CONVERGENCE THEOREMS FOR THE
CHOQUET-PETTIS INTEGRAL

Chun-Kee Park

Abstract. In this paper, we introduce the concept of Choquet-Pettis integral of Banach-valued functions using the Choquet integral of real-valued functions and investigate convergence theorems for the Choquet-Pettis integral.

1. Introduction

The fuzzy measure was introduced by Sugeno [9] and the Choquet integral of real-valued functions with respect to a fuzzy measure was introduced by Murofushi and Sugeno [5]. The Choquet integral is a generalization of the Lebesgue integral, since they coincide when $\mu$ is a classical $\sigma$–additive measure. The Choquet integral is a basic tool for the subjective evaluation and decision analysis. The convergence theorems are very important in classical integral theory and also Choquet integral theory. Narukawa, Murofushi and Sugeno [8] introduced the regular fuzzy measure on a locally compact Hausdorff space and showed the usefulness in the point of representation of some functional.

In this paper, we introduce the concept of Choquet-Pettis integral of Banach-valued functions using the Choquet integral of real-valued functions. The Choquet-Pettis integral is an extension of the Choquet integral.
integral for Banach-valued functions and this integral is also a generalization of the Pettis integral, since the Choquet integral and the Lebesgue integral coincide when $\mu$ is a classical $\sigma-$additive measure. We also investigate convergence theorems for this integral.

2. Preliminaries

Throughout this paper, $X$ denotes a real Banach space and $X^*$ its dual. Let $\Omega$ be a nonempty classical set, $\Sigma$ a $\sigma$-algebra formed by the subsets of $\Omega$ and $(\Omega, \Sigma)$ a measurable space.

**Definition 2.1.** [7,9]. A **fuzzy measure** on a measurable space $(\Omega, \Sigma)$ is an extended real-valued set function $\mu : \Sigma \rightarrow [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,  
(ii) $\mu(A) \leq \mu(B)$ whenever $A \subset B$, $A, B \in \Sigma$.

When $\mu(\Omega) < \infty$, we say that $\mu$ is **finite**. When $\mu$ is finite, we define the **conjugate** $\mu^c$ of $\mu$ by

$$\mu^c(A) = \mu(\Omega) - \mu(A^c),$$

where $A^c$ is the complement of $A \in \Sigma$.

A fuzzy measure $\mu$ is said to be **lower semi-continuous** if it satisfies

$$A_1 \subset A_2 \subset \cdots \text{ implies } \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n).$$

A fuzzy measure $\mu$ is said to be **upper semi-continuous** if it satisfies

$$A_1 \supset A_2 \supset \cdots \text{ and } \mu(A_1) < \infty \text{ implies } \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n).$$

A fuzzy measure $\mu$ is said to be **continuous** if it is both lower and upper semi-continuous.

If a fuzzy measure $\mu$ is lower(resp., upper) semi-continuous, then $\mu^c$ is upper(resp., lower) semi-continuous.

The class of real-valued measurable functions is denoted by $M$ and the class of nonnegative real-valued measurable functions is denoted by $M^+$. The class of non-negative upper semi-continuous real-valued functions with compact support is denoted by $USCC^+$ and the class of non-negative lower semi-continuous real-valued functions is denoted by $LSC^+$. 
Definition 2.2. [1,5]. (1) The Choquet integral of $f \in M^+$ with respect to a fuzzy measure $\mu$ on $A \in \Sigma$ is defined by

\[(C) \int_A f d\mu = \int_0^\infty \mu((f \geq r) \cap A)dr,\]

where the right-hand side integral is the Lebesgue integral and $(f \geq r) = \{\omega \in \Omega \mid f(\omega) \geq r\}$ for all $r \geq 0$.

If $(C) \int_A f d\mu < \infty$, then we say that $f$ is Choquet integrable on $A$ with respect to $\mu$. Instead of $(C) \int_\Omega f d\mu$, we will write $(C) \int f d\mu$.

(2) Suppose $\mu(\Omega) < \infty$. The Choquet integral of $f \in M^+$ with respect to a fuzzy measure $\mu$ on $A \in \Sigma$ is defined by

\[(C) \int_A f d\mu = (C) \int_A f^+ d\mu - (C) \int_A f^- d\mu,\]

where $f^+ = f \lor 0$ and $f^- = -(f \land 0)$. When the right-hand side is $\infty - \infty$, the Choquet integral is not defined. If $(C) \int_A f d\mu$ is finite, then we say that $f$ is Choquet integrable on $A$ with respect to $\mu$.

$L_1^+(\mu)$ denotes the class of nonnegative Choquet integrable functions. That is,

\[L_1^+(\mu) := \left\{ f \mid f \in M^+, (C) \int f d\mu < \infty \right\}.\]

The Choquet integral is a generalization of the Lebesgue integral, since they coincide when $\mu$ is a classical $\sigma$-additive measure. For each $f \in M^+$, we also have

\[(C) \int_A f d\mu = \int_0^\infty \mu((f > r) \cap A)dr, \forall A \in \Sigma,\]

where $(f > r) = \{\omega \in \Omega \mid f(\omega) > r\}$ for all $r \geq 0$.

Definition 2.3. [2]. Let $f, g \in M$. We say that $f$ and $g$ are comonotonic if $f(\omega) < f(\omega') \Rightarrow g(\omega) \leq g(\omega')$ for $\omega, \omega' \in \Omega$. We denote $f \sim g$ when $f$ and $g$ are comonotonic.

Definition 2.4. [3]. A sequence $(f_n)$ of real-valued measurable functions is said to converge to $f$ in distribution, in symbols $f_n \overset{D}{\to} f$, if

\[\lim_{n \to \infty} \mu((f_n \geq r)) = \mu(f \geq r) \quad \text{e.c.},\]

where “e.c.” stands “except at most countably many values of $r$”.

Convergence theorems for the Choquet-Pettis integral
3. Results

We introduce the concept of Choquet-Pettis integral of Banach-valued functions. The concept of Pettis integral and its properties may be found in [4].

**Definition 3.1.** A function $f : \Omega \to X$ is called **Choquet-Pettis integrable** if for each $x^* \in X^*$ the function $x^*f$ is Choquet integrable and for every $A \in \Sigma$ there exists $x_A \in X$ such that $x^*(x_A) = (C) \int_A x^* f d\mu$ for all $x^* \in X^*$. The vector $x_A$ is called the Choquet-Pettis integral of $f$ on $A$ and is denoted by $(CP) \int_A f d\mu$.

The Choquet-Pettis integral is a generalization of the Pettis integral, since the Choquet integral and the Lebesgue integral coincide when $\mu$ is a classical $\sigma$-additive measure.

**Definition 3.2.** 

1. Let $f : \Omega \to X$ and $g : \Omega \to X$ be weakly measurable. $f$ and $g$ are said to be **weakly comonotonic** if for each $x^* \in X^*$ $x^*f$ and $x^*g$ are comonotonic. We denote $f \sim_w g$ when $f$ and $g$ are weakly comonotonic.

2. A sequence $(f_n)$ of $X$-valued weakly measurable functions is said to **converge weakly to $f$ in distribution** on $\Omega$, in symbols $f_n \overset{wD}{\to} f$, if for each $x^* \in X^*$ $(x^* f_n)$ converges to $x^* f$ in distribution.

A set $N \in \Sigma$ is called a **null set** with respect to $\mu$ if $\mu(A \cup N) = \mu(A)$ for all $A \in \Sigma$ [6]. “$P(\omega)$ $\mu$-a.e. on $A$” means that there exists a null set $N$ such that $P(\omega)$ is true for all $\omega \in A - N$, where $P(\omega)$ is a proposition concerning the point of $A$.

**Theorem 3.3.** Let $f : \Omega \to X$ and $g : \Omega \to X$ be Choquet-Pettis integrable. Then

1. $af$ is Choquet-Pettis integrable and

   $$(CP) \int_A af d\mu = a(CP) \int_A f d\mu$$

   for all $A \in \Sigma$ and $a \geq 0$;

2. if $f \sim_w g$, then $f + g$ is Choquet-Pettis integrable and

   $$(CP) \int_A (f + g) d\mu = (CP) \int_A f d\mu + (CP) \int_A g d\mu$$

   for all $A \in \Sigma$;
(3) if $f = g \mu$-a.e. and $\mu^*$-a.e. on $\Omega$, then

$$\int_A f d\mu = \int_A g d\mu$$

for all $A \in \Sigma$.

Proof. (1) Since $f : \Omega \to X$ is Choquet-Pettis integrable, for each $x^* \in X^*$ $x^*f$ is Choquet integrable and for every $A \in \Sigma$ there exists $x_A \in X$ such that $x^*(x_A) = (C) \int_A x^*f d\mu$ for all $x^* \in X^*$. Hence for each $x^* \in X^*$ $x^*(af)$ is Choquet integrable and for every $A \in \Sigma$ $x^*(ax_A) = (C) \int_A x^*(af) d\mu$ for all $x^* \in X^*$. Thus $af$ is Choquet-Pettis integrable and $(CP) \int_A afd\mu = ax_A = a(CP) \int_A f d\mu$ for all $A \in \Sigma$ and $a \geq 0$.

(2) Since $f : \Omega \to X$ and $g : \Omega \to X$ are Choquet-Pettis integrable, for each $x^* \in X^*$ $x^*f$ and $x^*g$ are Choquet integrable and for every $A \in \Sigma$ there exist $x_A, y_A \in X$ such that $x^*(x_A) = (C) \int_A x^*f d\mu$ and $x^*(y_A) = (C) \int_A x^*g d\mu$ for all $x^* \in X^*$. Since $f \sim w g$, for each $x^* \in X^*$ $x^*(f + g)$ is Choquet integrable and for every $A \in \Sigma$ $x^*(x_A + y_A) = (C) \int_A x^*(f + g) d\mu$ for all $x^* \in X^*$. Thus $f + g$ is Choquet-Pettis integrable and $(CP) \int_A (f + g) d\mu = x_A + y_A = (CP) \int_A f d\mu + (CP) \int_A g d\mu$ for all $A \in \Sigma$.

(3) Since $f : \Omega \to X$ and $g : \Omega \to X$ are Choquet-Pettis integrable, for each $x^* \in X^*$ $x^*f$ and $x^*g$ are Choquet integrable and for every $A \in \Sigma$ there exist $x_A, y_A \in X$ such that $x^*(x_A) = (C) \int_A x^*f d\mu$ and $x^*(y_A) = (C) \int_A x^*g d\mu$ for all $x^* \in X^*$. Since $f = g \mu$-a.e. and $\mu^*$-a.e. on $\Omega$, $x^*f = x^*g$ $\mu$-a.e. and $\mu^*$-a.e. on $\Omega$ for all $x^* \in X^*$. Hence for every $A \in \Sigma$ $(CP) \int_A x^*f d\mu = (CP) \int_A x^*g d\mu$ i.e., $x^*(x_A) = x^*(y_A)$ for all $x^* \in X^*$. Hence $x_A = y_A$ i.e., $(CP) \int_A f d\mu = (CP) \int_A g d\mu$.

\[ \square \]

Theorem 3.4. Let $X$ be a reflexive Banach space and let $(f_n)$ be a sequence of Choquet-Pettis integrable $X$-valued functions on $\Omega$. If $(f_n)$ converges weakly to $f$ in distribution on $\Omega$ and if $g$ and $h$ are Choquet-Pettis integrable $X$-valued functions on $\Omega$ such that $\mu((x^*h \geq r)) \leq \mu((x^*f_n \geq r)) \leq \mu((x^*g \geq r))$ e.c. for $n = 1, 2, \ldots$ and $x^* \in X^*$, then $f$ is Choquet-Pettis integrable and $(CP) \int f_n d\mu \to (CP) \int f d\mu$ weakly.

Proof. Since $g$ and $h$ are Choquet-Pettis integrable, for each $x^* \in X$ $x^*g$ and $x^*h$ are Choquet integrable. Since $(f_n)$ converges weakly to $f$ in distribution, for each $x^* \in X$ $(x^*f_n)$ converges to $x^*f$ in distribution.
By hypothesis, \( \mu((x^h \geq r)) \leq \mu((x^g \geq r)) \) e.c. for \( n = 1, 2, \ldots \) and \( x^\ast \in X^\ast \). By [3, Theorem 8.9] \( x^\ast f \) is Choquet integrable and \( \lim_{n \to \infty} (C) \int_A x^\ast f_n d\mu = (C) \int_A x^\ast f d\mu \) for all \( A \in \Sigma \) and \( x^\ast \in X^\ast \). Since \( f_n \) is Choquet-Pettis integrable for \( n = 1, 2, \ldots \), for each \( A \in \Sigma \) there exists \( x_{n,A} \in X \) such that \( x^\ast(x_{n,A}) = (C) \int_A x^\ast f_n d\mu \) for all \( x^\ast \in X^\ast \), i.e., \( x_{n,A} = (CP) \int_A f_n d\mu \). Thus \( (x_{n,A}) \) is a weak Cauchy sequence in \( X \). Since \( X \) is a reflexive Banach space, the sequence \( (x_{n,A}) \) converges weakly to some \( x_A \in X \). Thus \( \lim_{n \to \infty} x^\ast(x_{n,A}) = x^\ast(x_A) \) for all \( x^\ast \in X^\ast \). Hence \( x^\ast(x_A) = (C) \int_A x^\ast f d\mu \) for all \( x^\ast \in X^\ast \). Thus \( f \) is Choquet-Pettis integrable and \( x_A = (CP) \int_A f d\mu \) for each \( A \in \Sigma \). In particular, \( (CP) \int f_n d\mu \to (CP) \int f d\mu \) weakly.

\[ \Box \]

**Theorem 3.5.** (1) Let \( \mu \) be a finite and lower semi-continuous fuzzy measure and let \( (f_n) \) be a sequence of real-valued measurable functions. If \( f_n \uparrow f \mu \)-a.e. and \( \mu^\ast \)-a.e. and there exists a Choquet integrable function \( g \) such that \( f_n^\ast \leq g \) on \( \Omega \), then \( f \) is Choquet integrable and \( (C) \int f_n d\mu \uparrow (C) \int f d\mu \).

(2) Let \( \mu \) be a finite and upper semi-continuous fuzzy measure and let \( (f_n) \) be a sequence of real-valued measurable functions. If \( f_n \downarrow f \mu \)-a.e. and \( \mu^\ast \)-a.e. and there exists a Choquet integrable function \( g \) such that \( f_n^\ast \leq g \) on \( \Omega \), then \( f \) is Choquet integrable and \( (C) \int f_n d\mu \downarrow (C) \int f d\mu \).

**Proof.** (1) Since \( f_n \uparrow f \mu \)-a.e. and \( \mu^\ast \)-a.e., \( f_n^+ \uparrow f^+ \mu \)-a.e. and \( f_n^- \downarrow f^- \mu \)-a.e. Since \( \mu \) is lower semi-continuous, by [11, Theorem 2.4] \( f^+ \) is Choquet integrable with respect to \( \mu \) and \( (C) \int f_n^+ d\mu \uparrow (C) \int f^+ d\mu \). Since \( \mu \) is lower semi-continuous, \( \mu^\ast \) is upper semi-continuous. Since there exists a Choquet integrable function \( g \) such that \( f_n^\ast \leq g \) on \( \Omega \), by [11, Theorem 2.4] \( f^- \) is Choquet integrable with respect to \( \mu^\ast \) and \( (C) \int f_n^- d\mu^\ast \downarrow (C) \int f^- d\mu^\ast \). Hence \( f \) is Choquet integrable and \( (C) \int f_n d\mu \uparrow (C) \int f d\mu \).

(2) The proof is similar to (1).

\[ \Box \]

**Theorem 3.6.** Let \( \mu \) be a finite and continuous fuzzy measure and let \( X \) be a reflexive Banach space and let \( (f_n) \) be a sequence of Choquet-Pettis integrable \( X \)-valued functions on \( \Omega \).

(1) If \( f_n \uparrow f \) weakly \( \mu \)-a.e. and \( \mu^\ast \)-a.e. and there exists a Choquet integrable function \( g \) such that \( (x^\ast f_n) \leq g \) on \( \Omega \) for all \( x^\ast \in X^\ast \),
then $f$ is Choquet-Pettis integrable and $(CP) \int f_n d\mu \uparrow (CP) \int f d\mu$ weakly.

(2) If $f_n \downarrow f$ weakly $\mu$-a.e. and $\mu^c$-a.e. and there exists a Choquet integrable function $g$ such that $(x^* f_n)^+ \leq g$ on $\Omega$ for all $x^* \in X^*$, then $f$ is Choquet-Pettis integrable and $(CP) \int f_n d\mu \downarrow (CP) \int f d\mu$ weakly.

Proof. (1) Let $A \in \Sigma$. Since $f_n \uparrow f$ weakly $\mu$-a.e. and $\mu^c$-a.e. and there exists a Choquet integrable function $g$ such that $(x^* f_n)^+ \leq g$ on $\Omega$ for all $x^* \in X^*$, by Theorem 3.5 $x^* f$ is Choquet integrable and $(CP) \int_A x^* f_n d\mu \uparrow (CP) \int_A x^* f d\mu$ for all $x^* \in X^*$. Since $f_n$ is Choquet-Pettis integrable for $n = 1, 2, \ldots$, there exists $x_{n,A} \in X$ such that $x^*(x_{n,A}) = (CP) \int_A x^* f_n d\mu$ for all $x^* \in X^*$ i.e., $x_{n,A} = (CP) \int_A f_n d\mu$. Thus $(x_{n,A})$ is a weak Cauchy sequence in $X$. Since $X$ is a reflexive Banach space, the sequence $(x_{n,A})$ converges weakly to some $x_A \in X$. Thus $x^*(x_{n,A}) \uparrow x^*(x_A)$ for all $x^* \in X^*$. Hence $x^*(x_A) = (CP) \int_A x^* f d\mu$ for all $x^* \in X^*$. Thus $f$ is Choquet-Pettis integrable and $x_A = (CP) \int_A f d\mu$. In particular, $(CP) \int f_n d\mu \uparrow (CP) \int f d\mu$ weakly.

(2) The proof is similar to (1).

THEOREM 3.7. Let $\mu$ be a finite and continuous fuzzy measure and let $X$ be a reflexive Banach space and let $(f_n)$ be a sequence of Choquet-Pettis integrable $X$-valued functions on $\Omega$. If $f_n \rightarrow f$ weakly $\mu$-a.e. and $\mu^c$-a.e. and there exist Choquet integrable functions $g$ and $h$ such that $h \leq x^* f_n \leq g$ on $\Omega$ for $n = 1, 2, \ldots$ and $x^* \in X^*$, then $f$ is Choquet-Pettis integrable and $(CP) \int f_n d\mu \rightarrow (CP) \int f d\mu$ weakly.

Proof. Let $A \in \Sigma$. Since $f_n \rightarrow f$ weakly $\mu$-a.e., $(x^* f_n)^+ \rightarrow (x^* f)^+$ $\mu$-a.e. for all $x^* \in X^*$. Since $x^* f_n \leq g$ on $\Omega$ for $n = 1, 2, \ldots$ and $x^* \in X^*$, $(x^* f_n)^+ \leq g^+$ on $\Omega$ for $n = 1, 2, \ldots$ and $x^* \in X^*$. By [11, Theorem 2.7] $(x^* f)^+$ is Choquet integrable with respect to $\mu$ and $\lim_{n \rightarrow \infty}(C) \int_A (x^* f_n)^+ d\mu = (C) \int_A (x^* f)^+ d\mu$ for all $x^* \in X^*$. Since $f_n \rightarrow f$ weakly $\mu^c$-a.e., $(x^* f_n)^- \rightarrow (x^* f)^- \mu^c$-a.e. for all $x^* \in X^*$. Since $h \leq x^* f_n$ on $\Omega$ for $n = 1, 2, \ldots$ and $x^* \in X^*$, $(x^* f_n)^- \leq h^-$ on $\Omega$ for $n = 1, 2, \ldots$ and $x^* \in X^*$. By [11, Theorem 2.7] $(x^* f)^-$ is Choquet integrable with respect to $\mu^c$ and $\lim_{n \rightarrow \infty}(C) \int_A (x^* f_n)^- d\mu^c = (C) \int_A (x^* f)^- d\mu^c$ for all $x^* \in X^*$. Hence $x^* f$ is Choquet integrable with
respect to $\mu$ and
\[
\lim_{n \to \infty} \left( (C) \int_A x^* f_n d\mu \right) = \lim_{n \to \infty} \left( (C) \int_A (x^* f_n)^+ d\mu - (C) \int_A (x^* f_n)^- d\mu^c \right)
\]
\[
= (C) \int_A (x^* f)^+ d\mu - (C) \int_A (x^* f)^- d\mu^c
\]
\[
= (C) \int_A x^* f d\mu
\]
for all $x^* \in X^*$. Since $f_n$ is Choquet-Pettis integrable for $n = 1, 2, \cdots$, there exists $x_{n,A} \in X$ such that $x^*(x_{n,A}) = (C) \int_A x^* f_n d\mu$ for all $x^* \in X^*$ i.e., $x_{n,A} = (CP) \int_A f_n d\mu$. Since $\lim_{n \to \infty} (C) \int_A x^* f_n d\mu = (C) \int_A x^* f d\mu$ for all $x^* \in X^*$, $(x_{n,A})$ is a weak Cauchy sequence in $X$. Since $X$ is a reflexive Banach space, the sequence $(x_{n,A})$ converges weakly to some $x_A \in X$. Thus $\lim_{n \to \infty} x^*(x_{n,A}) = x^*(x_A)$ for all $x^* \in X^*$. Hence $x^*(x_A) = (C) \int_A x^* f d\mu$ for all $x^* \in X^*$. Thus $f$ is Choquet-Pettis integrable and $x_A = (CP) \int_A f d\mu$. In particular, $(CP) \int f_n d\mu \to (CP) \int f d\mu$ weakly.

\[\square\]

In the sequel, we assume that $\Omega$ is a locally compact Hausdorff space, $\mathcal{B}$ is the class of Borel subsets of $\Omega$, $\mathcal{C}$ is the class of compact subsets of $\Omega$ and $\mathcal{O}$ is the class of open subsets of $\Omega$.

**Definition 3.8.**[8] Let $\mu$ be a fuzzy measure on the measurable space $(\Omega, \mathcal{B})$. $\mu$ is said to be outer regular if
\[
\mu(B) = \inf \{ \mu(O) | O \in \mathcal{O}, O \supset B \}
\]
for all $B \in \mathcal{B}$.

The outer regular fuzzy measure $\mu$ is said to be regular if
\[
\mu(O) = \inf \{ \mu(C) | C \in \mathcal{C}, C \subset O \}
\]
for all $O \in \mathcal{O}$.

The next theorem follows immediately from [8, Proposition 3.3].

**Theorem 3.9.** Let $\mu$ be a regular fuzzy measure.

1. If $f_n \in LSC^+$ for $n = 1, 2, \cdots$ and $f_n \uparrow f$ on $\Omega$, then $f$ is Choquet integrable and
\[
\lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu.
\]
(2) If $f_n \in USCC^+$ for $n = 1, 2, \cdots$ and $f_n \downarrow f$ on $\Omega$, then $f$ is Choquet integrable and

$$\lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu.$$ 

**Theorem 3.10.** Let $\mu$ be a finite and regular fuzzy measure. If $(f_n)$ is a sequence of continuous real-valued functions with compact support and $f_n \uparrow f$ on $\Omega$, then $f$ is Choquet integrable and

$$\lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu.$$ 

**Proof.** Since $(f_n)$ is a sequence of continuous real-valued functions with compact support and $f_n \uparrow f$ on $\Omega$, $f_n^+ \in LSC^+$ for $n = 1, 2, \cdots$ and $f_n^+ \uparrow f^+$. By Theorem 3.9,

$$\lim_{n \to \infty} (C) \int f_n^+ d\mu = (C) \int f^+ d\mu.$$ 

Since $(f_n)$ is a sequence of continuous real-valued functions with compact support and $f_n \uparrow f$ on $\Omega$, $f_n^- \in USCC^+$ for $n = 1, 2, \cdots$ and $f_n^- \downarrow f^-$. By Theorem 3.9,

$$\lim_{n \to \infty} (C) \int f_n^- d\mu = (C) \int f^- d\mu.$$ 

Hence $f$ is Choquet integrable and

$$\lim_{n \to \infty} (C) \int f_n d\mu = \lim_{n \to \infty} \left[ (C) \int f_n^+ d\mu - (C) \int f_n^- d\mu \right]$$

$$= \lim_{n \to \infty} (C) \int f_n^+ d\mu - \lim_{n \to \infty} (C) \int f_n^- d\mu$$

$$= (C) \int f^+ d\mu - (C) \int f^- d\mu$$

$$= (C) \int f d\mu.$$ 

**Theorem 3.11.** Let $\mu$ be a finite and regular fuzzy measure and let $X$ be a reflexive Banach space. If $(f_n)$ is a sequence of continuous Choquet-Pettis integrable $X$-valued functions with compact support and $f_n \uparrow f$
weakly on $\Omega$, then $f$ is Choquet-Pettis integrable and

$$\lim_{n \to \infty} (CP) \int f_n d\mu = (CP) \int f d\mu \text{ weakly.}$$

Proof. Let $A \in \Sigma$. Since $(f_n)$ is a sequence of continuous $X$-valued functions with compact support, $(x^* f_n)$ is a sequence of continuous real-valued functions with compact support for all $x^* \in X^*$. Since $f_n \uparrow f$ weakly on $\Omega$, $x^* f_n \uparrow x^* f$ on $\Omega$ for all $x^* \in X^*$. By Theorem 3.10, $x^* f$ is Choquet integrable for all $x^* \in X^*$ and $\lim_{n \to \infty} (C) \int_A x^* f_n d\mu = (C) \int_A x^* f d\mu$ for all $x^* \in X^*$. Since $f_n$ is Choquet-Pettis integrable for $n = 1, 2, \ldots$, there exists $x_{n,A} \in X$ such that $x^* (x_{n,A}) = (C) \int_A x^* f_n d\mu$ for all $x^* \in X^*$ and $n = 1, 2, \ldots$. That is, $x_{n,A} = (CP) \int_A f_n d\mu$ for $n = 1, 2, \ldots$. Thus $(x_{n,A})$ is a weak Cauchy sequence in $X$. Since $X$ is a reflexive Banach space, the sequence $(x_{n,A})$ converges weakly to some $x_A \in X$. Thus $\lim_{n \to \infty} x^* (x_{n,A}) = x^* (x_A)$ for all $x^* \in X^*$. Hence $x^* (x_A) = (C) \int_A x^* f d\mu$ for all $x^* \in X^*$. Thus $f$ is Choquet-Pettis integrable and $x_A = (CP) \int_A f d\mu$ for each $A \in \Sigma$. In particular,

$$\lim_{n \to \infty} (CP) \int f_n d\mu = (CP) \int f d\mu \text{ weakly.}$$

References

Convergence theorems for the Choquet-Pettis integral


Chun-Kee Park
Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea

*E-mail*: ckpark@kangwon.ac.kr