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THE GENERALIZED HYERS-ULAM STABILITY OF ADDITIVE FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN 2-NORMED SPACE

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Abstract. In this paper, we investigate the solution of the following functional inequality

$$\|f(x) + f(y) + f(az), w\| \leq \|f(x + y) - f(-az), w\|$$

for some fixed non-zero integer a, and prove the generalized Hyers-Ulam stability of it in non-Archimedean 2-normed spaces.

1. Introduction and preliminaries

The stability problems concerning group homomorphisms were raised by Ulam [18] in 1940, and answered affirmatively for Banach spaces by Hyers [10] in the next year. Rassias [17] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x) + f(y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \epsilon\|y\|^p)$$

and derived Hyer's theorem for the stability of the additive mapping(called *the generalized Hyers-Ulam stability of the additive mapping*). It should be remarked that a paper of Aoki [1] was published

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concerning Hyers' theorem of the additive mapping earlier than the paper of Rassias [17]. In 1994, a generalization of the Rassias theorem was obtained by Găvruta as follows [5].

The functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

is called *an additive functional equation*. In particular, every solution of the additive functional equation is said to be *an additive mapping*.

In [6], Gilányi showed that if a mapping $f : X \longrightarrow Y$ satisfies the following functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\|, \tag{1.2}$$

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

Gilányi [7] and Fechner [2] proved the generalized Hyers-Ulam stability of (1.2). Park, Cho, and Han [15] proved the generalized Hyers-Ulam stability of the following functional inequalities:

$$\|f(x) + f(y) + f(z)\| \le \|f(x+y+z)\|. \tag{1.3}$$

Hensel [9] has introduced a normed space which does not have the Archimedean property and Moslehian and Rassias [13] proved the generalized Hyers-Ulam stability of the additive functional equation and the quadratic functional equation in non-Archimedean spaces.

A *valuation* is a function $|\cdot|$ from a field $\mathbb{K}$ to $[0,\infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold :

(i) $|r| = 0$ if and only if $r = 0$,
(ii) $|rs| = |r||s|$, and
(iii) $|r+s| \le |r| + |s|$.

A field $\mathbb{K}$ endowed with a valuation $|\cdot|$ is called *a valuation field*. If the condition (iii) in the definition of a valuation is replaced with

(iv) $|r+s| \le \max\{|r|, |s|\}$,

then the valuation $|\cdot|$ is called *a non-Archimedean valuation*. If $|\cdot|$ is a non-Archimedean valuation on a filed $\mathbb{K}$, then clearly, $|-1| = |1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$.

Gähler [3], [4] introduced the concept of 2-normed spaces and Gähler and White [19] introduced the concept of 2-Banach spaces. In 1999 to 2003, Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces(see [11] and [12]). Recently, Park [14] investigated the stability ploblems of approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces.

Definition 1.1. Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$ with $\dim X > 1$. Then a mapping $\|\cdot,\cdot\| : X \times X \longrightarrow [0,\infty)$ is called *a non-Archimedean* 2*-norm on* X if it satisfies the following :

(AN1) $\|x,y\| = 0$ if and only if x, y are linearly dependent,
(AN2) $\|x,y\| = \|y,x\|$,
(AN3) $\|rx,y\| = |r|\|x,y\|$, and
(AN4) $\|x,y+z\| \le \max\{\|x,y\|, \|x,z\|\}$

for all $x, y, z \in X$ and all $r \in \mathbb{K}$. In case, $(X, \|\cdot,\cdot\|)$ is called *a non-Archimedean* 2*-normed space.*

Let $\{x_n\}$ be a sequence in a non-Archimedean 2-normed space $(X, \|\cdot,\cdot\|)$. The sequence $\{x_n\}$ is called *a Cauchy sequence* if, for any $\epsilon > 0$, there exists a positive integer k such that $\|x_n - x_m, w\| \le \epsilon$ for all $m, n \ge k$ and all $w \in X$. The sequence $\{x_n\}$ is called *convergent* to x in $(X, \|\cdot,\cdot\|)$, denoted by $\lim_{n\to\infty} x_n = x$, if for any $\epsilon > 0$, there exists a positive integer k such that $\|x_n - x, w\| \le \epsilon$ for all $n \ge k$ and all $w \in X$. By (AN4), we have

$$\|x_n - x_m, w\| \le max\{\|x_{j+1} - x_j, w\| \mid m \le j \le n-1\} \quad (n > m).$$

and hence $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot,\cdot\|)$ if and only if the sequence $\{x_{n+1} - x_n\}$ converges to zero in $(X, \|\cdot,\cdot\|)$.

Let $(X, \|\cdot,\cdot\|)$ be a non-Archimedean 2-normed space. By (AN2) and (AN4), $\|x, y+z\| \le \|x,y\| + \|x,z\|$ and $\|x+y, z\| \le \|x,z\| + \|y,z\|$. Hence $\|\cdot,\cdot\|$ is continuous in each component and so using these, we have the following lemma :

Lemma 1.2. *For any convergent sequence* $\{x_n\}$ *in a non-Archimedean* 2*-normed space* $(X, \|\cdot,\cdot\|)$,

$$\lim_{n\to\infty} \|x_n, w\| = \|\lim_{n\to\infty} x_n, w\|.$$

for all $w \in Y$.

A non-Archimedean 2-normed space in which every Cauchy sequence is a convergent sequence is called *a non-Archimedean* 2*-Banach* *space*. Recently, the generalized Hyers-Ulam stability on some non-Archimedean Banach spaces was proved([8], [16]).

In this paper, we investigate the solution of the following functional inequality

$$\|f(x)+f(y)+f(az),w\| \le \|f(x+y)-f(-az),w\| \tag{1.4}$$

for some fixed non-zero integer a, and prove the generalized Hyers-Ulam stability of it in non-Archimedean 2-Banach spaces.

2. Solutions and stability of (1.4)

In this section, let X be a non-Archimedean 2-normed space with $\dim X > 1$ and Y a non-Archimedean 2-Banach space with $\dim Y > 1$. Note that since $\dim X > 1$, $\|x,y\| = 0$ for all $y \in X$ if and only if $x = 0$ and that by (AN1), $\|x, 0\| = 0$ for all $x \in X$. Using these, we have the following theorem.

THEOREM 2.1. *A mapping $f : X \longrightarrow Y$ saisfies (1.4) if and only if f is an additive mapping.*

Proof. Suppose that f satisfies (1.4). Setting $x = y = z = 0$ in (1.4), we have $\|3f(0), w\| \le \|0, w\| = 0$ for all $w \in Y$ and so $\|3f(0), w\| = 0$ for all $w \in Y$. Hence we have

$$f(0) = 0. \tag{2.1}$$

Putting $y = -x$ and $z = 0$ in (1.4), we have $\|f(x)+f(-x), w\| \le \|0, w\|$ for all $w \in Y$ and so $\|f(x)+f(-x), w\| = 0$ for all $w \in Y$. Thus we have

$$f(-x) = -f(x) \tag{2.2}$$

for all $x \in X$. Replacing x, y, z by $-ax-ay$, ax, y in (1.4) respectively, by (2.2), we have

$$\|f(-ax-ay)+f(ax)+f(ay), w\| \le \|f(-ay)-f(-ay), w\| = \|0, w\| = 0 \tag{2.3}$$

for all $x, y \in X$ and $w \in Y$. Hence by (2.2) and (AN1), we have

$$f(ax+ay) = f(ax) + f(ay)$$

for all $x, y \in X$ and since $a \neq 0$, f is additive.

Suppose that f is additive. Since f is an odd mapping, we have

$$\|f(x)+f(y)+f(az),w\| = \|f(x+y)-f(-az),w\|$$

for all $x, y, z \in X$ and $w \in Y$ and so f satisfies (1.4). $\square$

Now, we will prove the generalized Hyers-Ulam stability of (1.4) in non-Archimedean 2-Banach spaces.

THEOREM 2.2. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$\lim_{n\to\infty} \frac{\phi((-2)^n x, (-2)^n y, (-2)^n z)}{|2|^n} = 0 \tag{2.4}$$

for all $x, y, z \in X$ and the limit

$$\lim_{n\to\infty} \max\{\frac{\phi((-2)^k x, (-2)^k x, (-2)^{k+1}\frac{x}{a})}{|2|^{k-1}} \mid 0 \le k \le n-1\} \tag{2.5}$$

exsits for all $x \in X$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\|f(x)+f(y)+f(az),w\| \le \|f(x+y)-f(-az),w\| + \phi(x,y,z). \tag{2.6}$$

for all $x, y, z \in X$ and $w \in Y$. Then there exists an additive mapping $A : X \longrightarrow Y$ such that

$$\begin{aligned} &\|f(x)-A(x),w\| \\ &\le \lim_{n\to\infty} \max\{\frac{\phi((-2)^k x, (-2)^k x, (-2)^{k+1}\frac{x}{a})}{|2|^{k+1}} \mid 0 \le k \le n-1\} \end{aligned} \tag{2.7}$$

for all $x \in X$ and $w \in Y$. Moreover, if $\phi : X^3 \longrightarrow [0,\infty)$ satisfies

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\{\frac{\phi((-2)^i x, (-2)^i x, (-2)^{i+1}\frac{x}{a})}{|2|^i} \mid k \le i \le k+n-1\} = 0 \tag{2.8}$$

for all $x \in X$, then A is a unique additive mapping satisfying (2.7).

Proof. Replacing x, y, z by $(-2)^n x$, $(-2)^n x$, $(-2)^{n+1}\frac{x}{a}$ in (2.6), respectively, and dividing (2.6) by $|2|^{n+1}$, we have

$$\|\frac{f((-2)^n x)}{(-2)^n} - \frac{f((-2)^{n+1}x)}{(-2)^{n+1}}, w\| \le \frac{1}{|2|} \cdot \frac{\phi((-2)^n x, (-2)^n x, (-2)^n \cdot \frac{-2x}{a})}{|2|^n}$$

for all $x \in X$, $w \in Y$, and all $n \in \mathbb{N}$. By (2.4), $\{\frac{f((-2)^n x)}{(-2)^n}\}$ is a Cauchy sequence in Y for all $x \in X$ and since Y is a non-Archimedean 2-Banach space, there is a function $A : X \longrightarrow Y$ such that

$$A(x) = \lim_{n\to\infty} \frac{f((-2)^n x)}{(-2)^n}$$

for all $x \in X$. Moreover for $0 \le m < n$, by (AN4), we have

$$\begin{aligned} &\|\frac{f((-2)^n x)}{(-2)^n} - \frac{f((-2)^m x)}{(-2)^m}, w\| \\ &\le \max\{\frac{\phi((-2)^k x, (-2)^k x, (-2)^{k+1}\frac{x}{a})}{|2|^{k+1}} \mid m \le k \le n-1\} \end{aligned} \tag{2.9}$$

for all $x \in X$ and $w \in Y$. Replacing x, y, z by $(-2)^n x$, $(-2)^n y$, $(-2)^n z$ in (2.6), respectively, and dividing (2.6) by $|2|^n$, by (AN3), we have

$$\begin{aligned} &\|\frac{f((-2)^n x)}{(-2)^n} + \frac{f((-2)^n y)}{(-2)^n} + \frac{f((-2)^n az)}{(-2)^n}, w\| \\ &\le \|\frac{f((-2)^n (x+y))}{(-2)^n} - \frac{f(-(-2)^n az)}{(-2)^n}, w\| + \frac{\phi((-2)^n x, (-2)^n y, (-2)^n z)}{|2|^n} \end{aligned} \tag{2.10}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.10), by Lemma 1.2 and (2.4), we have

$$\|A(x) + A(y) + A(az), w\| \le \|A(x+y) - A(-az), w\|$$

for all $x, y, z \in X$ and $w \in Y$. By Theorem 2.1, A is additive and by (2.5) and (2.9), we have (2.7).

Suppose that (2.8) holds. Now, we show the uniqueness of A. Let A_0 be an additive mapping with (2.7). Then for any postive integer n, $2^n A(x) = A(2^n x)$ and $2^n A_0(x) = A_0(2^n x)$ for all $x \in X$. Hence by (2.7), (AN3) and (AN4), we have

$$\begin{aligned}
&\|A(x) - A_0(x), w\| \\
&= \frac{\|A((-2)^k x) - A_0((-2)^k x), w\|}{|2|^k} \\
&\leq \max\{\frac{\|A((-2)^k x) - f((-2)^k x), w\|}{|2|^k}, \frac{\|A_0((-2)^k x) - f((-2)^k x), w\|}{|2|^k}\} \\
&\leq \lim_{n\to\infty} \max\{\frac{\phi((-2)^{i+k}x, (-2)^{i+k}x, (-2)^{i+k+1}\frac{x}{a})}{|2|^{i+k+1}} \mid 0 \leq i \leq n-1\} \\
&\leq \lim_{n\to\infty} \max\{\frac{\phi((-2)^{i}x, (-2)^{i}x, (-2)^{i+1}\frac{x}{a})}{|2|^{i+1}} \mid k \leq i \leq k+n-1\}
\end{aligned}$$

for all $x \in X$, $w \in Y$, and all $k \in \mathbb{N}$. Hence, letting $k \to \infty$ in the above inequality, by (2.8), we have

$$A(x) = A_0(x)$$

for all $x \in X$. □

Related with Theorem 2.2, we can also have the following theorem. And the proof is similar to that of Theorem 2.2.

Theorem 2.3. *Assume that* $\phi : X^3 \longrightarrow [0,\infty)$ *is a function such that*

$$\lim_{n\to\infty} |2|^n \phi(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}) = 0 \tag{2.11}$$

for all $x, y, z \in X$ *and for any* $x \in X$, *the limit*

$$\lim_{n\to\infty} \max\{|2|^{k-1}\phi(\frac{x}{(-2)^k}, \frac{x}{(-2)^k}, \frac{x}{(-2)^{k-1}a}) \mid 0 \leq k \leq n-1\}$$

exsits. Let $f : X \longrightarrow Y$ *be a mapping satisfying (2.6). Then there exists an additive mapping* $A : X \longrightarrow Y$ *such that*

$$\begin{aligned}
&\|f(x) - A(x), w\| \\
&\leq \lim_{n\to\infty} \max\{|2|^{k-1}\phi(\frac{x}{(-2)^k}, \frac{x}{(-2)^k}, \frac{x}{(-2)^{k-1}a}) \mid 0 \leq k \leq n-1\}
\end{aligned}$$

for all $x \in X$ *and* $w \in Y$. *Moreover, if* $\phi : X^3 \longrightarrow [0,\infty)$ *satisfies*

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\{|2|^{i-1}\phi(\frac{x}{(-2)^i}, \frac{x}{(-2)^i}, \frac{x}{(-2)^i a}) \mid k \leq i \leq k+n-1\} = 0$$

for all $x \in X$, then A is a unique additive mapping satisfying (2.7).

Proof. Replacing x, y, z by $\frac{x}{(-2)^n}$, $\frac{x}{(-2)^n}$, $\frac{x}{(-2)^{n-1}a}$ in (2.6), respectively and multiplying (2.6) by $|2|^{n-1}$, since $f(0)=0$, by (AN3), we have

$$\begin{aligned}&\|(-2)^n f(\frac{x}{(-2)^n}) - (-2)^{n-1} f(\frac{x}{(-2)^{n-1}}), w\|\\ &\leq |2|^{n-1}\phi(\frac{x}{(-2)^n}, \frac{x}{(-2)^n}, \frac{x}{(-2)^{n-1}a})\end{aligned}$$

for all $x \in X$, $w \in Y$, and all $n \in \mathbb{N}$. By (2.11), $\{(-2)^n f(\frac{x}{(-2)^n})\}$ is a Cauchy sequence in Y. Since Y is a non-Archimedean 2-Banach space, there is a function $A : X \longrightarrow Y$ such that

$$A(x) = \lim_{n\to\infty}(-2)^n f(\frac{x}{(-2)^n})$$

for all $x \in X$ and $w \in Y$. Further for $0 \leq m < n$, we have

$$\begin{aligned}&\|(-2)^n f(\frac{x}{(-2)^n}) - (-2)^m f(\frac{x}{(-2)^m}), w\|\\ &\leq \max\{|2|^{k-1}\phi(\frac{x}{(-2)^k}, \frac{x}{(-2)^k}, \frac{x}{(-2)^{k-1}a}) \mid m \leq k \leq n-1\}\end{aligned} \tag{2.12}$$

for all $x \in X$ and $w \in Y$. The rest of proof is similar to the proof of Theorem 2.2. □

As an example of $\phi(x,y,z)$ in Theorem 2.2 and Theorem 2.3, we can take $\phi(x,y,z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ for some positive real numbers ϵ and p. Then we can formulate the following corollary :

COROLLARY 2.4. *Let X be a non-Archimedean 2-normed space with $\dim X > 1$ and Y a non-Archimedean 2-Banach space with $\dim Y > 1$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned}&\|f(x) + f(y) + f(az), w\|\\ &\leq \|f(x+y) - f(-az), w\| + \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)\end{aligned}$$

for all $x, y, z \in X$, $w \in Y$, and some positive real numbesr ϵ, p with $p \neq 1$. Suppose that $|2| < 1$. Then there exists a unique additive mapping $A : X \longrightarrow Y$ such that A satifies (1.4) and

$$\|A(x) - f(x),\ \ w\| \leq \frac{\epsilon(2|a|^p + |2|^p)}{|2a|^p}\|x\|^p$$

for all $x \in X$ *and all* $w \in Y$.

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