CHARACTERIZATIONS OF GRADED PRÜFER ⋆-MULTIPLICATION DOMAINS

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Abstract. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain graded by an arbitrary grading torsionless monoid $\Gamma$, and $\star$ be a semistar operation on $R$. In this paper we define and study the graded integral domain analogue of $\star$-Nagata and Kronecker function rings of $R$ with respect to $\star$. We say that $R$ is a graded Prüfer $\star$-multiplication domain if each nonzero finitely generated homogeneous ideal of $R$ is $\star_f$-invertible. Using $\star$-Nagata and Kronecker function rings, we give several different equivalent conditions for $R$ to be a graded Prüfer $\star$-multiplication domain. In particular we give new characterizations for a graded integral domain, to be a PrMD.

1. Introduction

Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded (commutative) integral domain graded by an arbitrary grading torsionless monoid $\Gamma$, that is $\Gamma$ is a commutative cancellative monoid (written additively). Let $\langle \Gamma \rangle = \{a - b | a, b \in \Gamma\}$, be the quotient group of $\Gamma$, which is a torsion-free abelian group.

Let $H$ be the saturated multiplicative set of nonzero homogeneous elements of $R$. Then $R_H = \bigoplus_{\alpha \in \langle \Gamma \rangle} (R_H)_\alpha$, called the homogeneous quotient
field of $R$, is a graded integral domain whose nonzero homogeneous elements are units. For a fractional ideal $I$ of $R$ let $I_h$ denote the fractional ideal generated by the set of homogeneous elements of $R$ in $I$. It is known that if $I$ is a prime ideal, then $I_h$ is also a prime ideal (cf. [29, Page 124]). An integral ideal $I$ of $R$ is said to be homogeneous if $I = \bigoplus_{\alpha \in \Gamma}(I \cap R_{\alpha})$; equivalently, if $I = I_h$. A fractional ideal $I$ of $R$ is homogenous if $sI$ is an integral homogeneous ideal of $R$ for some $s \in H$ (thus $I \subseteq R_H$). For $f \in R_H$, let $C_R(f)$ (or simply $C(f)$) denote the fractional ideal of $R$ generated by the homogeneous components of $f$. For a fractional ideal $I$ of $R$ with $I \subseteq R_H$, let $C(I) = \sum_{f \in I}C(f)$. For more on graded integral domains and their divisibility properties, see [3,29].

Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ and $N_v(H) = \{f \in R | C(f)^v = R\}$. (Definitions related to the $v$-operation will be reviewed in the sequel.) Then $N_v(H)$ is a saturated multiplicative subset of $R$. The graded integral domain analogue of the well known Nagata ring is the ring $R_{N_v(H)}$. In [4], Anderson and Chang, studied relationships between the ideal-theoretic properties of $R_{N_v(H)}$ and the homogeneous ideal-theoretic properties of $R$. For example it is shown that if $R$ has a unit of nonzero degree, $\text{Pic}(R_{N_v(H)}) = 0$ and that $R$ is a $PvMD$ if and only if each ideal of $R_{N_v(H)}$ is extended from a homogeneous ideal of $R$, if and only if $R_{N_v(H)}$ is a Prüfer (or Bézout) domain [4, Theorems 3.3 and 3.4]. Also, they generalized the notion of Kronecker function ring, (for e.a.b. star operations on $R$) and then showed that this ring is a Bézout domain [4, Theorem 3.5]. For the definition and properties of semistar-Nagata and Kronecker function rings of an integral domain see the interesting survey article [21]. Recall that the Picard group (or the ideal class group) of an integral domain $D$, is $\text{Pic}(D) = \text{Inv}(D)/\text{Prin}(D)$, where $\text{Inv}(D)$ is the multiplicative group of invertible fractional ideals of $D$, and $\text{Prin}(D)$ is the subgroup of principal fractional ideal of $D$.

Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be an integral domain, and $\star$ be a semistar operation on $R$. In Section 2 of this paper we study the homogeneous elements of $Q\text{Spec}^\star(R)$ denoted by $hQ\text{Spec}^\star(R)$. We show that if $\star$ is a finite type semistar operation on $R$ which sends homogeneous fractional ideals to homogeneous ones, and such that $R^\star \subseteq R_H$, then each homogeneous quasi-$\star$-ideal of $R$, is contained in a homogeneous quasi-$\star$-prime ideal of $R$. One of key results in this paper is Proposition 2.3 which shows that if $R^\star \subseteq R_H$, the $\tilde{\star}$ sends homogeneous fractional ideals to homogeneous ones. We also define and study the Nagata ring of $R$ with
respect to $\star$. The $\star$-Nagata ring is defined by the quotient ring $R_{N_*(H)}$, where $N_*(H) = \{ f \in R | C(f)^\star = R^\star \}$. Among other things, it is shown that $\text{Pic}(R_{N_*(H)}) = 0$. In Section 3 we define and study the Kronecker function ring of $R$ with respect to $\star$. The Kronecker function ring, inspired by [20, Theorem 5.1], is defined by $\text{Kr}(R, \star) := \{ 0 \} \cup \{ f/g | 0 \neq f, g \in R \}$, and there is $0 \neq h \in R$ such that $C(f)C(h) \subseteq (C(g)C(h))^\star$. It is shown that if $\star$ sends homogeneous fractional ideals to fractional ones, then $\text{Kr}(R, \star)$ is a Bézout domain. In Section 3 we define the notion of graded Prüfer $\star$-multiplication domains and give several different equivalent conditions to be a graded $P\star$MD. A graded integral domain $R$, is called a graded Prüfer $\star$-multiplication domain (graded $P\star$MD) if every finitely generated homogeneous ideal of $R$ is a $\star$-invertible, i.e., $(II^{-1})^\star = R^\star$ for each finitely generated homogeneous ideal $I$ of $R$. Among other results we show that $R$ is a graded $P\star$MD if and only if $R_{N_*(H)}$ is a Prüfer domain if and only if $R_{N_*(H)}$ is a Bézout domain if and only if $R_{N_*(H)} = \text{Kr}(R, \tilde{\star})$ if and only if $\text{Kr}(R, \tilde{\star})$ is a flat $R$-module.

To facilitate the reading of the paper, we review some basic facts on semistar operations. Let $D$ be an integral domain with quotient field $K$. Let $\mathcal{F}(D)$ denote the set of all nonzero $D$-submodules of $K$. Let $\mathcal{F}(D)$ be the set of all nonzero fractional ideals of $D$; i.e., $E \in \mathcal{F}(D)$ if $E \in \mathcal{F}(D)$ and there exists a nonzero element $r \in D$ with $rE \subseteq D$. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of $D$. Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \mathcal{F}(D)$. As in [30], a semistar operation on $D$ is a map $\star : \mathcal{F}(D) \to \mathcal{F}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \mathcal{F}(D)$, the following three properties hold:

$\star_1 : (xE)^\star = xE^\star$;

$\star_2 : E \subseteq F$ implies that $E^\star \subseteq F^\star$;

$\star_3 : E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Let $\star$ be a semistar operation on the domain $D$. For every $E \in \mathcal{F}(D)$, put $E^{\star f} := \cup F^\star$, where the union is taken over all finitely generated $F \in f(D)$ with $F \subseteq E$. It is easy to see that $\star f$ is a semistar operation on $D$, and $\star f$ is called the semistar operation of finite type associated to $\star$. Note that $(\star f)_f = \star_f$. A semistar operation $\star$ is said to be of finite type if $\star = \star_f$; in particular $\star f$ is of finite type. We say that a nonzero ideal $I$ of $D$ is a quasi-\star-ideal of $D$, if $I^\star \cap D = I$; a quasi-\star-prime (ideal of $D$), if $I$ is a prime quasi-\star-ideal of $D$; and a quasi-\star-maximal (ideal of $D$), if $I$ is maximal in the set of all proper quasi-\star-ideals of $D$. Each quasi-\star-maximal ideal is a prime ideal. It was shown in [16, Lemma 4.20] that
if $D^* \neq K$, then each proper quasi-$\ast_f$-ideal of $D$ is contained in a quasi-$\ast_f$-maximal ideal of $D$. We denote by $\text{QMax}^*(D)$ (resp., $\text{QSpec}^*(D)$) the set of all quasi-$\ast$-maximal ideals (resp., quasi-$\ast$-prime ideals) of $D$.

If $\ast_1$ and $\ast_2$ are semistar operations on $D$, one says that $\ast_1 \leq \ast_2$ if $E^{\ast_1} \subseteq E^{\ast_2}$ for each $E \in \mathcal{F}(D)$ (cf. [30, page 6]). This is equivalent to saying that $(E^{\ast_1})^{\ast_2} = (E^{\ast_2})^{\ast_1}$ for each $E \in \mathcal{F}(D)$ (cf. [30, Lemma 16]). Obviously, for each semistar operation $\ast$ defined on $D$, we have $\ast_f \leq \ast$. Let $d_D$ (or, simply, $d$) denote the identity (semi)star operation on $D$. Clearly, $d_D \leq \ast$ for all semistar operations $\ast$ on $D$.

It has become standard to say that a semistar operation $\ast$ is stable if $(E \cap F)^\ast = E^\ast \cap F^\ast$ for all $E, F \in \mathcal{F}(D)$. (“Stable” has replaced the earlier usage, “quotient”, in [30, Definition 21].) Given a semistar operation $\ast$ on $D$, it is possible to construct a semistar operation $\tilde{\ast}$, which is stable and of finite type defined as follows: for each $E \in \mathcal{F}(D)$,

$$E^\tilde{\ast} := \{ x \in K | xJ \subseteq E, \text{ for some } J \subseteq R, J \in f(R), J^\ast = D^\ast \}.$$

It is well known that [16, Corollary 2.7]

$$E^\tilde{\ast} := \bigcap \{ ED_P | P \in \text{QMax}^*(D) \}, \text{ for each } E \in \mathcal{F}(D).$$

The most widely studied (semi)star operations on $D$ have been the identity $d, v, t := v_f$, and $w := \tilde{v}$ operations, where $A^v := (A^{-1})^{-1}$, with $A^{-1} := (R : A) := \{ x \in K | xA \subseteq D \}$.

Let $\ast$ be a semistar operation on an integral domain $D$. We say that $\ast$ is an e.a.b. (endlich arithmetisch brauchbar) semistar operation of $D$ if, for all $E, F, G \in \mathcal{F}(D)$, $(EF)^\ast \subseteq (EG)^\ast$ implies that $F^\ast \subseteq G^\ast$ ([20, Definition 2.3 and Lemma 2.7]). We can associate to any semistar operation $\ast$ on $D$, an e.a.b. semistar operation of finite type $\ast_a$ on $D$, called the e.a.b. semistar operation associated to $\ast$, defined as follows for each $F \in \mathcal{F}(D)$ and for each $E \in \mathcal{F}(D)$:

$$F^{\ast_a} := \bigcup \{ ((FH)^\ast : H^\ast) | H \in f(R) \},$$

$$E^{\ast_a} := \bigcup \{ F^{\ast_a} | F \subseteq E, F \in \mathcal{F}(D) \}.$$

[20, Definition 4.4 and Proposition 4.5] (note that $((FH)^\ast : H^\ast) = ((FH)^\ast : H)$). It is known that $\ast_f \leq \ast_a$ [20, Proposition 4.5(3)]. Obviously $(\ast_f)_a = \ast_a$. Moreover, when $\ast = \ast_f$, then $\ast$ is e.a.b. if and only if $\ast = \ast_a$ [20, Proposition 4.5(5)].

Let $\ast$ be a semistar operation on a domain $D$. Recall from [17] that, $D$ is called a Prüfer $\ast$-multiplication domain (for short, a $\text{P} \ast \text{MD}$) if each
Prüfer $\star$-multiplication domains

finitely generated ideal of $D$ is $\star_f$-invertible; i.e., if $(II^{-1})^\star = D^\star$ for all $I \in f(D)$. When $\star = v$, we recover the classical notion of PvMD; when $\star = d_D$, the identity (semi)star operation, we recover the notion of Prüfer domain.

2. Nagata ring

Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, $\star$ be a semistar operation on $R$, $H$ be the set of nonzero homogeneous elements of $R$. An overring $T$ of $R$, with $R \subseteq T \subseteq R_H$ will be called a homogeneous overring if $T = \bigoplus_{a \in \Gamma} (T \cap (R_H)_a)$. Thus $T$ is a graded integral domain with $T_a = T \cap (R_H)_a$.

In this section we study the homogeneous elements of $Q\text{Spec}^\star(R)$, denoted by $h\text{-}Q\text{Spec}^\star(R)$, and the graded integral domain analogue of $\star$-Nagata ring. Let $h\text{-}Q\text{Max}^\star(R)$ denote the set of ideals of $R$ which are maximal in the set of all proper homogeneous quasi-$\star$-ideals of $R$. The following lemma shows that, if $R^\star \subseteq R_H$ and $\star = \star_f$ sends homogeneous fractional ideals to homogeneous ones, then $h\text{-}Q\text{Max}^\star_f(R)$ is nonempty and each proper homogeneous quasi-$\star_f$-ideal is contained in a maximal homogeneous quasi-$\star_f$-ideal.

**Lemma 2.1.** Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, $\star$ a finite type semistar operation on $R$ which sends homogeneous fractional ideals to homogeneous ones, and such that $R^\star \subseteq R_H$. If $I$ is a proper homogeneous quasi-$\star$-ideal of $R$, then $I$ is contained in a proper homogeneous quasi-$\star$-prime ideal.

**Proof.** Let $X := \{I|I$ is a homogeneous quasi-$\star$-ideal of $R\}$. Then it is easy to see that $X$ is nonempty. Indeed, in this case $R^\star$ is a homogeneous overring of $R$, and if $u \in H$ is a nonunit in $R^\star$, then $uR^\star \cap R$ is a proper homogeneous quasi-$\star$-ideal of $R$. Also $X$ is inductive (see proof of [16, Lemma 4.20]). From Zorn’s Lemma, we see that every proper homogeneous quasi-$\star$-ideal of $R$ is contained in some maximal element $Q$ of $X$.

Now we show that $Q$ is actually prime. Take $f, g \in H \setminus Q$ and suppose that $fg \in Q$. By the maximality of $Q$ we have $(Q, f)^\star = R^\star$ (note that $(Q, f)^\star \cap R$ is a homogeneous quasi-$\star$-ideal of $R$ and properly contains $Q$). Since $\star$ is of finite type, we can find a finitely generated ideal $J \subseteq Q$
such that \((J,f)^* = R^*\). Then \(g \in gR^* \cap R = g(J,f)^* \cap R \subseteq Q^* \cap R = Q\), a contradiction. Thus \(Q\) is a prime ideal. \(\blacksquare\)

The following example shows that we cannot drop the condition that \(\ast\) sends homogeneous fractional ideals to homogeneous ones, in the above lemma.

**Example 2.2.** Let \(k\) be a field and \(X, Y\) be indeterminates over \(k\). Let \(R = k[X,Y]\), which is a \((\mathbb{N}_0)\)-graded Noetherian integral domain with \(\deg X = \deg Y = 1\). Set \(M := (X,Y + 1)\) which is a maximal non-homogeneous ideal of \(R\). Let \(T\) be a DVR \([14]\), with maximal ideal \(N\), dominating the local ring \(R_M\). If \(R_H \subseteq T\), then there exists a prime ideal \(P\) of \(R\) such that \(P \cap H = \emptyset\) and \(N \cap R_H = PR_H\). Thus \(M = N \cap R = N \cap R_H \cap R = PR_H \cap R = P\). Hence \(M \cap H = \emptyset\), which is a contradiction, since \(X \in M \cap H\). So that, \(R_H \not\subseteq T\). Let \(\ast\) be a semistar operation on \(R\) defined by \(E^\ast = ET \cap ER_H\) for each \(E \in \mathcal{F}(R)\). Then clearly \(\ast = \ast_f\) and \(R^\ast \subseteq R_H\). If \(P\) is a nonzero prime ideal of \(R\), such that \(P \cap H = \emptyset\), then \(P^\ast \cap R = PT \cap PR_H \cap R = PT \cap P = P\). Thus \(P\) is a quasi-\(\ast_f\)-prime ideal.

On the other hand if \(P\) is any nonzero prime ideal of \(R\) such that \(P \cap H \neq \emptyset\), then \(PT = N^k\), for some integer \(k \geq 1\). Therefore, if we assume that \(P\) is a quasi-\(\ast_f\)-ideal of \(R\), then we would have \(P = PT \cap PR_H \cap R = PT \cap R = N^k \cap R \supseteq M^k\), which implies that \(P = M\). Thus \(\text{QSpec}^\ast(R) = \{M\} \cup \{P \in \text{Spec}(R) | P \neq 0\} \text{ and } P \cap H = \emptyset\). Therefore by \([16]\) Lemma 4.1, Remark 4.5], we have \(\text{QSpec}^\ast(R) = \{Q \in \text{Spec}(R) | 0 \neq Q \subseteq M\} \cup \{P \in \text{Spec}(R) | P \neq 0\} \text{ and } P \cap H = \emptyset\). Hence in the present example we have \(h\text{-QSpec}^\ast(R) = h\text{-QMax}^\ast(R) = \emptyset\), and \(h\text{-QSpec}^\ast(R) = h\text{-QMax}^\ast(R) = \{(X)\}\). Note that in this example \(h\text{-QMax}^\ast(R) \not\subseteq \text{QMax}^\ast(R) = \text{QMax}^\ast(R)\).

From now on in this paper, we are interested and consider, the semistar operations \(\ast\) on \(R\), such that \(R^\ast \subseteq R_H\) and sends homogeneous fractional ideals to homogeneous ones. For any such semistar operation, if \(I\) is a homogeneous ideal of \(R\), we have \(I^\ast = R^\ast\) if and only if \(I \not\subseteq Q\) for each \(Q \in h\text{-QMax}^\ast(R)\). Also if \(P\) is a quasi-\(\ast\)-prime ideal of \(R\), then either \(P_h = 0\) or \(P_h\) is a quasi-\(\ast\)-prime ideal of \(R\). Indeed, if \(P_h \neq 0\), then \(P_h \subseteq (P_h)^* \cap R \subseteq P^* \cap R = P\), which implies that \(P_h = (P_h)^* \cap R\), since \((P_h)^* \cap R\) is a homogeneous ideal.

The following proposition is the key result in this paper.

**Proposition 2.3.** Let \(R = \bigoplus_{\alpha \in \Gamma} R_\alpha\) be a graded integral domain, and \(\ast\) be a semistar operation on \(R\) such that \(R^\ast \subseteq R_H\). Then, \(\ast\) sends
homogeneous fractional ideals to homogeneous ones. In particular \( h\text{-}QMax^*(R) \neq \emptyset \), and \( R^* \) is a homogeneous overring of \( R \).

**Proof.** Let \( E \) be a homogenous fractional ideal of \( R \). To show that \( E^\star \) is homogeneous let \( f \in E^\star \). Then \( fJ \subseteq E \) for some finitely generated ideal \( J \) of \( R \) such that \( J^* = R^* \). Suppose that \( J = (g_1, \ldots, g_n) \). Using [4, Lemma 1.1(1)], there is an integer \( m \geq 1 \) such that \( C(g_i)^{m+1}C(f) = C(g_i)C(fg_i) \) for all \( i = 1, \ldots, n \). Since \( E \) is a homogeneous fractional ideal and \( fg_i \in E \), we have \( C(fg_i) \subseteq E \). Thus we have \( C(g_i)^{m+1}C(f) \subseteq E \). Let \( J_0 := C(g_1)^{m+1} + \cdots + C(g_n)^{m+1} \). Thus \( J_0^* = R^* \). Since \( C(f)J_0 \subseteq E \), \( C(f) \subseteq E^\star \). Therefore \( E^\star \) is a homogeneous ideal.

**Lemma 2.4.** Let \( R = \bigoplus_{a \in \Gamma} R_a \) be a graded integral domain, \( \star \) a semistar operation on \( R \) which sends homogeneous fractional ideals to homogeneous ones. Then \( \star \) sends homogeneous fractional ideals to homogeneous ones.

**Proof.** Let \( E \) be a homogenous fractional ideal of \( R \). Let \( 0 \neq x \in E^{\star \prime} \). Then, there exists an \( F \in f(R) \) such that \( F \subseteq E \) and \( x \in F^\star \). Suppose that \( F \) is generated by \( y_1, \ldots, y_n \in R_H \). Let \( G \) be a homogeneous fractional ideal of \( R \), generated by homogeneous components of \( y_1, \ldots, y_n \). Note that \( F \subseteq G \subseteq E \). Thus \( \star \) sends homogeneous fractional ideals to homogeneous ones.

Note that the \( v \)-operation sends homogeneous fractional ideals to homogeneous ones by [3, Proposition 2.5]. Using the above two results, the \( t \) and \( w \)-operations also, send homogeneous fractional ideals to homogeneous ones.

It is well-known that \( \text{QMax}^{\star \prime}(R) = \text{QMax}^\star(R) \), see [5, Theorem 2.16], for star operation case, and [18, Corollary 3.5(2)], in general semistar operations. Although Example 2.2 shows that it may happen that \( h\text{-}\text{QMax}^{\star \prime}(R) \neq h\text{-}\text{QMax}^\star(R) \), we have the following proposition whose proof is almost the same as [4, Theorem 2.16].

**Proposition 2.5.** Let \( R = \bigoplus_{a \in \Gamma} R_a \) be a graded integral domain, \( \star \) a semistar operation on \( R \) such that \( R^\star \subseteq R_H \), which sends homogeneous fractional ideals to homogeneous ones. Then \( h\text{-}\text{QMax}^{\star \prime}(R) = h\text{-}\text{QMax}^\star(R) \).
Proof. Assume that \( Q \in h\text{-}Q\text{Max}^{*\ell}(R) \). Then since \( \tilde{\ast} \leq \ast_f \) by [18 Lemma 2.7(1)], we have \( Q \subseteq Q^* \cap R \subseteq Q^{*\ell} \cap R = Q \), that is \( Q \) is a quasi-\( \tilde{\ast} \)-ideal. Suppose that \( Q \notin h\text{-}Q\text{Max}^{\tilde{x}}(R) \). Then \( Q \) is properly contained in some \( P \in h\text{-}Q\text{Max}^{\tilde{x}}(R) \). So since \( Q \in h\text{-}Q\text{Max}^{*\ell}(R) \), using Lemma 2.1 we must have \( P^{*\ell} = R^{*\ell} \). Thus there is some finitely generated ideal \( F \subseteq P \) such that \( F^* = R^* \). So for any \( r \in R \), \( rF \subseteq F \subseteq P \). But then, \( r \in P^\ast \), so \( R \subseteq P^\ast \), which implies that \( P^\ast = R^\ast \), a contradiction. Therefore, we must have \( Q \in h\text{-}Q\text{Max}^{\tilde{x}}(R) \).

If \( Q \in h\text{-}Q\text{Max}^{\tilde{x}}(R) \), then \( Q = Q^* \cap R \subseteq Q^{*\ell} \cap R \subseteq R \). Suppose that \( Q^{*\ell} \cap R = R \), which implies that \( Q^{*\ell} = R^* \). Then there is a finitely generated ideal \( F \subseteq Q \) such that \( F^* = R^* \). Now for any \( r \in R \), \( rF \subseteq F \subseteq Q \). Therefore \( R \subseteq Q^\ast \), and so \( R = Q^\ast \cap R = Q \), which is a contradiction. So \( Q^{*\ell} \cap R \subseteq R \). Now, since \( Q^{*\ell} \cap R \) is a homogeneous quasi-\( \ast_f \)-ideal, there is \( P \in h\text{-}Q\text{Max}^{\ast\ell}(R) \) such that \( Q \subseteq P^{*\ell} \cap R \subseteq P \). From the first half of the proof, we know that \( P \in h\text{-}Q\text{Max}^{\tilde{x}}(R) \). So we must have \( P = Q \). Therefore \( Q \in h\text{-}Q\text{Max}^{\ast\ell}(R) \). \( \square \)

Park in [31, Lemma 3.4], proved that \( I^u = \bigcap_{P \in h\text{-}Q\text{Max}^{\ast\ell}(R)} IR_{H \setminus P} \) for each homogeneous ideal \( I \) of \( R \).

**Proposition 2.6.** Let \( R = \bigoplus_{n \in \mathbb{T}} R_n \) be a graded integral domain, \( \ast \) a semistar operation on \( R \) such that \( R^* \subseteq R_H \). Then
\[
\tilde{I}^\ast = \bigcap_{P \in h\text{-}Q\text{Max}^{\tilde{x}}(R)} IR_{H \setminus P}
\]
for each homogeneous ideal \( I \) of \( R \). Moreover
\[
\tilde{I}^\ast R_{H \setminus P} = IR_{H \setminus P} \quad \text{for all homogeneous ideal } I \text{ of } R \text{ and all } P \in h\text{-}Q\text{Max}^{\tilde{x}}(R).
\]

**Proof.** By Proposition 2.3, \( \tilde{I}^\ast \) is a homogeneous ideal. Also note that \( \bigcap_{P \in h\text{-}Q\text{Max}^{\ast\ell}(R)} IR_{H \setminus P} \) is a homogeneous ideal of \( R \). Let \( f \in \tilde{I}^\ast \) be homogeneous. Then \( fJ \subseteq I \) for some homogeneous finitely generated ideal \( J \) of \( R \) such that \( J^* = R^* \). It is easy to see that \( J^* = R^\ast \). Hence we have \( J \not\subseteq P \) for all \( P \in h\text{-}Q\text{Max}^{\tilde{x}}(R) \). Thus \( f \in IR_{H \setminus P} \) for all \( P \in h\text{-}Q\text{Max}^{\tilde{x}}(R) \). Conversely, let \( f \in \bigcap_{P \in h\text{-}Q\text{Max}^{\ast\ell}(R)} IR_{H \setminus P} \) be homogeneous. Then \( (I : f) \) is a homogeneous ideal which is not contained in any \( P \in h\text{-}Q\text{Max}^{\tilde{x}}(R) \). Therefore \( (I : f)^\ast = R^\ast \). So that there exist a finitely generated ideal \( J \subseteq (I : f) \) such that \( J^* = R^* \). Thus \( fJ \subseteq I \), i.e., \( f \in \tilde{I}^\ast \). The second assertion follows from the first one. \( \square \)

Let \( D \) be a domain with quotient field \( K \), and let \( X \) be an indeterminate over \( K \). For each \( f \in K[X] \), we let \( c_D(f) \) denote the content of
the polynomial \( f \), i.e., the (fractional) ideal of \( D \) generated by the coefficients of \( f \). Let \( \star \) be a semistar operation on \( D \). If \( N_\star := \{ g \in D[X] | g \neq 0 \text{ and } c_D(g)^\star = D^\star \} \), then \( N_\star = D[X] \setminus \cup \{ P[X] | P \in \text{QMax}^{\star\prime}(D) \} \) is a saturated multiplicative subset of \( D[X] \). The ring of fractions

\[
\text{Na}(D, \star) := D[X]_{N_\star}
\]

is called the \( \star \)-Nagata domain (of \( D \) with respect to the semistar operation \( \star \)). When \( \star = d \), the identity (semi)star operation on \( D \), then \( \text{Na}(D, d) \) coincides with the classical Nagata domain \( D(X) \) (as in, for instance [28 page 18], [23 Section 33] and [18]).

Let \( N_\star(H) = \{ f \in R(C(f)^\star = R^\star) \} \). It is easy to see that \( N_\star(H) \) is a saturated multiplicative subset of \( R \). Indeed assume \( f, g \in N_\star(H) \). Then \( C(f)^{n+1}C(g) = C(f)^nC(fg) \) for some integer \( n \geq 1 \) by [4 Lemma 1.1(2)], and \( C(fg) \subseteq C(f)C(g) \). Thus \( fg \in N_\star(H) \iff C(fg)^\star = R^\star \iff C(f)^\star = C(g)^\star = R^\star \iff f, g \in N_\star(H) \). Also it is easy to show that \( N_\star(H) = N_\star(H) = N_\star(H) \). We define the graded integral domain analogue of \( \star \)-Nagata ring, by the quotient ring \( R_{N_\star(H)} \). When \( \star = v \), \( R_{N_\star(H)} \) was studied in [4], denoted by \( R_{N_\star(H)} \).

**Lemma 2.7.** Let \( R = \bigoplus_{n \in I} R_n \) be a graded integral domain, and \( \star \) be a semistar operation on \( R \) such that \( R^\star \subseteq R_H \), which sends homogeneous fractional ideals to homogeneous ones.

1. \( N_\star(H) = R \setminus \bigcup_{Q \in h\text{-QMax}^{\star\prime}(R)} Q \).
2. \( \text{Max}(R_{N_\star(H)}) = \{ hR_{N_\star(H)} | Q \in h\text{-QMax}^{\star\prime}(R) \} \) if and only if \( R \) has the property that if \( I \) is a nonzero ideal of \( R \) with \( C(I)^\star = R^\star \), then \( I \cap N_\star(H) \neq \emptyset \).

**Proof.** (1) Let \( x \in R \). Then \( x \in N_\star(H) \iff C(x)^\star = R^\star \iff C(x) \nsubseteq Q \) for all \( Q \in h\text{-QMax}^{\star\prime}(R) \iff x \nsubseteq Q \) for all \( Q \in h\text{-QMax}^{\star\prime}(R) \iff x \in R \setminus \bigcup_{Q \in h\text{-QMax}^{\star\prime}(R)} Q \).

(2) \((\Rightarrow)\) Let \( I \) be a nonzero ideal of \( R \) with \( C(I)^\star = R^\star \). Then \( I \nsubseteq Q \) for all \( Q \in h\text{-QMax}^{\star\prime}(R) \), and hence \( IR_{N_\star(H)} = R_{N_\star(H)} \). Thus \( I \cap N_\star(H) \neq \emptyset \).

\((\Leftarrow)\) Let \( I \) be a nonzero ideal of \( R \) such that \( I \subseteq \bigcup_{Q \in h\text{-QMax}^{\star\prime}(R)} Q \). If \( C(I)^{\star\prime} = R^\star \), then, by assumption, there exists an \( f \in I \) with \( C(f)^\star = R^\star \). But, since \( I \subseteq \bigcup_{Q \in h\text{-QMax}^{\star\prime}(R)} Q \), we have \( f \in Q \) for some \( Q \in h\text{-QMax}^{\star\prime}(R) \), a contradiction. Thus \( C(I)^\star \nsubseteq R^\star \), and hence \( I \subseteq Q \) for some \( Q \in h\text{-QMax}^{\star\prime}(R) \). Thus \( \{ hQ_{N_\star(H)} | Q \in h\text{-QMax}^{\star\prime}(R) \} \) is the set of maximal ideals of \( R_{N_\star(H)} \) by [23 Proposition 4.8].

\[\square\]
We will say that $R$ satisfies property $(\#_*)$ if, for any nonzero ideal $I$ of $R$, $C(I)^* = R^*$ implies that there exists an $f \in I$ such that $C(f)^* = R^*$.

**Example 2.8.** Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, and let $*$ be a semistar operation on $R$. If $R$ contains a unit of nonzero degree, then $R$ satisfies property $(\#_*)$ (see [4, Example 1.6] for the case $* = t$).

The next result is a generalization of the fact that $\tilde{I}^* = I \cap K$, where $K$ is the quotient field of $R$ [18, Proposition 3.4(3)].

**Lemma 2.9.** Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, and $*$ be a semistar operation on $R$ such that $R^* \subsetneq R_H$, with property $(\#_*)$. Then $\tilde{I}^* = IR_{N_*(H)} \cap R_H$ and $I^* R_{N_*(H)} = IR_{N_*(H)}$ for each homogeneous ideal $I$ of $R$. In particular $\tilde{I}^*$ is integrally closed if and only if $R_{N_*(H)}$ is integrally closed.

**Proof.** If $\tilde{I}^* = IR_{N_*(H)} \cap R_H$, then it is easy to see that $\tilde{I}^* R_{N_*(H)} = IR_{N_*(H)}$. Hence it suffices to show that $\tilde{I}^* = IR_{N_*(H)} \cap R_H$.

$(\subseteq)$ Let $f \in I^*(\subseteq R_H)$, and let $J$ be a finitely generated ideal of $R$ such that $J^* = R^*$ and $f J \subseteq I$. Then $C(J)^* = R^*$, and since $R$ satisfies property $(\#_*)$, there exists an $h \in J$ with $C(h)^* = R^*$. Hence $h \in N_*(H)$ and $fh \in I$. Thus $f \in IR_{N_*(H)} \cap R_H$.

$(\supseteq)$ Let $f = \frac{g}{h} \in IR_{N_*(H)} \cap R_H$, where $g \in I$ and $h \in N_*(H)$. Then $fh = g \in I$, and since $C(h)^{m+1} C(f) = C(h)^m C(fh)$ for some integer $m \geq 1$ by [4, Lemma 1.1(1)], we have $f C(h)^{m+1} \subseteq C(f) C(h)^{m+1} = C(h)^m C(fh) = C(h)^m C(g) \subseteq I$. Also note that $(C(h)^{m+1})^* = R^*$, since $C(h)^* = R^*$. Thus $f \in I^*$.

For the in particular case, assume that $R_{N_*(H)}$ is integrally closed. Using [3, Proposition 2.1], $R_H$ is a GCD-domain, hence is integrally closed. Therefore $R^* = R_{N_*(H)} \cap R_H$ is integrally closed. Conversely, assume that $R^*$ is integrally closed. Then $R_Q$ is integrally closed by [14, Proposition 3.8] for all $Q \in \text{QSpec}^\ast(R)$. Let $Q R_{N_*(H)}$ be a maximal ideal of $R_{N_*(H)}$ for some $Q \in h\text{-QMax}^\ast(R)$. Then $(R_{N_*(H)})_{Q R_{N_*(H)}} = R_Q$ is integrally closed. Thus $R_{N_*(H)}$ is integrally closed. 

**Lemma 2.10.** Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, and $*$ be a semistar operation on $R$ such that $R^* \subsetneq R_H$, with property $(\#_*)$. Then for each nonzero finitely generated homogeneous ideal $I$ of $R$, $I$ is $f\text{-invertible if and only if, } IR_{N_*(H)}$ is invertible.
Proof. Let $I$ be nonzero finitely generated homogeneous ideal of $R$, such that $I$ is $\star_f$-invertible. Let $QR_{N,(H)} \in \text{Max}(R_{N,(H)})$, where $Q \in h$-QMax($R$) by Lemma 2.7(2). Thus by [22, Theorem 2.23],
\[(IR_{N,(H)})Q_{R_{N,(H)}} = IR_Q \text{ is invertible (is principal) in } R_Q. \text{ Hence } IR_{N,(H)} \text{ is invertible by [23, Theorem 7.3]. Conversely, assume that } I \text{ is finitely generated, and } IR_{N,(H)} \text{ is invertible. By flatness we have } I^{-1}R_{N,(H)} = (R : I)R_{N,(H)} = (R_{N,(H)} : IR_{N,(H)}) = (IR_{N,(H)})^{-1}. \text{ Therefore,}
\[(II^{-1})R_{N,(H)} = (IR_{N,(H)})(I^{-1}R_{N,(H)}) = (IR_{N,(H)})(IR_{N,(H)})^{-1} = R_{N,(H)}. \text{ Hence } II^{-1} \cap N_*(H) \neq \emptyset. \text{ Let } f \in II^{-1} \cap N_*(H). \text{ So that } R^* = C(f)^* \subseteq (II^{-1})^* \subseteq R^*. \text{ Thus } I \text{ is } \star_f\text{-invertible.} \]

**Corollary 2.11.** Let $R = \bigoplus_{\alpha \in \Gamma}R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on $R$ such that $R^* \subseteq R_H$, with property (\#), and $0 \neq f \in R$. Then the following conditions are equivalent:

1. $C(f)$ is $\star_f$-invertible.
2. $C(f)R_{N,(H)}$ is invertible.
3. $C(f)R_{N,(H)} = fR_{N,(H)}$.

**Proof.** Exactly is the same as [4, Corollary 1.9].

Let $\mathbb{Z}$ be the additive group of integers. Clearly, the direct sum $\Gamma \oplus \mathbb{Z}$ of $\Gamma$ with $\mathbb{Z}$ is a torsionless grading monoid. So if $y$ is an indeterminate over $R = \bigoplus_{\alpha \in \Gamma}R_\alpha$, then $R[y,y^{-1}]$ is a graded integral domain graded by $\Gamma \oplus \mathbb{Z}$. In the following proposition we use a technique for defining semistar operations on integral domains, due to Chang and Fontana [9], Theorem 2.3.6.

**Proposition 2.12.** Let $R = \bigoplus_{\alpha \in \Gamma}R_\alpha$ be a graded integral domain with quotient field $K$, let $y$, $X$ be two indeterminates over $R$ and let $\star$ be a semistar operation on $R$ such that $R^* \subseteq R_H$. Set $T := R[y,y^{-1}]$, $K_1 := K(y)$ and take the following subset of $\text{Spec}(T)$:

$\Delta^* := \{Q \in \text{Spec}(T) | Q \cap R = (0) \text{ or } Q = (Q \cap R)R[y,y^{-1}] \text{ and } (Q \cap R)^{\star} \subseteq R^* \}.$

Set $S^* := T[X]\setminus(\bigcup\{Q[X]|Q \in \Delta^*\})$ and:

$E^* := E[X]_{S^*} \cap K_1$, for all $E \in \overline{\mathcal{F}}(T)$.

(a) The mapping $\star : \overline{\mathcal{F}}(T) \to \overline{\mathcal{F}}(T)$, $E \mapsto E^*$ is a stable semistar operation of finite type on $T$, i.e., $\star \ast = \ast \star$.

(b) $(\ast \star f) = (\star \ast f) = \ast f$.

(c) $(ER[y,y^{-1}])^{\star} \cap K = E^\ast$ for all $E \in \overline{\mathcal{F}}(R)$. 

**Proof.**
(d) $(ER[y, y^{-1}])'' = E^*R[y, y^{-1}]$ for all $E \in \mathcal{F}(R)$.

(e) $T'' \subseteq T_{H'}$, where $H'$ is the set of nonzero homogeneous elements of $T$, and $\star t$ sends homogeneous fractional ideals to homogeneous ones.

(f) $\operatorname{QMax}^\star(T) = \{Q|Q \in \operatorname{Spec}(T)\text{ such that } Q \cap R = (0) \text{ and } c_R(Q)^\star = R^\star\} \cup \{PR[y, y^{-1}]|P \in \operatorname{QMax}^\star(R)\}$.

(g) $h\operatorname{QMax}^\star(T) = \{PR[y, y^{-1}]|P \in h\operatorname{QMax}^\star(R)\}$.

(h) $(w_R)t = (t_R)t = w_T$.

Proof. Set $\nabla^* := \{Q \in \operatorname{Spec}(T)| Q \cap R = (0) \text{ and } c_D(Q)^\star = R^\star\}$ or $Q = PR[y, y^{-1}]$ and $P \in \operatorname{QMax}^\star(D)$. Then it is easy to see that the elements of $\nabla^*$ are the maximal elements of $\Delta^*$ (see proof of [9, Theorem 2.3]). Thus

$$S^* := T[X]\setminus\bigcup\{Q[X]|Q \in \Delta^*\} = T[X]\setminus\bigcup\{Q[X]|Q \in \nabla^*\}.$$  

(a) It follows from [9, Theorem 2.1 (a) and (b)], that $\star t$ is a stable semistar operation of finite type on $T$.

(b) Since $\operatorname{QMax}^\star(D) = \operatorname{QMax}^\star(T)$, the conclusion follows easily from the fact that $S^* = S^\star = S^\ast$.

(c) and (d) Exactly are the same as proof of [9, Theorem 2.3(c) and (d)].

(e) From part (d) we have $T'' = R^2R[y, y^{-1}] \subseteq R_HR[y, y^{-1}] = T_{H'}$. The second assertion follows from Proposition 2.3 since $\ast t = \star t$ by (a).

(f) Follows from [9, Theorem 2.1(e)] and the remark in the first paragraph in the proof.

(g) Let $M \in h\operatorname{QMax}^\star(T)$. Since $y, y^{-1} \in T$, clearly we have $M \cap R \neq (0)$. Then by (f), there is $P \in \operatorname{QMax}^\star(R)$ such that $M \subseteq PR[y, y^{-1}]$. If $P \in h\operatorname{QMax}^\star(R)$, then $M = PR[y, y^{-1}]$ and we are done. So suppose that $P \notin h\operatorname{QMax}^\star(R)$. Then note that $P_h \in h\operatorname{QSpec}^\star(R)$ and $M \subseteq P_hR[y, y^{-1}] = (PR[y, y^{-1}])h$; hence $M = P_hR[y, y^{-1}]$, because $M$ is a homogeneous maximal quasi-$\star t$-ideal. Note that in this case $P_h \in h\operatorname{QMax}^\star(R)$ by [16, Lemma 4.1, Remark 4.5]. So that $M \in \{PR[y, y^{-1}]|P \in h\operatorname{QMax}^\star(R)\}$. The other inclusion is trivial.

(h) Suppose that $\ast t = t$. Note that if $M \in \operatorname{QMax}^\star(T)$, and $M \cap R \neq (0)$, then, $M = (M \cap R)[y, y^{-1}]$ and $M \cap R \in \operatorname{QMax}^\star(R)$. To see that $Q \cap R = (0)$, and $Q$ is a quasi-$t$-maximal ideal of $T$ if and only if $c_R(Q)^t = R$. Indeed, if $Q$ is a quasi-$t$-maximal ideal of $T$, and $c_R(Q)^t \subseteq R$, then there exists
a quasi-\( t \)-maximal ideal \( P \) of \( R \) such that \( c_R(Q)^t \subseteq P \). Hence \( Q \subseteq P[y, y^{-1}] \), and therefore \( Q = P[y, y^{-1}] \). Consequently \( (0) = Q \cap R = P[y, y^{-1}] \cap R = P \) which is a contradiction. Conversely assume that \( c_R(Q)^t = R \). Suppose \( Q \) is not a quasi-\( t \)-maximal ideal of \( T \), and let \( M \) be a quasi-\( t \)-maximal ideal of \( T \) which contains \( Q \). Since the containment is proper, we have \( M \cap R \neq (0) \). Thus \( M = (M \cap R)[y, y^{-1}] \) and \( M \cap R \in \text{QMax}^*(R) \) (cf. \[24, Proposition 1.1\]). Since \( Q \subseteq M \), \( c_R(Q) \) is contained in the quasi-\( t \)-ideal \( M \cap R \), so that \( c_R(Q)^t \neq R \) which is a contradiction. Thus we showed that \( \text{QMax}^*(T) = \{Q \mid Q \in \text{Spec}(T) \text{ such that } Q \cap R = (0) \text{ and } c_R(Q)^t = R^* \} \cup \{PR[y, y^{-1}]|P \in \text{QMax}^*(R)\} = \text{QMax}^*(T) \), where the second equality is by (f). Thus using (a) and (b), we obtain \( (w_R)^t = (t_R)^t = (v_R)^t = w_T \).

It is known that \( \text{Pic}(D(X)) = 0 \) \[1, Theorem 2\]. More generally, if \( * \) is a star operation on \( D \), then \( \text{Pic}(\text{Na}(D, *)) = 0 \) \[26, Theorem 2.14\]. Also in the graded case it is shown in \[4, Theorem 3.3\], that \( \text{Pic}(R_{N, (H)}) = 0 \), where \( R = \bigoplus_{a \in \Gamma} R_a \) is a graded integral domain containing a unit of nonzero degree. We next show in general that \( \text{Pic}(R_{N, (H)}) = 0 \).

**Theorem 2.13.** Let \( R = \bigoplus_{a \in \Gamma} R_a \) be a graded integral domain with a unit of nonzero degree, and \( * \) be a semistar operation on \( R \) such that \( R^* \subseteq R_H \). Then \( \text{Pic}(R_{N, (H)}) = 0 \).

**Proof.** Let \( y \) be an indeterminate over \( R \), and \( T = R[y, y^{-1}] \). Using Proposition 2.12(e) and (g) and Lemma 2.7, we deduce that \( \text{Max}(T_{N, (H)}) = \{QT_{N, (H)}|Q \in h-\text{QMax}^*(R)\} \). Next since \( \text{Max}((R_{N, (H)})(y)) = \{P(y)|P \text{ is a maximal ideal of } R_{N, (H)}\} \), \[23, Proposition 33.1\], we have \( \text{Max}((R_{N, (H)})(y)) = \{(QR_{N, (H)})(y)|Q \in h-\text{QMax}^*(R)\} \). Thus by a computation similar to the proof of \[4, Lemma 3.2\], we obtain the equality \( T_{N, (H)} = (R_{N, (H)})(y) \). The rest of the proof is exactly the same as proof of \[4, Theorem 3.3\], using Proposition 2.12.

Let \( D \) be a domain and \( T \) an overring of \( D \). Let \( * \) and \( *' \) be semistar operations on \( D \) and \( T \), respectively. One says that \( T \) is \( (*, *)' \)-linked overring of \( D \) if

\[ F* = D* \Rightarrow (FT)'* = T*' \]

for each nonzero finitely generated ideal \( F \) of \( D \). (The preceding definition generalizes the notion of “\( t \)-linked overring” which was introduced
It is shown in [15, Theorem 3.8], that $T$ is a $(\star, \star')$-linked overring of $D$ if and only if $N_a(D, \star) \subseteq N_a(T, \star')$. We need a graded analogue of linkedness.

Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, and $T$ be a homogeneous overring of $R$. Let $\star$ and $\star'$ be semistar operations on $R$ and $T$, respectively. We say that $T$ is homogeneously $(\star, \star')$-linked overring of $R$ if

$$F^* = D^* \Rightarrow (FT)^{\star'} = T^{\star'}$$

for each nonzero homogeneous finitely generated ideal $F$ of $R$. We say that $T$ is homogeneously $t$-linked overring of $R$ if $T$ is homogeneously $(t, t)$-linked overring of $R$. Also it can be seen that $T$ is homogeneously $(\star, \star')$-linked overring of $R$ if and only if $T$ is homogeneously $(\tilde{\star}, \star')$-linked overring of $R$ (cf. [15, Theorem 3.8]).

**Example 2.14.** Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, and let $\star$ be a semistar operation on $R$ such that $R^* \subseteq R_H$. Let $P \in h$-QSpec$(R)$. Then, $R_{H\setminus P}$ is a homogeneously $(\star, \star')$-linked overring of $R$, for all semistar operation $\star'$ on $R_{H\setminus P}$. Indeed assume that $F$ is a nonzero finitely generated homogeneous ideal of $R$ such that $F^* = R^*$. Then we have $F^* = R^*$. Thus using Proposition 2.6, we have $F R_{H\setminus P} = F^* R_{H\setminus P} = R^* R_{H\setminus P} = R_{H\setminus P}$.

**Lemma 2.15.** Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain with a unit of nonzero degree, and let $T$ be a homogeneous overring of $R$. Let $\star$ (resp. $\star'$) be a semistar operation on $R$ (resp. on $T$). Then, $T$ is a homogeneously $(\star, \star')$-linked overring of $R$ if and only if $R_{N(\star)} \subseteq T_{N(\star')}$.

**Proof.** Let $f \in R$ such that $C_R(f)^* = R^*$. Then by assumption $C_T(f)^{\star'} = (C_R(f)T)^{\star'} = R^{\star'}$. Hence $R_{N(\star)} \subseteq T_{N(\star')}$. Conversely let $F$ be a nonzero homogeneous finitely generated ideal of $R$ such that $F^* = R^*$. Since $R$ has a unit of nonzero degree we can choose an element $f \in R$ such that $C_R(f) = F$. From the fact that $C_R(f)^* = R^*$, we have that $f$ is a unit in $R_{N(\star)}$ and so by assumption, $f$ is a unit in $T_{N(\star')}$. This implies that $C_T(f)^{\star'} = (C_R(f)T)^{\star'} = T^{\star'}$, i.e., $(FT)^{\star'} = T^{\star'}$. 

3. Kronecker function ring

Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, $\star$ an e.a.b. star operation on $R$. The graded analogue of the well known Kronecker
function ring (see [23 Theorem 32.7]) of $R$ with respect to $*$ is defined by
\[
Kr(R, *) := \left\{ \frac{f}{g} \mid f, g \in R, \ g \neq 0, \ \text{and} \ C(f) \subseteq C(g)^* \right\}
\]
in [4]. The following lemma is proved in [4 Theorems 2.9 and 3.5], for an e.a.b. star operation $*$. We need to state it for e.a.b. semistar operations. Since the proof is exactly the same as star operation case, we omit the proof.

**Lemma 3.1.** Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, $*$ an e.a.b. semistar operation on $R$, and
\[
Kr(R, *) := \left\{ \frac{f}{g} \mid f, g \in R, \ g \neq 0, \ \text{and} \ C(f) \subseteq C(g)^* \right\}.
\]
Then

(1) $Kr(R, *)$ is an integral domain.

In addition, if $R$ has a unit of nonzero degree, then,

(2) $Kr(R, *)$ is a Bézout domain.

(3) $I Kr(R, *) \cap R_H = I^*$ for every nonzero finitely generated homogeneous ideal $I$ of $R$.

Inspired by the work of Fontana and Loper in [20], we can generalize this definition of $Kr(R, *)$ to all semistar operations on $R$ which send homogeneous fractional ideals, to homogeneous ones, provided that $R$ has a unit of nonzero degree. Before doing that we need a lemma.

**Lemma 3.2.** Let $R = \bigoplus_{a \in \Gamma} R_a$ be a graded integral domain, $*$ a semistar operation on $R$ which sends homogeneous fractional ideals to homogeneous ones. Suppose that $a \in R$ is homogeneous and $B, F \in f(R)$, with $B$ homogeneous and $F \subseteq R_H$, such that $aF \subseteq (BF)^*$. Then there exists a homogeneous $T \in f(R)$ such that $aT \subseteq (BT)^*$.

**Proof.** Suppose that $F$ is generated by $y_1, \cdots, y_n \in R_H$. Let $y_i = \sum t_{ij}$ be the decomposition of $y_i$ to homogeneous elements for $i = 1, \cdots, n$. Then $ay_i \in (BF)^* = (\sum y_i B)^* \subseteq (\sum t_{ij} B)^*$. Since $(\sum t_{ij} B)^*$ is homogeneous we have $at_{ij} \in (\sum t_{ij} B)^*$. Let $T$ be the fractional ideal of $R$, generated by all homogeneous elements $t_{ij}$. So that $aT \subseteq (BT)^*$ and $T \in f(R)$ is homogeneous.
Theorem 3.3. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, $\star$ a semistar operation on $R$ which sends homogeneous fractional ideals to homogeneous ones, and

$$\text{Kr}(R, \star) := \left\{ \frac{f}{g} \mid f, g \in R, g \neq 0, \text{ and there is } 0 \neq h \in R \text{ such that } C(f)C(h) \subseteq (C(g)C(h))^\star \right\}.$$  

Then

1. $\text{Kr}(R, \star) = \text{Kr}(R, \star_a)$.
2. $\text{Kr}(R, \star)$ is a Bézout domain.
3. $I \text{Kr}(R, \star) \cap R_H = I^* \star a$ for every nonzero finitely generated homogeneous ideal $I$ of $R$.
4. If $f, g \in R$ are nonzero such that $C(f + g)^\star = (C(f) + C(g))^\star$, then $(f, g) \text{Kr}(R, \star) = (f + g) \text{Kr}(R, \star)$. In particular, $f \text{Kr}(R, \star) = C(f) \text{Kr}(R, \star)$ for all $f \in R$.

Proof. It is clear from the definition that $\text{Kr}(R, \star) = \text{Kr}(R, \star_a)$. Thus using Lemma 2.4, we can assume, without loss of generality, that $\star$ is a semistar operation of finite type.

Parts (2) and (3) are direct consequences of (1) using Lemma 3.1. For the proof of (1) we have two cases:

Case 1: Assume that $\star$ is an e.a.b. semistar operation of finite type. In this case, for $f, g, h \in R \setminus \{0\}$ we have

$$C(f)C(h) \subseteq (C(g)C(h))^\star \Leftrightarrow C(f) \subseteq C(g)^\star.$$  

Therefore $\text{Kr}(R, \star)$ -as defined in this theorem- coincides with $\text{Kr}(R, \star)$ of an e.a.b. semistar operation $\star$, as defined in Lemma 3.1. Also in this case $\star = \star_a$ by [20, Proposition 4.5(5)]. Hence in this case (1) is true.

Case 2: General case. Let $\star$ be a semistar operation of finite type on $R$. By definition it is easy to see that, given two semistar operations on $R$ with $\star_1 \leq \star_2$, then $\text{Kr}(R, \star_1) \subseteq \text{Kr}(R, \star_2)$. Using [20, Proposition 4.5(3)] we have $\star \leq \star_a$. Therefore $\text{Kr}(R, \star) \subseteq \text{Kr}(R, \star_a)$. Conversely let $f/g \in \text{Kr}(R, \star_a)$. Then, by Case 1, $C(f) \subseteq C(g)^{**}$. Set $A := C(f)$ and $B := C(g)$. Then $A \subseteq B^{**} = \bigcup\{(BH)^* : H \in f(R)\}$. Suppose that $A$ is generated by homogeneous elements $x_1, \ldots, x_n \in R$. Then there is $H_i \in f(R)$, such that $x_iH_i \subseteq (BH_i)^*$ for $i = 1, \ldots, n$. Choose $0 \neq r_i \in R$ such that $F_i = r_iH_i \subseteq R$. Thus $x_iF_i \subseteq (BF_i)^*$. Therefore Lemma 3.2 gives a homogeneous $T_i \in f(R)$ such that $x_iT_i \subseteq (BT_i)^*$. Now set $T := T_1T_2 \cdots T_n$ which is a finitely generated homogeneous
fractional ideal of $R$ such that $AT \subseteq (BT)^\star$. Now since $R$ has a unit of nonzero degree, we can find an element $h \in R$ such that $C(h) = T$. Then $C(f)C(h) \subseteq (C(g)C(h))^\star$. This means that $f/g \in \text{Kr}(R, \star)$ to complete the proof of (1).

The proof of (4) is exactly the same as [4, Theorem 2.9(3)]. □

4. Graded P•MDs

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, $\star$ be a semistar operation on $R$, $H$ be the set of nonzero homogeneous elements of $R$, and $N_\star(H) = \{ f \in R \mid C(f)^\star = R^\star \}$. In this section we define the notion of graded Prüfer $\star$-multiplication domain (graded P$\star$MD for short) and give several characterization of it.

We say that a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ with a semistar operation $\star$, is a graded Prüfer $\star$-multiplication domain (graded P$\star$MD) if every nonzero finitely generated homogeneous ideal of $R$ is $\star$-invertible, i.e., $(I^{-1})^\star = R^\star$ for every nonzero finitely generated homogeneous ideal $I$ of $R$. It is easy to see that a graded P$\star$MD is the same as a graded P$\star_f$MD by definition, and is the same as a graded P$\star_v$MD by [22, Proposition 2.18]. When $\star = v$ we recover the classical notion of a graded Prüfer $v$-multiplication domain (graded P$v$MD) [2]. It is known that $R$ is a graded P$v$MD if and only if $R$ is a P$v$MD [2, Theorem 6.4].

Also when $\star = d$, a graded PdMD is called a graded Prüfer domain [4]. It is clear that every graded Prüfer domain is a graded P$v$MD and hence a P$v$MD. In particular every graded Prüfer domain is an integrally closed domain. Although $R$ is a graded P$v$MD if and only if $R$ is a P$v$MD, Anderson and Chang in [4, Example 3.6] provided an example of a graded Prüfer domain which is not Prüfer. It is known that if $A, B, C$ are ideals of an integral domain $D$, then $(A+B)(A+C)(B+C) = (A+B+C)(AB + AC + BC)$. Thus $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded Prüfer domain if and only if every nonzero ideal of $R$ generated by two homogeneous elements is invertible. We use this result in this section without comments.

The following proposition is inspired by [23, Theorem 24.3].

**Proposition 4.1.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following conditions are equivalent:

1. $R$ is a graded Prüfer domain.
Each finitely generated nonzero homogeneous ideal of $R$ is a cancelation ideal.

If $A, B, C$ are finitely generated homogeneous ideals of $R$ such that $AB = AC$ and $A$ is nonzero, then $B = C$.

$R$ is integrally closed and there is a positive integer $n > 1$ such that $(a, b)^n = (a^n, b^n)$ for each $a, b \in H$.

$R$ is integrally closed and there exists an integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)$ for each $a, b \in H$.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ and $(4) \Rightarrow (5)$ are clear.

$(3) \Rightarrow (4)$ By the same argument as in the proof of part $(2) \Rightarrow (3)$, in [23, Proposition 24.1], we have that $R$ is integrally closed in $R_H$. Therefore by [3, Proposition 5.4], $R$ is integrally closed. Now if $a, b \in H$ we have $(a, b)^3 = (a, b)(a^2, b^2)$. Thus by (3) we obtain that $(a, b)^2 = (a^2, b^2)$.

$(5) \Rightarrow (1)$ If (5) holds then [23, Proposition 24.2], implies that each nonzero homogeneous ideal generated by two homogeneous elements is invertible. Therefore $R$ is a graded Prüfer domain.

The ungraded version of the following theorem is due to Gilmer (see [23, Corollary 28.5]).

**Theorem 4.2.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then $R$ is a graded Prüfer domain if and only if $C(f)^nC(g) = C(fg)$ for all $f, g \in R_H$.

**Proof.** $(\Rightarrow)$ Let $f, g \in R_H$. Then by [4, Lemma 1.1(1)], there exists some positive integer $n$ such that $C(f)^nC(g) = C(fg)^nC(fg)$. Now since $R$ is a graded Prüfer domain, the homogeneous fractional ideal $C(f)^n$ is invertible. Thus $C(f)^nC(g) = C(fg)$ for all $f, g \in R_H$.

$(\Leftarrow)$ Let $\alpha \in H$ be a unit of nonzero degree. Assume that $C(f)^nC(g) = C(fg)$ for all $f, g \in R_H$. Hence $R$ is integrally closed by [2, Theorem 3.7]. Now let $a, b \in H$ be arbitrary. We can choose a positive integer $n$ such that $\deg(a) \neq \deg(a^n b)$. So that $C(a + a^n b) = (a, b)$. Hence, since $(a + a^n b)(a - a^n b) = a^2 - (a^n b)^2$, we have $(a, b)(a, -b) = (a^2, -b^2)$. Consequently $(a, b)^2 = (a^2, b^2)$. Thus by Proposition 4.1, we see that $R$ is a graded Prüfer domain.

**Lemma 4.3.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and $P$ be a homogeneous prime ideal. Then, the following statements are equivalent:

(2) Each finitely generated nonzero homogeneous ideal of $R$ is a cancelation ideal.

(3) If $A, B, C$ are finitely generated homogeneous ideals of $R$ such that $AB = AC$ and $A$ is nonzero, then $B = C$.

(4) $R$ is integrally closed and there is a positive integer $n > 1$ such that $(a, b)^n = (a^n, b^n)$ for each $a, b \in H$.

(5) $R$ is integrally closed and there exists an integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)$ for each $a, b \in H$.
(1) $R_{H\setminus P}$ is a graded Prüfer domain.

(2) $R_P$ is a valuation domain.

(3) For each nonzero homogeneous $u \in R_H$, $u$ or $u^{-1}$ is in $R_{H\setminus P}$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $R_{H\setminus P}$ is a graded Prüfer domain. In particular $R_{H\setminus P}$ is a (graded) PrMD and each nonzero homogeneous ideal of $R_{H\setminus P}$ is a $t$-ideal. So that $h$-$\text{QMax}^i(R_{H\setminus P}) = \{ PR_{H\setminus P} \}$. Thus by [10, Lemma 2.7], we see that $(R_{H\setminus P})_{PR_{H\setminus P}} = R_P$ is a valuation domain.

(2) $\Rightarrow$ (3) Let $0 \neq u \in R_H$. Thus by the hypothesis $u$ or $u^{-1}$ is in $R_P$. Thus $u$ or $u^{-1}$ is in $R_{H\setminus P}$.

(3) $\Rightarrow$ (1) Let $I, J$ be two nonzero homogeneous ideals of $R_{H\setminus P}$ and assume that $I \not\subseteq J$. So there is a homogeneous element $a \in I \setminus J$. For each $b \in J$, we have $\frac{a}{b} \notin R_{H\setminus P}$, since otherwise we have $a = (\frac{a}{b}) b \in J$. Thus by the hypothesis $\frac{b}{a} \in R_{H\setminus P}$. Hence $b = (\frac{b}{a}) a \in I$. Thus we showed that $J \subseteq I$, and so every two homogeneous ideal are comparable.

Now let $(a, b)$ be an ideal generated by two homogeneous elements of $R_{H\setminus P}$. Now by the first paragraph $(a, b) = (a)$ or $(a, b) = (b)$. Thus $(a, b)$ is invertible. Hence $R_{H\setminus P}$ is a graded Prüfer domain. \hfill $\square$

**Theorem 4.4.** Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain, and $*$ be a semistar operation on $R$ such that $R^* \subsetneq R_H$. Then, the following statements are equivalent:

(1) $R$ is a graded $P*$-$\text{MD}$.

(2) $R_{H\setminus P}$ is a graded Prüfer domain for each $P \in h$-$\text{QSpec}^\ast(R)$.

(3) $R_{H\setminus P}$ is a graded Prüfer domain for each $P \in h$-$\text{QMax}^\ast(R)$.

(4) $R_P$ is a valuation domain for each $P \in h$-$\text{QSpec}^\ast(R)$.

(5) $R_P$ is a valuation domain for each $P \in h$-$\text{QMax}^\ast(R)$.

**Proof.** (2) $\Rightarrow$ (3) is trivial, and, (2) $\iff$ (4) and (3) $\iff$ (5), follow from Lemma [13].

(1) $\Rightarrow$ (2) Let $I$ be a nonzero finitely generated homogeneous ideal of $R$. Then $I$ is $\tilde{\ast}$-invertible. Therefore, for each $P \in h$-$\text{QSpec}^\ast(R)$, since $II^{-1} \not\subseteq P$, we have $R_{H\setminus P} = (II^{-1})R_{H\setminus P} = IR_{H\setminus P}I^{-1}R_{H\setminus P} = (IR_{H\setminus P})(IR_{H\setminus P})^{-1}$. So that $IR_{H\setminus P}$ is invertible. Thus $R_{H\setminus P}$ is a graded Prüfer domain for each $P \in h$-$\text{QSpec}^\ast(R)$.

(3) $\Rightarrow$ (1) Let $I$ be a nonzero finitely generated homogeneous ideal of $R$. Suppose that $I$ is not $\tilde{\ast}$-invertible. Hence there exists $P \in h$-$\text{QMax}^\ast(R)$ such that $II^{-1} \not\subseteq P$. Thus $R_{H\setminus P} = (IR_{H\setminus P})(IR_{H\setminus P})^{-1} = II^{-1}R_{H\setminus P} \subseteq PR_{H\setminus P}$, which is a contradiction. So that $II^{-1} \not\subseteq P$ for
each $P \in h\text{-}{\operatorname{QMax}}^\natural(R)$. Therefore $(II^{-1})^\natural = R^\natural$, that is $I$ is $\natural$-invertible, and hence $R$ is a graded $P\starMD$. \hfill \Box

The ungraded version of the following theorem is due to Chang in the star operation case \cite[Theorem 3.7]{6}, and is due to Anderson, Fontana, and Zafrullah in the case of semistar operations \cite[Theorem 1.1]{6}.

**Theorem 4.5.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and $\star$ be a semistar operation on $R$ such that $R^\star \subsetneq R_H$. Then $R$ is a graded $P\starMD$ if and only if $(C(f)C(g))^\star = C(fg)^\star$ for all $f, g \in R_H$.

**Proof.** ($\Rightarrow$) Let $f, g \in R_H$. Choose a positive integer $n$ such that $C(f)^{n+1}C(g) = C(f)^nC(fg)$ by \cite[Lemma 1.1(1)]{4}. Thus $(C(f)^{n+1}C(g))^\star = (C(f)^nC(fg))^\star$. Since $R$ is a graded $P\starMD$, the homogeneous fractional ideal $C(f)^n$ is $\natural$-invertible. Thus $(C(f)C(g))^\star = C(fg)^\star$ for all $f, g \in R_H$.

($\Leftarrow$) Assume that $(C(f)C(g))^\star = C(fg)^\star$ for all $f, g \in R_H$. Let $P \in h\text{-}{\operatorname{QMax}}^\natural(R)$. Then using Proposition 2.6, we have $C(f)R_H\setminus PC(g)R_H\setminus P = C(f)C(g)R_H\setminus P = (C(f)C(g))^\star R_H\setminus P = C(fg)^\star R_H\setminus P$. Since $R_H\setminus P$ has a unit of nonzero degree, Theorem 4.4 shows that $R_H\setminus P$ is a graded Prüfer domain. Now Theorem 4.4 implies that $R$ is a graded $P\starMD$. \hfill \Box

We now recall the notion of $\star$-valuation overring (a notion due essentially to P. Jaffard \cite[page 46]{25}). For a domain $D$ and a semistar operation $\star$ on $D$, we say that a valuation overring $V$ of $D$ is a $\star$-valuation overring of $D$ provided $F^\star \subseteq FV$, for each $F \in f(D)$.

**Remark 4.6.** (1) Let $\star$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. Recall that for each $F \in f(R)$ we have

$$F^{\star*} = \bigcap \{FV | V \text { is a } \star - \text{valuation overring of } R\},$$

by \cite[Propositions 3.3 and 3.4 and Theorem 3.5]{19}.

(2) We have $N_\star(H) = N_{\natural_a}(H)$. Indeed, since $\star \leq \natural_a$ by \cite[Proposition 4.5]{20}, we have $N_\star(H) = N_{\natural_a}(H) \subseteq N_{\natural_a}(H)$. Now if $f \in R \setminus N_\star(H)$ then, $C(f)^\star \not\subseteq R^\natural$. Thus there is a homogeneous quasi-$\natural$-prime ideal $P$ of $R$ such that $C(f) \subseteq P$. Let $V$ be a valuation domain dominating $R_P$ with maximal ideal $M$ \cite[Corollary 19.7]{23}. Therefore $V$ is a $\natural$-valuation overring of $R$ by \cite[Theorem 3.9]{18}, and $C(f)V \subseteq M$; so $C(f)^{(\natural)_a} \not\subseteq R^{(\natural)_a}$ and $f \notin N_{\natural_a}(H)$. Thus we obtain that $N_\star(H) = N_{\natural_a}(H)$. 

In the following theorem we generalize a characterization of $P\star$MDs
proved by Arnold and Brewer [7, Theorem 3]. It also generalizes [8, Theorem 3.7], [4, Theorems 3.4 and 3.5], and [17, Theorem 3.1].

**Theorem 4.7.** Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with
a unit of nonzero degree, and $\star$ be a semistar operation on $R$ such that
$\overline{R}^* \subseteq R_H$. Then, the following statements are equivalent:

1. $R$ is a graded $P\star$MD.
2. Every ideal of $R_{N_\star(H)}$ is extended from a homogeneous ideal of $R$.
3. Every principal ideal of $R_{N_\star(H)}$ is extended from a homogeneous ideal of $R$.
4. $R_{N_\star(H)}$ is a Prüfer domain.
5. $R_{N_\star(H)}$ is a Bézout domain.
6. $R_{N_\star(H)} = \text{Kr}(R, \overline{\star})$.
7. $\text{Kr}(R, \overline{\star})$ is a quotient ring of $R$.
8. $\text{Kr}(R, \overline{\star})$ is a flat $R$-module.
9. $I^\sharp = I^{\star\star}$ for each nonzero homogeneous finitely generated ideal of $R$.

In particular if $R$ is a graded $P\star$MD, then $\overline{R}^*$ is integrally closed.

**Proof.** By Proposition 2.3 and Theorem 3.3, we have $\text{Kr}(R, \overline{\star})$ is well-defined and is a Bézout domain.

(1) $\Rightarrow$ (2) Let $0 \neq f \in R$. Then $C(f)$ is $\overline{\star}$-invertible, because $R$ is a graded $P\star$MD, and thus $f R_{N_\star(H)} = C(f) R_{N_\star(H)}$ by Corollary 2.11. Hence if $A$ is an ideal of $R_{N_\star(H)}$, then $A = IR_{N_\star(H)}$ for some ideal $I$ of $R$, and thus $A = \bigoplus_{f \in I} C(f) R_{N_\star(H)}$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Is the same as part (3) $\Rightarrow$ (1) in [4, Theorem 3.4].

(1) $\Rightarrow$ (4) Let $A$ be a nonzero finitely generated ideal of $R_{N_\star(H)}$. Then by Corollary 2.11, $A = IR_{N_\star(H)}$ for some nonzero finitely generated homogeneous ideal $I$ of $R$. Since $R$ is a graded $P\star$MD, $I$ is $\overline{\star}$-invertible, and thus $A = IR_{N_\star(H)}$ is invertible by Lemma 2.10.

(4) $\Rightarrow$ (5) Follows from Theorem 2.13.

(5) $\Rightarrow$ (6) Clearly $R_{N_\star(H)} \subseteq \text{Kr}(R, \overline{\star})$. Since $R_{N_\star(H)}$ is a Bézout domain, then $\text{Kr}(R, \overline{\star})$ is a quotient ring of $R_{N_\star(H)}$, by [23, Proposition 27.3]. If $Q \in h\text{-QMax}(R)$, then $Q \text{Kr}(R, \overline{\star}) \subseteq \text{Kr}(R, \overline{\star})$. Otherwise $Q \text{Kr}(R, \overline{\star}) = \text{Kr}(R, \overline{\star})$, and hence there is an element $f \in Q$, such that $f \text{Kr}(R, \overline{\star}) = \text{Kr}(R, \overline{\star})$. Thus $\frac{1}{f} \in \text{Kr}(R, \overline{\star})$. Therefore $R = C(1) \subseteq C(f) \subseteq \overline{R}^*$, so that $C(f) \overline{\star} = R_{N_\star(H)}$. Hence $f \in N_{\overline{\star}}(H) = N_\star(H)$.
by Remark 4.6(2). This means that $Q^x = R^x$, a contradiction. Thus $Q \text{Kr}(R, \tilde{x}) \subsetneq \text{Kr}(R, \tilde{x})$, and so there is a maximal ideal $M$ of $\text{Kr}(R, \tilde{x})$ such that $Q \text{Kr}(R, \tilde{x}) \subseteq M$. Hence $M \cap R_{N,(H)} = Q R_{N,(H)}$, by Lemma 2.7. Consequently $R_Q \subseteq \text{Kr}(R, \tilde{x})_M$, and since $R_Q$ is a valuation domain, we have $R_Q = \text{Kr}(R, \tilde{x})_M$. Therefore $R_{N,(H)} = \bigcap_{Q \in h^{-1}\text{QMax}(R)} R_Q \supseteq \bigcap_{M \in \text{Max}(\text{Kr}(R, \tilde{x})))} \text{Kr}(R, \tilde{x})_M$. Hence $R_{N,(H)} = \text{Kr}(R, \tilde{x})$.

$(6) \Rightarrow (7)$ and $(7) \Rightarrow (8)$ are clear.

$(8) \Rightarrow (6)$ Recall that an overring $T$ of an integral domain $S$ is a flat $S$-module if and only if $T_M = S_{M \cap S}$ for all $M \in \text{Max}(T)$ by [32, Theorem 2].

Let $A$ be an ideal of $R$ such that $A \text{Kr}(R, \tilde{x}) = \text{Kr}(R, \tilde{x})$. Then there exists an element $f \in A$ such that $f \text{Kr}(R, \tilde{x}) = \text{Kr}(R, \tilde{x})$ using Theorem 3.3, so $\frac{1}{f} \in \text{Kr}(R, \tilde{x}) = \text{Kr}(R, \tilde{x})$. Thus $R = C(1) \subseteq C(f)^{\ast} \subseteq R^{\ast}$. Hence $C(f)^{\ast} = R^{\ast}$. Therefore $f \in A \cap N_4(H) \neq \emptyset$. Hence, if $P_0$ is a homogeneous maximal quasi-$\tilde{x}$-ideal of $R$, then $P_0 \text{Kr}(R, \tilde{x}) \subseteq \text{Kr}(R, \tilde{x})$, and since $P_0 R_{N,(H)}$ is a maximal ideal of $R_{N,(H)}$, there is a maximal ideal $M_0$ of $\text{Kr}(R, \tilde{x})$ such that $M_0 \cap R = (M_0 \cap R_{N,(H)}) \cap R = P_0 R_{N,(H)} \cap R = P_0$. Thus by $(8)$, $\text{Kr}(R, w)_{M_0} = R_{P_0} = (R_{N,(H)})_{P_0 R_{N,(H)}}$.

Let $M_1$ be a maximal ideal of $\text{Kr}(R, \tilde{x})$, and let $P_1$ be a homogeneous maximal quasi-$\tilde{x}$-ideal of $R$ such that $M_1 \cap R_{N,(H)} \subseteq P_1 R_{N,(H)}$. By the above paragraph, there is a maximal ideal $M_2$ of $\text{Kr}(R, \tilde{x})$ such that $\text{Kr}(R, \tilde{x})_{M_2} = (R_{N,(H)})_{P_1 R_{N,(H)}}$. Note that $\text{Kr}(R, \tilde{x})_{M_2} \subseteq \text{Kr}(R, \tilde{x})_{M_1}$, $M_1$ and $M_2$ are maximal ideals, and $\text{Kr}(R, \tilde{x})$ is a Prüfer domain; hence $M_1 = M_2$ (cf. [23, Theorem 17.6(c)]) and $\text{Kr}(R, \tilde{x})_{M_1} = (R_{N,(H)})_{P_1 R_{N,(H)}}$. Thus

$$\text{Kr}(R, \tilde{x}) = \bigcap_{M \in \text{Max}(\text{Kr}(R, \tilde{x}))} \text{Kr}(R, \tilde{x})_M = \bigcap_{P \in h^{-1}\text{QMax}(R)} (R_{N,(H)})_{PR_{N,(H)}} = R_{N,(H)}.$$  

$(6) \Rightarrow (9)$ Assume that $R_{N,(H)} = \text{Kr}(R, \tilde{x})$. Let $I$ be a nonzero homogeneous finitely generated ideal of $R$. Then by Lemma 2.9 and Theorem 3.3(3), we have $I^x = IR_{N,(H)} \cap R_H = I \text{Kr}(R, \tilde{x}) \cap R_H = I^{\ast}$.  

$(9) \Rightarrow (1)$ Let $a$ and $b$ be two nonzero homogeneous elements of $R$. Then $((a, b)^2)^{\ast} = ((a, b)(a^2, b^2))^{\ast}$ which implies that $((a, b)^2)^{\ast} = (a^2, b^2)^{\ast}$. Hence $((a, b)^2)^{\ast} = (a^2, b^2)^{\ast}$ and so $(a, b)^2 R_{H \setminus P} = (a^2, b^2) R_{H \setminus P}$ for each homogeneous maximal quasi-$\tilde{x}$-ideal $P$ of $R$. On the other hand $R^x = R^{\ast}$ by $(9)$. Hence $R^x$ is integrally closed. Thus $R^x R_{H \setminus P} = R_{H \setminus P}$ is
integrally closed. Therefore by Proposition 4.1, \( R_{H \setminus P} \) is a graded Prüfer domain for each homogeneous maximal quasi-\( \ast \)-ideal of \( R \). Thus \( R \) is a graded \( \mathbb{P} \ast \text{MD} \) by Theorem 4.4.

The following theorem is a graded version of a characterization of Prüfer domains proved by Davis [12, Theorem 1]. It also generalizes [13, Theorem 2.10], in the \( t \)-operation, and [15, Theorem 5.3], in the case of semistar operations.

**Theorem 4.8.** Let \( R = \bigoplus_{\alpha \in \Gamma} R_\alpha \) be a graded integral domain with a unit of nonzero degree, and \( \ast \) be a semistar operation on \( R \) such that \( R^* \subsetneq R_H \). Then, the following statements are equivalent:

1. \( R \) is a graded \( \mathbb{P} \ast \text{MD} \).
2. Each homogeneously \( (\ast, t) \)-linked overring of \( R \) is a \( \mathbb{P} \ast \text{MD} \).
3. Each homogeneously \( (\ast, d) \)-linked overring of \( R \) is a graded Prüfer domain.
4. Each homogeneously \( (\ast, t) \)-linked overring of \( R \), is integrally closed.
5. Each homogeneously \( (\ast, d) \)-linked overring of \( R \), is integrally closed.

**Proof.**

(1) \( \Rightarrow \) (2) Let \( T \) be a homogeneously \( (\ast, t) \)-linked overring of \( R \). Thus by Lemma 2.15, we have \( R_{N_{\ast}(H)} \subseteq T_{N_{\ast}(H)} \). Since \( R \) is a graded \( \mathbb{P} \ast \text{MD} \), by Theorem 4.7, we have \( R_{N_{\ast}(H)} \) is a Prüfer domain. Thus by [23, Theorem 26.1], we have \( T_{N_{\ast}(H)} \) is a Prüfer domain. Hence, again by Theorem 4.7, we have \( T \) is a graded \( \mathbb{P} \ast \text{MD} \). Therefore using [2, Theorem 6.4], \( T \) is a \( \mathbb{P} \ast \text{MD} \).

(2) \( \Rightarrow \) (4) \( \Rightarrow \) (5) and (3) \( \Rightarrow \) (5) are clear.

(5) \( \Rightarrow \) (1) Let \( P \in h\text{-QMax}^\ast(R) \). For a nonzero homogeneous \( u \in R_H \), let \( T = R[u^2, u^3]_{H \setminus P} \). Then \( R_{H \setminus P} \) and \( T \) are homogeneous \( (\ast, d) \)-linked overring of \( R \) by Example 2.14. So that \( R_{H \setminus P} \) and \( T \) are integrally closed. Hence \( u \in T \), and since \( T = R_{H \setminus P}[u^2, u^3] \), there exists a polynomial \( \gamma \in R_{H \setminus P}[X] \) such that \( \gamma(u) = 0 \) and one of the coefficients of \( \gamma \) is a unit in \( R_{H \setminus P} \). So \( u \) or \( u^{-1} \) is in \( R_{H \setminus P} \) by [27, Theorem 67]. Therefore by Lemma 4.3, \( R_{H \setminus P} \) is a graded Prüfer domain. Thus \( R \) is a graded \( \mathbb{P} \ast \text{MD} \) by Theorem 4.4.

(1) \( \Rightarrow \) (3) Is the same argument as in part (1) \( \Rightarrow \) (2).

The next result gives new characterizations of \( \mathbb{P} \ast \text{MDs} \) for graded integral domains, which is the special cases of Theorems 4.4, 4.5, 4.7, and 4.8, for \( \ast = v \).
Corollary 4.9. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with a unit of nonzero degree. Then, the following statements are equivalent:

1. $R$ is a (graded) $P$-vMD.
2. $R_{H,P}$ is a graded Prüfer domain for each $P \in h\text{-}\text{QMax}^t(R)$.
3. $R_P$ is a valuation domain for each $P \in h\text{-}\text{QMax}^t(R)$.
4. Every ideal of $R_{N,e(H)}$ is extended from a homogeneous ideal of $R$.
5. $R_{N,e(H)}$ is a Prüfer domain.
6. $R_{N,e(H)}$ is a Bézout domain.
7. $R_{N,e(H)} = \text{Kr}(R,w)$.
8. $\text{Kr}(R,w)$ is a quotient ring of $R$.
9. $\text{Kr}(R,w)$ is a flat $R$-module.
10. Each homogeneously $t$-linked overring of $R$ is a $P$-vMD.
11. Each homogeneously $t$-linked overring of $R$, is integrally closed.
12. $(C(f)C(g))^w = C(fg)^w$ for all $f, g \in R_H$.
13. $I^w = I^{ws}$ for each nonzero homogeneous finitely generated ideal of $R$.

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