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A NOTE ON SOME UNIFORM GEOMETRICAL PROPERTIES IN BANACH SPACES

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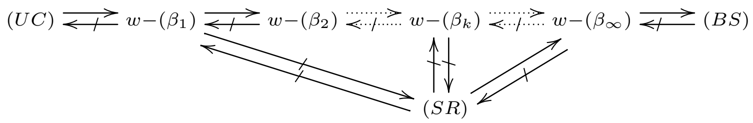
Abstract. In this paper, we investigate relationship between superreflexivity and weak property (β_k). Indeed, we get the following diagram.



1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By B_X and S_X, we denote the closed unit ball of X and the unit sphere of X, respectively. Denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and real numbers, respectively.

A Banach space is said to be reflexive if the natural embedding $\eta : X \to X^{**}$ is onto. A Banach space Y is said to be finitely representable in a Banach space X if for every $\epsilon > 0$ and for every finite-dimensional subspace F of Y there exists an isomorphism T from F into X satisfying

$$(1-\epsilon)\|y\| \leq \|Ty\| \leq (1+\epsilon)\|y\|.$$

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A Banach space X is said to be superreflexive (SR) if every Banach space Y finitely representable in X is reflexive. We shall say that a Banach space is uniformly convexifiable if it is isomorphic to a uniformly convex space, that is, if it can be endowed with an equivalent uniformly convex norm. It is well known that superreflexivity and uniform convexifiability are equivalent [1]. A Banach space X is said to have Banach-Saks property (BS) if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. In similar way, we say that a Banach space X has weak Banach-Saks property (w-BS) if any weakly convergent sequence in the space admits a subsequence whose arithmetic means converges in norm. Since any weakly convergent sequence is norm bounded, it follows that Banach-Saks property implies weak Banach-Saks property. We note that weak Banach-Saks property and Banach-Saks property coincide in the reflexive Banach space. A Banach space X is said to be uniformly convex (UC) if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in B_X$ and $\|x - y\| \geq \epsilon$, $\frac{1}{2}\|x + y\| \leq 1 - \delta$.

S. Kakutani [4] showed that unform convexity implies Banach-Saks property. T. Nishiura and D. Waterman [5] proved that Banach-Saks property implies reflexivity in Banach spaces.

A Banach space X has the weak property (β_k) if it is reflexive and there exists $\delta > 0$ such that for any $x \in B_X$ and any weakly null sequence $(x_n) \in B_X$ there exist $n_i \in \mathbb{N},\ i = 1, 2, \cdots, k$ with $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k+1} \left(x + \sum_{i=1}^{k} x_{n_i} \right) \right\| \leq 1 - \delta.$$

We say that X has the weak property (β_∞) if it has the weak property (β_k) for some $k \in \mathbb{N}$. K.G. Cho and C.S. Lee [2] introduced the notion of weak property (β_k) and show the following strict implications.

$$(UC) \Rightarrow w-(\beta_1) \Rightarrow w-(\beta_2) \Rightarrow \cdots \Rightarrow w-(\beta_\infty) \Rightarrow (BS)$$

In this paper, we study relationship between superreflexivity and weak property (β_k). The techniques are similar with [3].

2. The weak property (β_k) and supererflexivity in Banach spaces

We begin with the following theorem.

THEOREM 2.1. *Superreflexive Banach spaces have the weak property* (β_∞).

Proof. Let $(X, \|\cdot\|)$ be a superreflexive Banach space. Then there exists uniformly convex norm $|\cdot|$ and $M \geq m > 0$ such that $m\|x\| \leq |x| \leq M\|x\|$, for all $x \in X$. Suppose that $x \in B_{(X,\|\cdot\|)}$ and (x_n) is a weakly null sequence in $B_{(X,\|\cdot\|)}$. Then (x_n) is weakly null sequence in $(X, |\cdot|)$ and $|x|, |x_n| \leq M$. Since uniformly convexity implies the weak property (β_1) [2], there exists $0 < \delta_0 < 1$(independent on (x_n)) such that for $\frac{x}{M} \in B_{(X,|\cdot|)}$ and a weakly null sequence $\left(\frac{x_n}{M}\right)_{n\geq 1}$ in $B_{(X,|\cdot|)}$, there exist $n_1 \geq 1$ with

$$\left|\frac{1}{2}\left(\frac{x}{M} + \frac{x_{n_1}}{M}\right)\right| \leq 1 - \delta_0.$$

Letting $n_2 = n_1+1$, for $\frac{x_{n_2}}{M} \in B_{(X,|\cdot|)}$ and a weakly null sequence $\left(\frac{x_n}{M}\right)_{n>n_2}$ in $B_{(X,|\cdot|)}$, there exist $n_3 \geq n_2 + 1$ with

$$\left|\frac{1}{2}\left(\frac{x_{n_2}}{M} + \frac{x_{n_3}}{M}\right)\right| \leq 1 - \delta_0,$$

$$\vdots$$

Continuing this process, we get a subsequence (x_{n_i}) of (x_n) with

$$|x+x_{n_1}| \leq 2M(1-\delta_0) \quad \text{and} \quad |x_{n_{2i}}+x_{n_{2i+1}}| \leq 2M(1-\delta_0), \quad \text{for all } i \in \mathbb{N}.$$

Let (x^1_m) be the sequence defined by

$$2x^1_1 = x + x_{n_1} \quad \text{and} \quad 2x^1_{m+1} = x_{n_{2m}} + x_{n_{2m+1}}, \quad \text{for all } m \in \mathbb{N}.$$

Then (x^1_n) is weakly null and $|x^1_n| \leq M(1-\delta_0)$. Since $(X, |\cdot|)$ has the weak property (β_1), for $\frac{x^1_1}{M(1-\delta_0)} \in B_{(X,|\cdot|)}$ and a weakly null sequence $\left(\frac{x^1_n}{M(1-\delta_0)}\right)_{n\geq 2}$ in $B_{(X,|\cdot|)}$ there exists $n_1 > 1$ such that

$$\left|\frac{1}{2}\left(\frac{x^1_1}{M(1-\delta_0)} + \frac{x^1_{n_1}}{M(1-\delta_0)}\right)\right| \leq 1 - \delta_0.$$

Letting $n_2 = n_1 + 1$, for $\frac{x^1_{n_2}}{M(1-\delta_0)} \in B_{(X,|\cdot|)}$ and a weakly null sequence $\left(\frac{x^1_n}{M(1-\delta_0)}\right)_{n\geq n_2}$ in $B_{(X,|\cdot|)}$, there exist $n_3 \geq n_2 + 1$ with

$$\left|\frac{1}{2}\left(\frac{x^1_{n_2}}{M(1-\delta_0)} + \frac{x^1_{n_3}}{M(1-\delta_0)}\right)\right| \leq 1 - \delta_0.$$

$$\vdots$$

Continuing this process, we get a subsequence $(x^1_{n_i})$ of (x^1_n) with

$$|x^1_1+x^1_{n_1}| \le 2M(1-\delta_0)^2 \quad \text{and} \quad |x^1_{n_{2i}}+x^1_{n_{2i+1}}| \le 2M(1-\delta_0)^2, \text{ for all } i \in \mathbb{N}.$$

Without loss of generality, we may assume that $(x^1_{n_i}) = (x^1_{i+1})$. Continue this process, for all $k \in \mathbb{N}$, we get a subsequence (x^k_n) such that

$$|x^k_{2i-1} + x^k_{2i}| \le 2M(1-\delta_0)^{k+1}, \quad \text{where } i \in \mathbb{N}.$$

For a sufficiently large $N \in \mathbb{N}$, choose $\delta > 0$ such that

$$(1-\delta_0)^{N+1}\frac{M}{m} < 1-\delta.$$

Since

$$|x^N_1 + x^N_2| \le 2M(1-\delta_0)^{N+1} < 2m(1-\delta)$$

and

$$\begin{aligned} x^N_1 + x^N_2 &= \frac{1}{2}(x^{N-1}_1 + x^{N-1}_2) + \frac{1}{2}(x^{N-1}_3 + x^{N-1}_4) \\ &= \frac{1}{4}(x^{N-2}_1 + x^{N-2}_2 + \cdots + x^{N-2}_8) \\ &\vdots \\ &= \frac{1}{2^{N-1}}(x^1_1 + x^1_2 + \cdots + x^1_{2^N}) \\ &= \frac{1}{2^N}(x + x_{n_1} + x_{n_2} + x_{n_3} + \cdots + x_{2^{N+1}-2} + x_{2^{N+1}-1}), \end{aligned}$$

we get

$$\left| \frac{1}{2^{N+1}} \left(x + \sum_{i=1}^{2^{N+1}-1} x_{n_i} \right) \right| < (1-\delta)m.$$

Thus,

$$\left\| \frac{1}{2^{N+1}} \left(x + \sum_{i=1}^{2^{N+1}-1} x_{n_i} \right) \right\| < (1-\delta).$$

Since N and δ depend only on X, it follows that X has the weak property $(\beta_{2^{N+1}-1})$, hence the weak property (β_∞). □

The following is the example that satisfies the weak property (β_k) without the weak property (β_{k-1}).

EXAMPLE 1. For $x = (a_n) \in l_2$, we define a norm $\|x\|_{(k)}$ by

$$\|x\|_{(k)} = \left[\sup_{n_1 < n_2 < \cdots < n_k} \left(\sum_{i=1}^{k} |a_{n_i}| \right)^2 + \sum_{n \neq n_1, n_2, \cdots, n_k} |a_n|^2 \right]^{\frac{1}{2}}$$

Then $\|x\|_2 \leq \|x\|_{(k)} \leq \sqrt{k}\|x\|_2$. Let $X_k = \left(l_2, \|\cdot\|_{(k)}\right)$. Then X_k has the weak property (β_k) but no the weak property (β_{k-1}) [2].

Since X_k is isomorphic to l_2, X_k is superreflexive. We get the following proposition.

PROPOSITION 2.2. *Superreflexivity does not imply the weak property* (β_{k-1}), *for all* $k \geq 2$.

We now consider the converse of Theorem 2.1 and Proposition 2.2.

THEOREM 2.3. *Let Y be a Banach space with a basis (e_n) and with a norm such that if $0 \leq |a_n| \leq |b_n|$, then*

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| \leq \left\| \sum_{n=1}^{\infty} b_n e_n \right\|.$$

Let (Y_n) be a family of finite dimensional spaces. Let

$$Z = \left\{ x = (x_n) \in \prod_{n=1}^{\infty} Y_n : \sum_{n=1}^{\infty} \|x_n\| e_n \in Y \right\},$$

with the norm

$$\|x\| = \left\| \sum_{n=1}^{\infty} \|x_n\| e_n \right\|.$$

If Y has the weak property (β_1), then Z has the weak property (β_1).

Proof. We first note that Z is reflexive. Let δ_0 be chosen according to the definition of weak property (β_1) in Y with $0 < \delta_0 < 1$. Let $z = (z_n) \in B_Z$ and $(z^{(i)}) = ((z_n^{(i)}))$ be a weakly null sequence in B_Z. Then $(z_n^{(i)})$ is weakly null in Y_n as $i \to \infty$, for each $n \in \mathbb{N}$. Since Y_n is finite dimensional, $(z_n^{(i)})$ is norm null in Y_n as $i \to \infty$, for each $n \in \mathbb{N}$. Let $x = \sum_{n=1}^{\infty} \|z_n\| e_n$ and $x_i = \sum_{n=1}^{\infty} \|z_n^{(i)}\| e_n$. Then $\|x\| = \|z\| \leq 1$ and $\|x_i\| = \|z^{(i)}\| \leq 1$. Since the weak property (β_1) implies reflexivity, there exists a weakly convergent subsequence of (x_i) (which we still call (x_i)), say $x_i \to y = \sum_{n=1}^{\infty} a_n e_n$ weakly in Y. For each $n \in \mathbb{N}$, $a_n = e_n^*(x) =$

$\lim_{i\to\infty} e_n^*(x_i) = \lim_{i\to\infty} \|z_n^{(i)}\| = 0$. This means that (x_i) is weakly null in Y. Since Y has the weak property (β_1) with δ_0, there exists i_1 such that

$$\frac{1}{2}\left\|\sum_{n=1}^{\infty}\left(\|z_n\| + \left\|z_n^{(i_1)}\right\|\right) e_n\right\| = \frac{1}{2}\left\|x + x_{i_1}\right\| \le 1-\delta_0.$$

Thus,

$$\begin{aligned}\frac{1}{2}\left\|z + z^{(i_1)}\right\| &= \frac{1}{2}\left\|\sum_{n=1}^{\infty}\left\|z_n + z_n^{(i_1)}\right\| e_n\right\| \\ &\le \frac{1}{2}\left\|\sum_{n=1}^{\infty}\left(\|z_n\| + \left\|z_n^{(i_1)}\right\|\right) e_n\right\| \le 1-\delta_0.\end{aligned}$$

This implies that Z has the weak property (β_1). $\square$

It is well known that $\left(\prod_{n\ge 1} l_1^n\right)_2$ is not superreflexive but reflexive (see, e.g., [1, p.225]). By Theorem 2.3, we get the following.

Corollary 2.4. *The weak property (β_1) does not imply superreflexvity.*

By Theorem 2.1, 2.2, Corollary 2.4 and [2], we get the following diagram.

$$\begin{tikzcd}
(UC) \arrow[r, shift left] & w-(\beta_1) \arrow[l, shift left, "/" marking] \arrow[r, shift left] \arrow[drr, shift left, "/" marking] & w-(\beta_2) \arrow[l, shift left, "/" marking] \arrow[r, dotted, shift left] & w-(\beta_k) \arrow[l, dotted, shift left, "/" marking] \arrow[r, dotted, shift left] \arrow[d, shift left, "/" marking] & w-(\beta_\infty) \arrow[l, dotted, shift left, "/" marking] \arrow[r, shift left] \arrow[dl, shift left, "/" marking] & (BS) \arrow[l, shift left, "/" marking] \\
 & & & (SR) \arrow[ull, shift left, "/" marking] \arrow[u, shift left, "/" marking] \arrow[ur, shift left] & &
\end{tikzcd}$$

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References

[1] B. Beauzamy, *Introduction to Banach spaces and their geometry*, Mathematics Studies, **68**, Noth-Holland, Amsterdam, 1982.

[2] K.G. Cho and C.S. Lee, *Weak property (β_k)*, Korean J. Math. **20** (2012), 415–422.

[3] K.G. Cho and C.S. Lee, *Superreflexivity and property (D_k) in Banach spaces*, J. Appl. Math. Inform. **29** (2011), 1001–1006.

[4] S. Kakutani, *Weak convergence in uniformly convex spaces*, Tôhoku Math. J. **45** (1938), 347–354.

[5] T. Nishiura and D. Waterman, *Reflexivity and summability*, Studia Math. **23** (1963), 53–57.

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