AN ADDITIVE FUNCTIONAL INEQUALITY

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Abstract. In this paper, we solve the additive functional inequality
\[ \|f(x) + f(y) + f(z)\| \leq \|\rho f(s(x + y + z))\|, \]
where \( s \) is a nonzero real number and \( \rho \) is a real number with \(|\rho| < 3\).

Moreover, we prove the Hyers-Ulam stability of the above additive functional inequality in Banach spaces.

1. Introduction and preliminaries


In [5], Gilányi showed that if \( f \) satisfies the functional inequality
\[ \|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \]
then $f$ satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$


In Section 2, we solve the additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|\rho f(s(x + y + z))\|,$$

and prove the Hyers-Ulam stability of the additive functional inequality (2).

Park, Cho and Han [8] investigated the additive functional inequalities for the case $\rho = s = 1$, and the case $\rho = 2$ and $s = \frac{1}{2}$.

Throughout this paper, let $X$ be a normed space with norm $\|\cdot\|$ and $Y$ a Banach space with norm $\|\cdot\|$. Assume that $s$ is a nonzero real number and that $\rho$ is a real number with $|\rho| < 3$.

### 2. The additive functional inequality (2)

**Lemma 2.1.** If a mapping $f : X \to Y$ satisfies

$$\|f(x) + f(y) + f(z)\| \leq \|\rho f(s(x + y + z))\|$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

**Proof.** Letting $x = y = z = 0$ in (3), we get

$$\|3f(0)\| \leq \|\rho f(0)\|.$$

So $f(0) = 0$.

Letting $z = -x$ and $y = 0$ in (3), we get

$$\|f(x) + f(-x)\| \leq \|\rho f(0)\| = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ in (3), we get

$$\|f(x) + f(y) + f(-x - y)\| \leq \|\rho f(0)\| = 0$$

for all $x, y \in X$. So $f(x) + f(y) = -f(-x - y) = f(x+y)$ for all $x, y \in X$, as desired. \(\square\)

**Corollary 2.2.** If a mapping $f : X \to Y$ satisfies

$$f(x) + f(y) + f(z) = \rho f(s(x + y + z))$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.
Now, we prove the Hyers-Ulam stability of the additive functional inequality (2) in Banach spaces.

**Theorem 2.3.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping such that

\[
\|f(x) + f(y) + f(z)\| \leq \|\rho f(s(x + y + z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( h : X \to Y \) such that

\[
\|f(x) - h(x)\| \leq \frac{2 + 3 \cdot 2^r}{2r - 2} \theta \|x\|^r
\]

for all \( x \in X \).

**Proof.** Letting \( x = y = z = 0 \) in (5), we get \( f(0) = 0 \).

Letting \( y = -x \) and \( z = 0 \) in (5), we get

\[
\|f(x) + f(-x)\| \leq 2\theta \|x\|^r
\]

for all \( x \in X \). So

\[
\|f(2x) + f(-2x)\| \leq 2 \cdot 2^r \theta \|x\|^r
\]

for all \( x \in X \).

Letting \( y = x \) and \( z = -2x \) in (5), we get

\[
\|2f(x) + f(-2x)\| \leq (2 + 2^r)\theta \|x\|^r
\]

for all \( x \in X \). It follows from (7) and (8) that

\[
\|2f(x) - f(2x)\| \leq (2 + 3 \cdot 2^r)\theta \|x\|^r
\]

for all \( x \in X \). So

\[
\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{2 + 3 \cdot 2^r}{2r} \theta \|x\|^r
\]

for all \( x \in X \). Hence

\[
\left\|2^j f\left(\frac{x}{2^j}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|
\]

\[
\leq \frac{2 + 3 \cdot 2^r}{2r} \sum_{j=l}^{m-1} \frac{2^j}{2^r} \theta \|x\|^r
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (10) that the sequence \( \{2^m f\left(\frac{x}{2^m}\right)\} \) is a Cauchy sequence for
all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^n f(x/2^n)\} \) converges. So one can define the mapping \( h : X \to Y \) by
\[
h(x) := \lim_{n \to \infty} 2^n f(x/2^n)
\]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (10), we get (6).

It follows from (5) that
\[
\|h(x) + h(y) + h(z)\| = \lim_{n \to \infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\|
\]
\[
\leq \lim_{n \to \infty} 2^n \left\| \rho f\left(\frac{s(x+y+z)}{2^n}\right)\right\|
\]
\[
+ \lim_{n \to \infty} \frac{2^n \theta}{2^n r} (\|x\|^r + \|y\|^r + \|z\|^r)
\]
\[
= \|\rho h(s(x+y+z))\|
\]
for all \( x, y, z \in X \). So
\[
\|h(x) + h(y) + h(z)\| \leq \|\rho h(s(x+y+z))\|
\]
for all \( x, y, z \in X \). By Lemma 2.1, the mapping \( h : X \to Y \) is additive.

Now, let \( T : X \to Y \) be another additive mapping satisfying (6). Then we have
\[
\|h(x) - T(x)\| = 2^n \left\| h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|
\]
\[
\leq 2^n \left(\left\| h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right)
\]
\[
\leq \frac{2(2 + 3 \cdot 2^r)2^n}{(2^r - 2)2^n - \theta \|x\|^r},
\]
which tends to zero as \( n \to \infty \) for all \( x \in X \). So we can conclude that \( h(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( h \). Thus the mapping \( h : X \to Y \) is a unique additive mapping satisfying (6).

**Theorem 2.4.** Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (5). Then there exists a unique additive mapping \( h : X \to Y \) such that
\[
\|f(x) - h(x)\| \leq \frac{2 + 3 \cdot 2^r}{2 - 2^r} \theta \|x\|^r
\]
for all \( x \in X \).
Proof. It follows from (9) that
\[ \left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2 + 3 \cdot 2^r}{2} \theta \|x\|^r \]
for all \( x \in X \). Hence
\[ \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\| \]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (12) that the sequence \( \left\{ \frac{1}{2^n} f(2^n x) \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \left\{ \frac{1}{2^n} f(2^n x) \right\} \) converges. So one can define the mapping \( h : X \rightarrow Y \) by
\[ h(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (12), we get (11).

The rest of the proof is similar to the proof of Theorem 2.3. \( \square \)

By the triangle inequality, we have
\[ \| f(x) + f(y) + f(z) - \rho f(s(x + y + z)) \| \leq \| f(x) + f(y) + f(z) - \rho f(s(x + y + z)) \| . \]
As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the additive functional equation (4) in Banach spaces.

Corollary 2.5. Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \rightarrow Y \) be a mapping such that
\[ \| f(x) + f(y) + f(z) - \rho f(s(x + y + z)) \| \leq \theta (\|x\|^r + \|y\|^r + \|z\|^r) \]
for all \( x, y, z \in X \). Then there exists a unique additive mapping \( h : X \rightarrow Y \) satisfying (6).

Corollary 2.6. Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \rightarrow Y \) be a mapping satisfying (13). Then there exists a unique additive mapping \( h : X \rightarrow Y \) satisfying (11).
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