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A CONVERGENCE OF C_0 SEMIGROUPS IN THE WEAK TOPOLOGY

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Abstract. In this paper, we establish convergence of contraction C_0 semigroups in the weak topology on a general Banach space. We remove the restriction on a Banach space X and weaken the condition on resolvents of generators in the previous results [4, 5].

1. Introduction

Trotter-Kato type approximation theorem shows a relation among the convergence of C_0 semigroups $\{T_n(t) : t \geq 0\}$ on a Banach space X and the convergence of the generators A_n and their resolvents $R(\lambda,\ A_n)$ for the strong operator topology. In [2], Eisner and Serény ask the validity of Trotter-Kato theorem in the weak operator topology and show that the weak convergence of resolvents of generators does not imply the weak convergence of the corresponding semigroups. So we need certain conditions on the generators or their resolvents to obtain the weak convergence of corresponding semigroups. If X is a Hilbert space, it is proved in [5] that the weak convergence of the powers of resolvents and a restriction on generators imply the weak convergence of corresponding semigroups. This result can be extended to the Banach space X whose dual space X^* is uniformly convex [4]. The restriction on a Banach

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space X with uniform convexity of X^* seems to be essential to prove the result because we need the uniform continuity of the dual mapping in the proof.

In this paper we establish the weak convergence of semigroups with weaker conditions in the previous result in a general Banach space. We do not assume any restriction on a Banach space. To obtain the convergence of semigroups, we use the fact that the convergence of functions is equivalent to the convergence of their Laplace transforms and the equicontinuity of functions. Since the resolvents of generators are the Laplace transforms of the corresponding semigroups, we need the weak convergence of resolvents and the the equicontinuity of semigroups to have the weak convergence of semigroups. So the weak convergence of powers of the resolvents can be weakened by the weak convergence of the resolvents in the previous result and we still need certain condition that makes an equicontinuity of semigroups. In the case of strong operator topology, the equicontinuity of semigroups can be obtained by the denseness of the domain of the generator and the convergence of their resolvents. But this is not valid in the weak operator topology. So we need a restriction on generators to have the equicontinuity of semigroups in the weak operator topology.

2. Main Results

Let X be a Banach space. A family $\{T(t) : t \geq 0\}$ of bounded linear operators from X into itself is called a contraction C_0 semigroup on X if $T(0) = I$, $T(t+s) = T(t)T(s)$ for $t, s \geq 0$, for each $x \in X$ $T(t)x$ is continuous in $t \geq 0$ and $\|T(t)x\| \leq \|x\|$ for $t \geq 0$ and $x \in X$.

The linear operator A, defined by

$$Ax = \lim_{h \to 0} \frac{1}{h}(T(h)x - x)$$

for $x \in D(A) = \{x \in X : \lim_{h\to 0}(T(h)x - x)/h \text{ exists}\}$, is called the generator of a contraction C_0 semigroup $\{T(t) : t \geq 0\}$ and $D(A)$ is the domain of A.

The resolvent set of A is denoted by $\rho(A)$ and for $\lambda \in \rho(A)$ $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator of A. For more information about C_0 semigroups and their generators, we refer [3, 6].

The following theorem states that the convergence of functions is equivalent to the convergence of their Laplace transforms and the equicontinuity of functions. This result is appeared in [7] and Theorem 1 is quoted in [1]

THEOREM 1. *Let $f_n : [0,\ \infty) \to X$ be a continuous function with $\|f_n(t)\| \leq Me^{\omega t}$ for some $M > 0$, $\omega \in R$ and all $n \in N$. Then the followings are equivalent.*

(i) *The Laplace transform $\hat{f}_n(\lambda)$ of $f_n(t)$ converges pointwise on $(\omega,\ \infty)$ and the sequence $\{f_n : n \in N\}$ is equicontinuous at $t \geq 0$.*
(ii) *The function f_n converges uniformly on bounded subsets of $[0,\ \infty)$.*

We denote the value $\phi(x)$ of $\phi \in X^*$ at $x \in X$ by $< x,\ \phi >$. Now, we present our main theorem.

THEOREM 2. *Let X be a Banach space. Let $\{T_n(t) : t \geq 0\}$ be a sequence of contraction C_0 semigroups on X with generators A_n and let $\{T(t) : t \geq 0\}$ be a contraction C_0 semigroup on X with generator A. Suppose that*

$$w - \lim_{n\to\infty} R(\lambda, A_n)x = R(\lambda, A)x,$$

for $x \in X$ and

$$D = \{x \in \cap_{n=1}^{\infty} D(A_n) : \sup_{n\geq 1} \|A_n x\| < \infty\}$$

is dense in X. Then

$$w - \lim_{n\to\infty} T_n(t)x = T(t)x$$

for all $x \in X$ and the convergence is uniform on bounded $t-$ intervals.

Proof. Let $x \in D$ and $0 \leq t \leq T$. Then there exists $M > 0$ such that $\|A_n x\| \leq M$ for all n. For $h > 0$, we have

$$\begin{aligned}
\|T_n(h)x - x\| &= \|\int_0^h \frac{d}{ds} T_n(s)xds\| \\
&= \|\int_0^h T_n(s)A_n xds\| \\
&\leq \int_0^h \|A_n x\|ds = h\|A_n x\| \leq Mh,
\end{aligned}$$

since $x \in D$ and $\|T_n(t)\| \leq 1$ for all $t \geq 0$. Since $\|T(t+h)x - T(t)x\| \leq \|T(t)T(h)x - T(t)x\| \leq \|T(h)x - x\|$ for $h > 0$ and $\|T(t+h)x - T(t)x\| \leq$

$\|T(t+h)x-T(t+h)T(-h)x\| \leq \|x-T(-h)x\|$ for $h<0$, $\{T_n(t) : n \in N\}$ is equicontinuous.

For each $\phi \in X^*$, $\{< T_n(t)x,\ \phi >: n \in N\}$ is equicontinuous and $< R(\lambda, A_n)x,\ \phi >= \int_0^\infty e^{-\lambda t} < T_n(t)x,\ \phi > dt$ converges to $< R(\lambda, A)x,\ \phi >= \int_0^\infty e^{-\lambda t} < T(t)x,\ \phi > dt$, by hypothesis. By Theorem 1, we have $\lim_{n\to\infty} < T_n(t)x,\ \phi >=< T(t)x,\ \phi >$, uniformly on bounded t-intervals.

Let $x \in X$. By the density of D, there exists $x_k \in D$ such that $\lim_{k\to\infty} x_k = x$. Since $T_n(t)$ and $T(t)$ are contractions, we have

$$\begin{aligned}
&| < T_n(t)x - T(t)x,\ \phi > | \\
&\qquad \leq | < T_n(t)x - T_n(t)x_k,\ \phi > | + | < T_n(t)x_k - T(t)x_k,\ \phi > | \\
&\qquad\qquad\qquad + | < T(t)x_k - T(t)x,\ \phi > | \\
&\qquad \leq 2\|x - x_k\|\|\phi\| + | < T_n(t)x_k - T(t)x_k,\ \phi > |.
\end{aligned}$$

Let $\varepsilon > 0$. Then there exists k_0 such that $2\|x - x_{k_0}\|\|\phi\| < \varepsilon/2$. Since $x_{k_0} \in D$, there exists an integer n_0 such that $| < T_n(t)x_{k_0} - T(t)x_{k_0},\ \phi > | < \varepsilon/2$ for all $n \geq n_0$. Hence we have $| < T_n(t)x - T(t)x,\ \phi > | < \varepsilon$ for all $n \geq n_0$ and $x \in X$. $\square$

Remark. In the case of strong operator topology, the convergence of generators is equivalent to the convergence of their resolvents. But this is not valid in the weak operator topology. For example, if $w - \lim_{n\to\infty} A_n x = Ax$, then we have the boundedness of $\|A_n x\|$ but we do not have the convergence of their resolvents even if all A_n are bounded (see Example 2.1 in [2]). So the condition that D is dense in X seems to be essential to show the equicontinuity of semigroups.

Consider the inverse case. That is, if we have the weak convergence of semigroups, is $\{\|A_n x\|\}$ bounded? Since $R(\lambda, A_n)x = \int_0^\infty e^{-\lambda t} T_n(t)x dt$ and $w - \lim_{n\to\infty} T_n(t)x = T(t)x$, we have $w - \lim_{n\to\infty} R(\lambda, A_n)x = R(\lambda, A)x$, by the dominated convergence theorem. For $x \in D(A)$, $x = R(\lambda, A)y$ for some $y \in X$. Let $x_n = R(\lambda, A_n)y$. Then $w - \lim_{n\to\infty} x_n = x$ and $w - \lim_{n\to\infty} A_n x_n = Ax$, since $A_n x_n = A_n R(\lambda, A_n)y = \lambda R(\lambda, A_n)y - y$. Suppose that all A_n are bounded, uniformly in n, that is $\|A_n\| \leq K$ for some $K > 0$. Then we have the boundedness of $\|A_n x\|$, since $\|A_n x\| \leq ||A_n(x - x_n)\| + \|A_n x_n\|$. Thus the inverse of Theorem 2 does not seem to be true without the restrictions on generators A_n.

Corollary 3. *Let $\{T_n(t) : t \geq 0\}$ be a sequence of contraction C_0 semigroups on a Banach space X with generators A_n. Let A be a densely* *defined closed linear operator in* X *with* $(0,\ \infty) \subset \rho(A)$. *Suppose that*

$$w - \lim_{n\to\infty} R(\lambda, A_n)x = R(\lambda, A)x,$$

for $x \in X$ *and*

$$D = \{x \in \cap_{n=1}^{\infty} D(A_n) : \sup_{n\geq 1} \|A_n x\| < \infty\}$$

is dense in X. *Then* A *is the generator of a contraction* C_0 *semigroup* $\{T(t) : t \geq 0\}$ *and*

$$w - \lim_{n\to\infty} T_n(t)x = T(t)x$$

for all $x \in X$ *and the convergence is uniform on bounded* $t-$ *intervals.*

Proof. Since $w - \lim_{n\to\infty} R(\lambda, A_n)x = R(\lambda, A)x$, we have

$$\|R(\lambda,\ A)x\| \leq \liminf_{n\to\infty} \|R(\lambda,\ A_n)x\|.$$

Since A_n is the generator of a contraction C_0 semigroup, $\|R(\lambda,\ A_n)x\| \leq 1/\lambda\|x\|$ for all $n \in N$ and so we have $\|R(\lambda,\ A)\| < 1/\lambda$. By Hille-Yosida Theorem, A is the generator of a contraction C_0 semigroup $\{T(t) : t \geq 0\}$. By Theorem 2, the result follows. □

Consider the following inhomogeneous initial value problem.

$$\frac{du}{dt}(t) = Au(t) + f(t), \quad u(0) = x$$

For $x \in X$ and $f \in L^1([0,\ T], X)$, the mild solution is given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq T,$$

where A is the generator of a contraction C_0 semigroup $\{T(t) : t \geq 0\}$ (see [6]).

Corollary 4. *Let* $\{A_n : n \in N\}$ *be a sequence of generators of contraction* C_0 *semigroups* $\{T_n(t) : t \geq 0\}$ *on a Banach space satisfying the hypotheses of Corollary 3. Then*

$$w - \lim_{n\to\infty} u_n(t) = u(t), \quad \textit{uniformly in } [0,\ T],$$

where u_n *is the mild solution of* $du_n/dt = Au_n + f$, $u_n(0) = x$.

Proof. Since u_n is a mild solution, $u_n(t) = T_n(t)x + \int_0^t T_n(t-s)f(s)ds$. And

$$\begin{aligned} |< u_n(t) - u(t),\ \phi >| \quad &\leq \quad |< T_n(t)x - T(t)x,\ \phi >| \\ &\quad + |\int_0^t < T_n(t-s)f(s) - T(t-s)f(s),\ \phi > ds|. \end{aligned}$$

Since $\lim_{n\to\infty} < T_n(t-s)f(s) - T(t-s)f(s),\ \phi >= 0$ and $| < T_n(t-s)f(s) - T(t-s)f(s),\ \phi > | \leq 2\|f(s)\|\|\phi\|$, we have

$$\lim_{n\to\infty} \int_0^t < T_n(t-s)f(s) - T(t-s)f(s),\ \phi > ds = 0$$

by the dominated convergence theorem. Hence the result follows. □

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