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A PROOF ON POWER-ARMENDARIZ RINGS

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Abstract. Power-Armendariz is a unifying concept of Armendariz and commutative. Let R be a ring and I be a proper ideal of R such that R/I is a power-Armendariz ring. Han et al. proved that if I is a reduced ring without identity then R is power-Armendariz. We find another direct proof of this result to see the concrete forms of various kinds of subsets appearing in the process.

1. Introduction

Throughout this note every ring is associative with identity unless otherwise stated. $\mathbb{Z}$ denotes the ring of integers. Denote the n by n upper triangular matrix ring over R by $U_n(R)$. We use $R[x]$ to denote the polynomial ring with an indeterminate x over R. For $f(x) \in R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. For $n \geq 2$, define

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in U_n(R) \mid a, a_{ij} \in R \right\}.$$

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A ring (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. For a reduced ring R and $f(x), g(x) \in R[x]$, Armendariz [1, Lemma 1] proved that

$$ab = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)} \text{ whenever } f(x)g(x) = 0.$$

Rege and Chhawchharia [4] called a ring (possibly without identity) *Armendariz* if it satisfies this property. So reduced rings are clearly Armendariz. According to Han et al. [2], a ring R (possibly without identity) is called *power-Armendariz* if whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$, there exist $m, n \geq 1$ such that

$$a^m b^n = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)}.$$

It is obvious that $a^m b^n = 0$ for some $m, n \geq 1$ if and only if $a^\ell b^\ell = 0$ for some $\ell \geq 1$, in the preceding definition. Armendariz rings are clearly power-Armendariz, but the converse need not be true. In fact, letting $A = D_2(\mathbb{Z}))$, $D_3(A)$ is power-Armendariz by [2, Theorem], but $D_3(A)$ is not Armendariz by [3, Proposition 2.8].

2. Main result

Han et al. proved the following.

[2, Theorem 1.11(4)] Let R be a ring and I be a proper ideal of R such that R/I is a power-Armendariz ring. If I is a reduced ring without identity, then R is power-Armendariz.

We state here another direct proof of this theorem to see the concrete forms of various kinds of subsets appearing in the process.

Another proof of [2, Theorem 1.11(4)] The first basic part of this proof is almost a restatement of one of [2, Theorem 1.11(1, 2, 3)]. Suppose that I is a reduced ring, and let $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{\ell} b_j x^j \in R[x]$. Since R/I is power-Armendariz, there exists $s \geq 1$ such that $a_i^s b_j^s \in I$ for all i, j. Without loss of generality, we let $m = \ell$ by using zero coefficients if necessary.

Suppose $r_1 r_2 = 0$ for $r_1, r_2 \in R$. Then $(r_2 I r_1)^2 = 0$, but $r_2 I r_1 \subseteq I$ implies $r_2 I r_1 = 0$ since I is reduced. Similarly we get

(1) $\quad r_4 S r_3 = 0$ for all $S \subseteq I$ whenever $r_3 I r_4 = 0$ for some $r_3, r_4 \in R$,

through the computation of

$$(r_4 S r_3)^3 \subseteq (r_4 S r_3) I (r_4 S r_3) = r_4 S (r_3 I r_4) S r_3 = 0.$$

Summarizing, we have that

$$r_1r_2 = 0 \text{ implies } r_1Ir_2 = 0 \text{ and } r_2Ir_1 = 0 \tag{2}$$

by help of (1).

Suppose that $r_1r_2\cdots r_n = 0$ for $r_i \in R$ and $n \geq 2$.
Then $r_1Ir_2I\cdots Ir_n = 0$ by using (2) repeatedly, and so we furthermore have

$$r_{\sigma(1)}Ir_{\sigma(2)}I\cdots Ir_{\sigma(n)} = 0 \tag{3}$$

for any permutation σ of the set $\{1, 2, \ldots, n\}$ from the computation of

$$(r_{\sigma(1)}Ir_{\sigma(2)}I\cdots Ir_{\sigma(n)})^{2n} \subseteq Rr_1Ir_2I\cdots Ir_nR = 0,$$

using the condition that I is reduced. Especially we have $a_0Ib_0 = 0$ and $b_0Ia_0 = 0$ from $a_0b_0 = 0$. We will use freely the condition that I is reduced.

Consider $a_0b_1Ia_0b_1$.
Since $a_0b_1 = -a_1b_0$, we have $a_0b_1Ia_0b_1 = -a_0b_1Ia_1b_0 = 0$ from $a_0Ib_0 = 0$. This yields $b_1b_1Ia_0a_0 = 0$ by the computation of

$$\begin{aligned}(b_1b_1Ia_0a_0)^3 =&(b_1b_1Ia_0a_0)(b_1b_1Ia_0a_0)(b_1b_1Ia_0a_0)\\ =&(b_1b_1Ia_0)(a_0b_1b_1Ia_0a_0b_1)(b_1Ia_0a_0)\\ \subseteq&(b_1b_1Ia_0)(a_0b_1Ia_0b_1)(b_1Ia_0a_0) = 0.\end{aligned}$$

This also yields $a_0a_0Ib_1b_1 = 0$ by result (1); hence $a_0^{s+2}b_1^{s+2} = 0$ because $a_0^sb_1^s \in I$. Similarly we get $a_1^2Ib_0^2 = 0$ and $a_1^{s+2}b_0^{s+2} = 0$ also from $a_0b_0 = 0$ and $a_0b_1 + a_1b_0 = 0$, by exchanging the roles of a_0 and b_0.

Consider $a_0b_2Ia_0b_2$. Since $a_0b_2 = -a_1b_1 - a_2b_0$, we have $a_0b_2Ia_0b_2 = a_0b_2I(-a_1b_1 - a_2b_0) = -a_0b_2Ia_1b_1$ from $a_0Ib_0 = 0$. But (2) implies

$$(a_0b_2Ia_1b_1)^3 = (a_0b_2Ia_1b_1)(a_0b_2Ia_1b_1)(a_0b_2Ia_1b_1) \subseteq a_0Ia_0Ib_1Ib_1 = 0$$

since $a_0^2Ib_1^2 = 0$, entailing $a_0b_2Ia_0b_2 = 0$. So we get $a_0a_0Ib_2b_2 = 0$ and $a_0^{s+2}b_2^{s+2} = 0$ by a similar method to one above.

We will proceed by induction on m. Assume that $a_0b_hIa_0b_h = 0$ (then $a_0a_0Ib_hb_h = 0$ and $a_0Ia_0Ib_hIb_h = 0$ by (3) and the method above) for all $h < k$, where $1 \leq k \leq m$. Consider $a_0b_kIa_0b_k$. Since $a_0b_k = -a_1b_{k-1} - \cdots - a_kb_0$, we have $a_0b_kIa_0b_k = a_0b_kI(-a_1b_{k-1} - \cdots - a_{k-1}b_1)$

from $a_0Ib_0 = 0$. But (3) implies

$$
\begin{aligned}
&(a_0b_kIa_0b_k)^{2k+3} = (a_0b_kI(-a_1b_{k-1} - \cdots - a_{k-1}b_1))^{2k+3} \\
&=(a_0b_kI(-a_1b_{k-1} - \cdots - a_{k-1}b_1)) \times (a_0b_kI(-a_1b_{k-1} - \cdots - a_{k-1}b_1)) \\
&\qquad\qquad\qquad\qquad \times (a_0b_kI(-a_1b_{k-1} - \cdots - a_{k-1}b_1))^{2k+1} \\
&\subseteq a_0Ia_0I(I(-a_1b_{k-1} - \cdots - a_{k-1}b_1))^{2k+1} \\
&\subseteq a_0Ia_0I(I(-a_1b_{k-1} - \cdots - a_{k-1}b_1)I)^kI \\
&\subseteq a_0Ia_0I(b_{k-1}Ib_{k-1}I + \cdots + b_1Ib_1I) = 0
\end{aligned}
$$

since $a_0Ia_0Ib_hIb_h = 0$ for all $h = 0, 1, \ldots, k-1$, entailing $a_0b_kIa_0b_k = 0$. So we get $a_0a_0Ib_kb_k = 0$ and $a_0^{s+2}b_k^{s+2} = 0$ by a similar method to one above. This implies $a_0^2Ib_t^2 = 0$ and $a_0^{s+2}b_t^{s+2} = 0$ for all $t = 0, 1, \ldots, m$.

We similarly get $a_t^2Ib_0^2 = 0$ and $a_t^{s+2}b_0^{s+2} = 0$ for all $t = 0, 1, \ldots, m$, by exchanging the roles of a_0 and b_0. Summarizing, we now have

(4) $$a_0b_tIa_0b_t = 0, a_0^2Ib_t^2 = 0, a_0^{s+2}b_t^{s+2} = 0,$$
$$\text{and } a_tb_0Ia_tb_0 = 0, a_t^2Ib_0^2 = 0, a_t^{s+2}b_0^{s+2} = 0 \text{ for all } t = 0, 1, \ldots, m.$$

Next consider $a_1b_1Ia_1b_1$. Since $a_1b_1 = -a_0b_2 - a_2b_0$, we have $a_1b_1Ia_1b_1 = a_1b_1I(-a_0b_2 - a_2b_0)$. But

$$
\begin{aligned}
&(a_1b_1Ia_1b_1)^6 = (a_1b_1I(-a_0b_2 - a_2b_0))^6 \subseteq ((a_1b_1Ia_0b_2 + a_1b_1Ia_2b_0)I)^3 \\
&\subseteq (a_1b_1Ia_0b_2I + a_1b_1Ia_2b_0I)^3 = (Ia_0b_2I)^2 + (Ia_2b_0I)^2 = 0
\end{aligned}
$$

by help of (4). So we get $a_1b_1Ia_1b_1 = 0$, $a_1a_1Ib_1b_1 = 0$ and $a_1^{s+2}b_1^{s+2} = 0$ by the method above.

Consider $a_1b_2Ia_1b_2$. Since $a_1b_2 = -a_0b_3 - a_2b_1 - a_3b_0$, we have $a_1b_2Ia_1b_2 = a_1b_2I(-a_0b_3 - a_2b_1 - a_3b_0)$. Then $a_1b_1Ia_1b_1 = 0$ and (4) yield

$$
\begin{aligned}
(a_1b_2Ia_1b_2)^8 &= (a_1b_2I(-a_0b_3 - a_2b_1 - a_3b_0))^8 \\
&\subseteq ((a_1b_2I(-a_0b_3 - a_2b_1 - a_3b_0))I)^4 \\
&\subseteq ((a_1b_2Ia_0b_3 + a_1b_2Ia_2b_1 + a_1b_2Ia_3b_0)I)^4 \\
&\subseteq (Ia_0Ib_3I)^2 + (Ia_1Ib_1I)^2 + (Ia_0Ib_3I)^2 = 0
\end{aligned}
$$

by help of (3), entailing $a_1b_2Ia_1b_2 = 0$, $a_1a_1Ib_2b_2 = 0$, and $a_1^{s+2}b_2^{s+2} = 0$.

We will proceed by induction on m. Assume that $a_1b_hIa_1b_h = 0$ (then $a_1a_1Ib_hb_h = 0$ and $a_1Ia_1Ib_hIb_h = 0$ by (3) and the method above) for all $h < k$, where $1 \leq k \leq m$. Consider $a_1b_kIa_1b_k$. Since $a_1b_k =$

$-a_2b_{k-1} - \cdots - a_kb_1$, we have $a_1b_kIa_1b_k = a_1b_kI(-a_2b_{k-1} - \cdots - a_kb_1)$. But (3) implies

$$\begin{aligned}
&(a_1b_kIa_1b_k)^{2k+3} = (a_1b_kI(-a_2b_{k-1} - \cdots - a_kb_1))^{2k+3} \\
=&(a_1b_kI(-a_2b_{k-1} - \cdots - a_kb_1)) \times (a_1b_kI(-a_2b_{k-1} - \cdots - a_kb_1)) \\
&\qquad\qquad\qquad \times (a_1b_kI(-a_2b_{k-1} - \cdots - a_kb_1))^{2k+1} \\
\subseteq& \; a_1Ia_1I(I(-a_2b_{k-1} - \cdots - a_kb_1))^{2k+1} \\
\subseteq& \; a_1Ia_1I(I(-a_2b_{k-1} - \cdots - a_kb_1)I)^kI \\
\subseteq& \; a_1Ia_1I(b_{k-1}Ib_{k-1}I + \cdots + b_1Ib_1I) = 0
\end{aligned}$$

since $a_1Ia_1Ib_hIb_h = 0$ for $h = 1, \ldots, k-1$, entailing $a_1b_kIa_1b_k = 0$. So we get $a_1a_1Ib_kb_k = 0$ and $a_1^{s+2}b_k^{s+2} = 0$ by a similar method to one above. This implies $a_1^2Ib_t^2 = 0$ and $a_1^{s+2}b_t^{s+2} = 0$ for all $t = 0, 1, \ldots, m$. We similarly obtain $a_t^2Ib_1^2 = 0$ and $a_t^{s+2}b_1^{s+2} = 0$ for all $t = 0, 1, \ldots, m$.

Lastly we will show that $a_ub_hIa_ub_h = 0$ if $a_tb_hIa_tb_h = 0$ for all $t < u$ and $h = 1, \ldots, m$, where $1 \leq u \leq m$. We will proceed by induction on m. Assume that $a_tb_hIa_tb_h = 0$ (then $a_ta_tIb_hb_h = 0$ and $a_tIa_tIb_hIb_h = 0$ by (3) and the method above) for all $t < u$ and $h = 1, \ldots, m$, where $1 \leq u \leq m$. Consider $a_ub_hIa_ub_h$. From $\sum_{i+j=u+h} a_ib_j = 0$, we have $a_ub_hIa_ub_h = (-a_{u-1}b_{h+1} - \cdots - a_hb_u)Ia_ub_h$ by assumption. So we can let $u \geq h$. Let w be the number of monomials of degree $u+h$. But (3) implies

$$\begin{aligned}
&(a_ub_hIa_ub_h)^{2w+3} = ((-a_{u-1}b_{h+1} - \cdots - a_hb_u)Ia_ub_h)^{2w+3} \\
\subseteq& \; ((-a_{u-1}b_{h+1} - \cdots - a_hb_u)Ia_ub_h)^{2w+1} \times ((-a_{u-1}b_{h+1} - \cdots - a_hb_u)Ia_ub_h) \\
&\qquad\qquad\qquad \times ((-a_{u-1}b_{h+1} - \cdots - a_hb_u)Ia_ub_h) \\
\subseteq& \; (I(-a_{u-1}b_{h+1} - \cdots - a_hb_u)I)^wIb_hIb_h \\
\subseteq& \; I(a_{u-1}Ia_{u-1}I + \cdots + a_hIa_hI)b_hIb_h = 0
\end{aligned}$$

since $a_pIa_pIb_hIb_h = 0$ for all $p < u$, entailing $a_ub_hIa_ub_h = 0$. So we get $a_ua_uIb_hb_h = 0$ and $a_u^{s+2}b_h^{s+2} = 0$ by the method above. This implies that $a_i^{s+2}b_j^{s+2} = 0$ for all i, j. Therefore R is power-Armendariz. $\square$

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