number of $Y$ without the leaf 1 is

$$\binom{n}{m} \binom{p}{m} m! (kp - (n-m) + 1) \cdot y(n-m, k).$$

Since $m$ may be the number from 1 to $p$ and substituting $m = 0$ in (6) yields (5), we have the recursion (2).
A refined enumeration of $p$-ary labeled trees

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Table 1. $y(n, k)$ with $p = 2$

Since $\mathcal{Y}^{(p)}_{n, k}$ is the set of decreasing $p$-ary trees on $[n]$, the equation (3) holds (see [1]). For an arbitrary tree $Y \in \mathcal{Y}^{(p)}_{n, k}$, MD($Y$) consists of $k$ vertices. So MD($Y$) has $pk - (k - 1)$ locations to attach $n - k$ increasing leaves. Thus, if the inequality $pk - (k - 1) < n - k$ holds, then $\mathcal{Y}^{(p)}_{n, k}$ should be empty. For $n \geq 1$ and $k = 0$, it is obvious that $\mathcal{Y}^{(p)}_{n, k}$ is also empty. These give the equation (4).

The sequence $y(n, k)$ with $p = 2$ is shown in Table 1.

Now we calculate $f(n, k)$ which is the number of forests on $[n]$ consisting of $k$ $p$-ary trees, where the $k$ components are not ordered. Here we use the convention that the empty product is 1.

**Lemma 2.2.** For $0 \leq k \leq n$, we have

\[
 f(n, k) = \binom{n}{k} pk \prod_{i=1}^{n-k-1} (pn - i) \quad \text{if } n > k, \]

else $f(n, n) = 1$.

**Proof.** Consider a forest $F$ in $\mathcal{F}^{(p)}_{n,k}$. The forest $F$ consists of (non-ordered) $p$-ary trees $T_1, \ldots, T_k$ with roots $r_1, r_2, \ldots, r_k$, where $r_1 < r_2 < \cdots < r_k$. The number of ways for choosing roots $r_1, r_2, \cdots, r_k$ from $[n]$ is equal to $\binom{n}{k}$. From the reverse Prüfer algorithm (RP Algorithm) in [3], the number of ways for adding $n - k$ vertices successively to $k$ roots $r_1, r_2, \cdots, r_k$ is equal to

\[
 pk(pn - 1)(pn - 2) \cdots (pn - n + k + 1) \]
for $0 < k < n$, thus the equation (7) holds. For $0 = k < n$, $\mathcal{F}_{n,0}^{(p)}$ is empty, so $f(n,0) = 0$ included in (7). For $0 \leq k = n$, $\mathcal{F}_{n,n}^{(p)}$ is the set of forests with no edges, so $f(n,n) = 1$.

Since the number $y(n,k)$ is determined by the recurrence relation (2) in Lemma 2.1, we can count the number $t(n,k)$ with the following theorem.

**Theorem 2.3.** For $n \geq 1$, we have

$$(8) \quad t(n,k) = \sum_{m=k}^{n} \binom{n}{m} \frac{m-k}{n-k} (pn-pk)_{(n-m)} \cdot y(m,k) \quad \text{if} \quad 1 \leq k < n,$$

else $t(n,n) = \prod_{j=0}^{n-1} (pj - j + 1)$, where $a(\ell):=a(a-1)\cdots(a-\ell+1)$ is a falling factorial.

**Proof.** Given a $p$-ary tree $T$ in $\mathcal{T}_{n,k}^{(p)}$, let $Y$ be the subtree of $T$ consisting of $\text{MD}(T)$ and its increasing children. If $Y$ has $m$ vertices, then $Y$ is a subtree of $T$ with $(m-k)$ increasing leaves. Also, the induced subgraph $Z$ of $T$ generated by the $(n-k)$ vertices not belonging to $\text{MD}(T)$ is a (non-ordered) forest consisting of $(m-k)$ $p$-ary trees whose roots are increasing leaves of $Y$. Figure 4 illustrates the subgraphs $Y$ and $Z$ of a given ternary tree $T$.

Now let us count the number of $p$-ary trees $T \in \mathcal{T}_{n,k}^{(p)}$ with $|V(Y)| = m$ where $V(Y)$ is the set of vertices in $Y$. First of all, the number of ways for selecting a set $V(Y) \subset [n]$ is equal to $\binom{n}{m}$. By attaching $(m-k)$
increasing leaves to a decreasing $p$-ary tree with $k$ vertices, we can make a $p$-ary tree on $V(Y)$. So there are exactly $y(m, k)$ ways for making such a $p$-ary subtree on $V(Y)$. Since all the roots of $Z$ are determined by $Y$, by the definition of $\mathcal{F}_{n,k}^{(p)}$ and Lemma 2.2, the number of ways for constructing the other parts on $V(T) \setminus V(\text{MD}(T))$ is equal to
\[
f(n-k, m-k) \bigg/ \binom{n-k}{m-k} = \frac{m-k}{n-k} (pn - pk)_{(n-m)}.
\]
Since the range of $m$ is $k \leq m \leq n$, the equation (8) holds.

Finally, $\mathcal{T}^{(p)}(n, n)$ is the set of decreasing $p$-ary trees on $[n]$, so
\[
t(n, n) = y(n, n) = \prod_{j=0}^{n-1} (pj - j + 1)
\]
holds for $n \geq 1$.

The sequence $t(n, k)$ with $p = 2$ is listed in Table 2. Note that each row sum is equal to $n!C_n^{(p)}$ with $p = 2$.

**Remark.** Due to Lemma 2.1 and Theorem 2.3, we can calculate $t(n, k)$ for all $n, k$. In particular we express $t(n, k)$ as a linear combination of $y(k, k), y(k+1, k), \ldots, y(n, k)$. However a closed form, a recurrence relation, or a (double) generating function of $t(n, k)$ have not been found yet.

### References


Department of Mathematics Education  
Kangwon National University  
Chuncheon 200-701, Korea  
*E-mail*: shyunseo@kangwon.ac.kr

Department of Mathematics  
Inha University  
Incheon 402-751, Korea  
*E-mail*: shin@inha.ac.kr