THE BASES OF PRIMITIVE NON-POWERFUL COMPLETE SIGNED GRAPHS

BYUNG CHUL SONG AND BYEONG MOON KIM∗

Abstract. The base of a signed digraph S is the minimum number k such that for any vertices u, v of S, there is a pair of walks of length k from u to v with different signs. Let K be a signed complete graph of order n, which is a signed digraph obtained by assigning +1 or −1 to each arc of the n-th order complete graph Kn considered as a digraph. In this paper we show that for n ≥ 3 the base of a primitive non-powerful signed complete graph K of order n is 2, 3 or 4.

1. Introduction

A sign pattern matrix M is a square matrix with entries in \{1, 0, −1\}. In multiplying two sign pattern matrices, we use the operating rules of entries that continues to hold the signs of the usual addition and multiplication, that is

1+1 = 1; (−1)+(−1) = −1; 1+0 = 0+1 = 1; (−1)+0 = 0+(−1) = −1;
0·a = a·0 = 0; 1·1 = (−1)·(−1) = 1; 1·(−1) = (−1)·1 = −1 for any a ∈ \{1, 0, −1\}.

∗Corresponding author.
This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.
This is an Open Access article distributed under the terms of the Creative commons Attribution Non-Commercial License (http://creativecommons.org/licenses/by-nc/3.0/) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.
In this case we contact the ambiguous situations $1 + (-1)$ and $(-1) + 1$, which we will use the notation \( \# \) as in [2]. Define the addition and multiplication which involving the symbol \( \# \) as follows: For any \( a \in \Gamma = \{1, 0, -1, \#\} \),

\[
(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \#
\]

\[
0 \cdot \# = \# \cdot 0 = 0; \quad a \cdot \# = \# \cdot a = \# \quad (\text{when } a \neq 0).
\]

Matrices with entries in \( \Gamma \) are called \textit{generalized sign pattern matrices}. The addition and multiplication of the entries of generalized sign pattern matrices are defined in the usual way such that they coincide with the operations in sign pattern matrices.

**Definition 1.** A square generalized sign pattern matrix \( M \) is \textit{powerful} if each power of \( M \) contains no \( \# \) entry. A square generalized sign pattern matrix \( M \) is called \textit{non-powerful} if it is not powerful.

**Definition 2.** Let \( M \) be a square generalized sign pattern matrix of order \( n \). The smallest number \( l \) such that \( M^l = M^{l+p} \) for some \( p \) is called the \textit{(generalized) base} of \( M \) and denoted by \( l(M) \). The least positive integer \( p \) such that \( M^l = M^{l+p} \) for \( l = l(M) \) is called to be the \textit{(generalized) period} of \( M \) and denote it by \( p(M) \).

We introduce some graph theoretic concepts of generalized sign pattern matrices.

A \textit{signed digraph} \( S = (V, A, f) \) is a digraph with vertex set \( V \), arc set \( A \) and a sign function \( f \) defined on \( A \) with its value \( 1, -1 \). For \( v, w \in V \) we say \( f(vw) \) the \textit{sign} of an arc \( vw \), and we denote it by \( \text{sgn}(vw) \). The \textit{sign} of a (directed) walk \( W \) in \( S \), denoted by \( \text{sgn}(W) \) or \( f(W) \), is the product of signs of all arcs in \( W \). For example if \( W = v_1v_2v_3v_4 \), then \( \text{sgn}(W) = f(W) = f(v_1v_2)f(v_2v_3)f(v_3v_4) \). If two walks \( W_1 \) and \( W_2 \) have the same initial points, the same terminal points, the same lengths and different signs, then we say that \( W_1 \) and \( W_2 \) are a \textit{pair of SSSD walks}.

A (signed) digraph \( S \) is \textit{primitive} if there is a positive integer \( k \) such that for all vertices \( v, w \) of \( S \) there is a walk of length \( k \) from \( v \) to \( w \). A signed digraph \( S \) is \textit{powerful} if \( S \) contains no pair of SSSD walks. Also \( S \) is \textit{non-powerful} if it is not powerful. Hence every non-powerful primitive signed digraph contains a pair of SSSD walks. Let \( M = M(S) = [a_{ij}] \) be the adjacency matrix of a signed digraph \( S \), that is, the arc \((i, j)\) has sign \( \text{sgn}(i, j) = \alpha \) if and only if \( a_{ij} = \alpha \) with \( \alpha = 1 \), or \(-1 \).
the adjacency (signed) matrix $M$ of a signed digraph $S$ is a sign pattern matrix which satisfies that the $(i, j)$-entry of $M^k$ is 0 if and only if $S$ contains no walk of length $k$ from $i$ to $j$. Also $(i, j)$-entry of $M^k$ is 1 (or $-1$) if and only if all walks of length $k$ from $i$ to $j$ in $S$ are of sign 1 (or, $-1$). The $(i, j)$-entry of $M^k$ is $\sharp$ if and only if $S$ contains a pair of SSSD walks of length $k$ from $i$ to $j$. We see from the above relations between matrices and digraphs that each power of a signed digraph $S$ contains no pair of SSSD walks if and only if the adjacency matrix $M$ is powerful. Henceforth we may also say that a signed digraph $S$ is powerful or non-powerful if its adjacency sign pattern matrix $M$ is powerful or non-powerful respectively.

From now on we assume that $S = (V, A, f)$ is a primitive non-powerful signed digraph of order $n$. For each pair of vertices $u, v$ of $S$, we define the local base $l_S(u, v)$ from $u$ to $v$ to be the smallest integer $l$ such that for each $k \geq l$, there is a pair of SSSD walks of length $k$ from $u$ to $v$ in $S$. The base $l(S)$ of $S$ is defined to be $\max \{ l_S(u, v) | u, v \in V(S) \}$. It follows directly from the definitions that $l(S) = l(M)$ where $M$ is the adjacency matrix of $S$.

The upper bounds for the bases of primitive non-powerful sign pattern matrices are found by You et al. [5]. They also characterized extremal cases completely. Gao et al.[1], Shao and Gao[4] and Li and Liu [3] studied the base and the local base of a primitive non-powerful signed symmetric digraphs with loops.

Let us assume that $K$ is a complete non-powerful signed digraph of order $n$ which is the $n$-th order complete graph (considered as a digraph) by assigning signs to each arc such that it becomes a non-powerful signed digraph. In this paper we prove that the base of $K$ is less than or equals to 4. As a consequence if all the entries of a non-powerful sign pattern matrix $A$ are nonzero except diagonals, then the all entries of $A^4$ are $\sharp$. We also provide the examples when the base of $K$ is 2, 3 and 4 respectively.

2. Main theorems

Let $K = (V, A, f)$ be a complete non-powerful signed digraph of order $n$. That is, $K$ is the $n$-th order digraph which has unique arc for each ordered pair of vertices of $K$ and signs are assigned to each arc such that $K$ becomes a non-powerful signed digraph. Let $v_1, v_2, \cdots, v_r$
be vertices of $K$. If $C$ is a directed walk from $v_1$ to $v_r$ which goes through $v_2, v_3, \ldots, v_{r-1}$, then we denote $C$ by $v_1v_2\cdots v_{r-1}v_r$ and the sign $f(v_1v_2)f(v_2v_3)\cdots f(v_{r-1}v_r)$ of $C$ by $f(C) = f(v_1v_2\cdots v_{r-1}v_r) = \text{sgn}(C) = \text{sgn}(v_1v_2\cdots v_{r-1}v_r)$. Throughout this paper we use the notation $u \xrightarrow{k} v$ if there is a walk of length $k$ from a vertex $u$ to another vertex $v$. The sum $W_1 + W_2$ of two walks $W_1 = v_1v_2\cdots v_n$ and $W_2 = w_1w_2\cdots w_m$ such that $v_n = w_1$ and the inverse $-W_1$ of $W_1$ are defined by $W_1 + W_2 = v_1v_2\cdots v_nw_2\cdots w_m$ and $-W_1 = v_nv_{n-1}\cdots v_1$.

**Theorem 1.** The base $l(K)$ of the complete non-powerful signed digraph $K$ of order $n \geq 4$ is less than or equals to 4.

*Proof.* It suffices to show that there is a pair of SSSD walks of common length 4 from $u$ to $v$. Let $u, v$ be vertices of $K$. Since $n \geq 4$, we can choose a vertex $w$ of $K$ different from $u$ and $v$. Let $\sigma$ be the sign of the walk $uvw$. If there is a vertex $x$ of $K$ such that $x \neq u$ and the sign of the walk $uxu$ is $-\sigma$, then $uvwuv$ and $uxuwv$ are a pair of SSSD walks of length 4 from $u$ to $v$.

If the sign of the walk $uxu$ is $\sigma$ for any vertex $x$ of $K$ and there are distinct vertices $y, z$ of $K$ such that $z \neq u$ and the sign of the walk $zzy$ is $-\sigma$, then both $y$ and $z$ are different from $u$. If $y \neq v$, then $uvwuv$ and $uyzuv$ are a pair of SSSD walks with common length 4 from $u$ to $v$. If $y = v$, then since $z \neq v$, $uwuzv$ and $uzyuv$ are desired pair of SSSD walks with common length 4 from $u$ to $v$.

Assume that the sign of the walk $zzy$ is $\sigma$ for all distinct vertices $y, z$.

If $\sigma = -1$, then

\[
\text{sgn}(uvwuv)\text{sgn}(uvw) = f(uv)f(vw)f(wu)f(uw)f(u)\text{sgn}(uv) = (f(uv)f(vw)f(uw)f(uw)f(uw))(f(uv))(f(uv)) = \text{sgn}(uvw)\text{sgn}(vw)\text{sgn}(uw) = \sigma^3 = -1.
\]

Hence $uvwuv$ and $uvw$ are a pair of SSSD walks with common length 4 from $u$ to $v$.

If $\sigma = 1$, then since $K$ is non-powerful, there is an even cycle of sign $-1$, or there are two odd cycles with different signs. Assume that there is an even cycle $x_1x_2\cdots x_kx_1$ with sign $-1$. If $x_i \neq u$ for all $i = 1, 2, \cdots, k$, then
then since
\[
\text{sgn}(ux_1x_2u)\text{sgn}(ux_2x_3u)\cdots \text{sgn}(ux_{k-1}x_ku)\text{sgn}(ux_kx_1u) \\
= (f(ux_1)f(x_1x_2)f(x_2u))(f(ux_2)f(x_2x_3)f(x_3u)) \\
\cdots (f(ux_{k-1})f(x_{k-1}x_k)f(x_ku))(f(ux_k)f(x_kx_1)f(x_1u)) \\
= f(x_1x_2)f(x_2x_3)\cdots f(x_{k-1}x_k)f(x_kx_1) \\
= \text{sgn}(x_1x_2\cdots x_kx_1) = -1,
\]

among the walks \(ux_1x_2u, ux_2x_3u, \ldots, ux_{k-1}x_ku, ux_kx_1u\), there are two walks \(C_1, C_2\) with different signs. Thus \(C_1 + uv\) and \(C_2 + uv\) are a pair of SSSD walks SSSD walks of common length 4 from \(u\) to \(v\).

Let \(x_1 = u\) for some \(i\). Similarly among the walks
\[
ux_1x_2u, ux_2x_3u, \ldots, ux_{i-2}x_{i-1}u, ux_{i+1}x_{i+2}u, ux_{i+2}x_{i+3}u, \ldots, ux_{k-1}x_ku,
\]
we can find a pair, say \(C'_1\) and \(C'_2\), of SSSD walks. As a consequence, we have a pair \(C'_1 + uv\) and \(C'_2 + uv\) of SSSD walks of common length 4 from \(u\) to \(v\).

Let us assume that there are two odd cycles \(y_1y_2\cdots y_{l}y_1\) and \(z_1z_2\cdots z_{m}z_1\) with signs 1 and \(-1\) respectively. We want to show that there is a walk \(C_3 = uy_{l+1}u\) (or \(C_3 = uy_{l+1}u\)) of sign +1. If \(u \neq y_i\) for all \(i = 1, 2, \ldots, l\), then since
\[
\text{sgn}(uy_1y_2u)\text{sgn}(uy_2y_3u)\cdots \text{sgn}(uy_{l-1}y_{l}u)\text{sgn}(uy_{l}y_{1}u) = \text{sgn}(y_1y_2\cdots y_{l}y_1) = -1,
\]
among the walks \(uy_1y_2u, uy_2y_3u, \ldots, uy_{l-1}y_{l}u, uy_{l}y_{1}u\), there is a walk \(C_3\) with sign +1. If \(u = y_i\) for some \(i\), then since
\[
\text{sgn}(uy_1y_2u)\text{sgn}(uy_2y_3u)\cdots \text{sgn}(uy_{l-2}y_{l-1}u) \\
\text{sgn}(uy_{l+1}y_{l+2}u)\cdots \text{sgn}(uy_{l-1}y_{l}u)\text{sgn}(uy_{l}y_{1}u) \\
= \text{sgn}(y_1y_2\cdots y_{l}y_1) = 1,
\]
we have a walk from \(u\) to \(v\) of length 3 with sign 1.

Similarly among the walks \(uz_1z_2u, uz_2z_3u, \ldots, uz_{m-1}z_{m}u, uz_{m}z_1u\), there is a walk \(C_4\) of sign \(-1\). Thus \(C_3 + uv\) and \(C_4 + uv\) are a pair of SSSD walks with common length 4 from \(u\) to \(v\). As a consequence, we have \(l(K) \leq 4\).

We will show the upper bound 4 in Theorem 1 is extremal by constructing a complete nonpowerful signed digraph of base at least 4.
Theorem 2. Let $V = \{v_1, v_2, \cdots, v_n\}$, $A = \{(v_i, v_j)|1 \leq i, j \leq n, i \neq j\}$ and $f : A \rightarrow \{-1, 1\}$ such that

$$f(v_i, v_j) = \begin{cases} -1, & \text{if } j = 3 \text{ and } i \neq 1, \text{ or } (i, j) = (3, 2); \\ 1, & \text{otherwise.} \end{cases}$$

The signed digraph $G = (V, A, f)$ is primitive non-powerful and $l(G) \geq 4$.

Proof. Let $W$ be a walk of length 3 from $v_1$ to $v_2$. Then $W = v_i v_j v_k$ for some $i, j$. If $i = 2$, then for all $j \neq 2$ since $f(v_2 v_j v_k) = f(v_2 v_j) f(v_j v_k) = 1$, we have $\text{sgn}(v_2 v_j v_k) = 1$. If $i = 3$, then $j \neq 3$. Hence $f(v_2 v_j v_k) = f(v_2 v_j) f(v_j v_k) = 1$. If $i \geq 4$ and $j = 3$, then $\text{sgn}(v_2 v_j v_k) = f(v_2 v_j) f(v_j v_k) = 1(-1)(-1) = 1$. If $i \geq 4$ and $j \neq 3$, then $\text{sgn}(v_2 v_j v_k) = f(v_2 v_j) f(v_j v_k) = 1$. Hence the sign of a walk of length 3 from $v_1$ and $v_2$ is always 1. We have $l(v_1, v_2) \geq 4$, and hence $l(G) \geq 4$. By Theorem 1, we conclude that $l(G) = 4$.  

We can easily see that the base of a primitive non-powerful digraph is at least 2. In the following examples we provide two complete signed graphs of order $n \geq 4$ with base 2 and 3 respectively. As a result, the possible base of a complete signed graph of order $n \geq 4$ is 2, 3 and 4.

Example 1. Let $n \geq 4$, $V = \{v_1, v_2, \cdots, v_n\}$, $A = \{(v_i, v_j)|1 \leq i, j \leq n, i \neq j\}$ and $f : A \rightarrow \{-1, 1\}$ such that

$$f(v_i v_j) = \begin{cases} -1, & \text{if } j = 3 \text{ and } i \neq 1, \text{ or } (i, j) = (3, 2), \text{ or } (i, j) = (1, 2); \\ 1, & \text{otherwise} \end{cases}$$

We find a pair of SSSD walks of length 2 from $v_i$ to $v_j$ as follows for each $i$ and $j$.

- $v_1 v_2 v_1$ and $v_1 v_3 v_1$ if $i = 1$ and $j = 2$,
- $v_1 v_3 v_2$ and $v_1 v_4 v_2$ if $i = 1$ and $j = 2$,
- $v_1 v_2 v_3$ and $v_1 v_4 v_3$ if $i = 1$ and $j = 3$,
- $v_1 v_2 v_j$ and $v_1 v_3 v_j$ if $i = 1$ and $j \geq 4$,
- $v_1 v_3 v_1$ and $v_2 v_4 v_1$ if $i = 2$ and $j = 1$,
- $v_2 v_1 v_2$ and $v_2 v_3 v_2$ if $i = 1$ and $j = 2$,
- $v_2 v_1 v_3$ and $v_2 v_4 v_3$ if $i = 2$ and $j = 3$,
- $v_2 v_1 v_j$ and $v_2 v_3 v_j$ if $i = 2$ and $j \geq 4$. 


As a consequence, the signed digraph $G = (V, A, f)$ is primitive non-powerful and $l(G) = 2$.

**Example 2.** Let $n \geq 4$, $V = \{v_1, v_2, \cdots, v_n\}$, $A = \{(v_i, v_j) | 1 \leq i, j \leq n, i \neq j\}$ and $f : A \rightarrow \{-1, 1\}$ such that

$$f(v_i v_j) = \begin{cases} -1, & \text{if } (i, j) = (1, 2), \\ 1, & \text{otherwise} \end{cases}.$$ 

We can see for each walk of length 2 from $v_1$ to $v_2$ is of sign $+1$. Thus $l(G) \geq 3$. By the same method used in above example, there are a pair of SSSD walks of length 3 from $v_i$ to $v_j$ as follows for each $i$ and $j$. It follows that the signed digraph $G = (V, A, f)$ is primitive non-powerful and $l(G) = 3$.

A consequence of the above theorems and examples is that the base of a sign pattern matrix such that every diagonal entry is zero and every non diagonal entries is of sign $1$ or $-1$ is $2$, $3$ and $4$. Also we can consider the sign pattern matrix without zero entries. The corresponding digraph is a complete graph with loops on each vertices. In this case we have the following theorem.

**Theorem 3.** If $n \geq 3$ and $K$ is a non-powerful signed digraph over $n$-th order complete graph with loops on each vertices, then $l(K) \leq 3$.

**Proof.** Suppose that $l(K) \geq 4$. There are $v, w \in V$ and $\sigma \in \{+1, -1\}$ such that the sign of every walk from $v$ to $w$ of length 3 is always $\sigma$. Let $\tau$ be the sign of the loop incident on $v$. For all $x \in V$, since $\text{sgn}(vuwx) = \text{sgn}(vu)\text{sgn}(uxw) = \tau\text{sgn}(uxw) = \sigma$, we have $\text{sgn}(xwx) = \sigma$. Since $\text{sgn}(uxwx) = f(vx)f(xx)f(wx) = f(xx)\text{sgn}(uxw) = f(xx)\sigma\tau = \sigma$, we have $f(xx) = \tau$. 

\[
v_3 v_2 v_1 \text{ and } v_3 v_4 v_1 \quad \text{if } i = 3 \text{ and } j = 1,
\]
\[
v_3 v_1 v_2 \text{ and } v_3 v_4 v_2 \quad \text{if } i = 1 \text{ and } j = 2,
\]
\[
v_3 v_1 v_3 \text{ and } v_3 v_4 v_3 \quad \text{if } i = 3 \text{ and } j = 2,
\]
\[
v_3 v_1 v_j \text{ and } v_3 v_2 v_j \quad \text{if } i = 3 \text{ and } j \geq 4,
\]
\[
v_3 v_2 v_1 \text{ and } v_3 v_3 v_1 \quad \text{if } i = 1 \text{ and } j = 1,
\]
\[
v_3 v_1 v_2 \text{ and } v_3 v_3 v_2 \quad \text{if } i \geq 4 \text{ and } j = 2,
\]
\[
v_3 v_1 v_3 \text{ and } v_3 v_3 v_3 \quad \text{if } i \geq 4 \text{ and } j = 3,
\]
\[
v_3 v_1 v_j \text{ and } v_3 v_3 v_j \quad \text{if } i \geq 4 \text{ and } j \geq 4.
\]
Let $C = x_1x_2 \cdots x_kx_1$ be a cycle of length $k$ in $K$. We have
\[
\sigma^k = \text{sgn}(vx_1x_2w)\text{sgn}(vx_2x_3w)\cdots\text{sgn}(vx_kx_1w)
\]
\[
= (f(vx_1)f(x_1x_2)f(x_2w))(f(vx_2)f(x_2x_3)f(x_3w))
\]
\[
\cdots (f(vx_k)f(x_kx_1)f(x_1w))
\]
\[
= (f(vx_1)f(x_1w))(f(vx_2)f(x_2w))\cdots (f(vx_k)f(x_kw))f(x_1x_2)f(x_2x_3)
\]
\[
\cdots f(x_kx_1)
\]
\[
= (\sigma\tau)^k\text{sgn}(x_1x_2\cdots x_kx_1)
\]
\[
= \sigma^k\tau^k f(C).
\]
Thus the signs of all even and odd cycles are 1 and $\tau$ respectively. Therefore $K$ is powerful. This is a contradiction. Hence $l(K) \leq 3$. \hfill \Box

**Remark 1.** Let $n = 3$, $V = \{v_1, v_2, v_3\}$ and $A = \{(v_i, v_j)| i \neq j\}$. Since $v_1v_2v_3$ is the only $v_1 \xrightarrow{2} v_2$ walk in $K$, we have $l(K) \geq 3$. If $\text{sgn}(v_1v_2v_1) = \text{sgn}(v_2v_3v_2) = \text{sgn}(v_3v_1v_3) = 1$, then every 2-cycle in $G$ is of sign 1. Since
\[
\text{sgn}(v_1v_2v_3v_1)\text{sgn}(v_2v_3v_2v_1)
\]
\[
= f(v_1v_2)f(v_2v_3)f(v_3v_1)f(v_1v_3)f(v_3v_2)f(v_2v_3)f(v_1v_1)
\]
\[
= (f(v_1v_2)f(v_2v_1))(f(v_2v_3)f(v_3v_2))(f(v_3v_1)f(v_1v_3))
\]
\[
= \text{sgn}(v_1v_2v_1)\text{sgn}(v_2v_3v_2)\text{sgn}(v_3v_1v_3) = 1
\]
all 3-cycles in $K$ are of the same sign. It follows that $K$ is powerful.

If $\text{sgn}(v_1v_2v_1) = \text{sgn}(v_2v_3v_2) = \text{sgn}(v_3v_1v_3) = -1$ for all $v_i, v_j \in V$, then there is a $v_1 \xrightarrow{2} v_j$ walk $W$ in $K$. Since
\[
\text{sgn}(v_1v_2v_3v_1)\text{sgn}(v_3v_1v_3v_1) = f(v_1v_2v_1)f(v_2v_3v_2)f(v_3v_1v_3) = -1,
\]
there are two $v_i \xrightarrow{3} v_i$ walks $W_1$ and $W_2$ in $K$ with different signs. Thus we see that $W + W_1$ and $W + W_2$ are a pair of SSSD walks with length 5. We have $l(K) \leq 5$. Let $W = w_0w_1w_2w_3w_4$ be a $v_1 \xrightarrow{4} v_1$ walk in $K$. Hence we have $w_0 = w_4 = v_1$. We may assume that $w_1 = v_2$. If $w_2 = v_1$, then $f(W) = f(v_1v_2v_1)f(v_1w_4v_1) = 1$. If $w_2 = v_3$, then since $w_3 = v_2$, we have $f(W) = 1$. Therefore there is no $v_1 \xrightarrow{4} v_1$ walk in $K$ with sign $-1$. Thus $l(K) = 5$.

If the signs of $f(v_1v_2v_1), f(v_2v_3v_2)$ and $f(v_3v_1v_3)$ are not equal, then we may assume that $f(v_1v_2v_1) = f(v_2v_3v_2) = -f(v_3v_1v_3)$. Let $v_i, v_j \in V$. Hence there is a $v_i \xrightarrow{2} v_j$ walk $W = v_1v_kv_j$ in $K$. If $i \neq 2$, then there
are two $v_i \xrightarrow{2} v_i$ walks $W_1$ and $W_2$ in $K$ with different signs. It is clear that $W_1 + W$ and $W_2 + W$ are a pair of SSSD walks with length 4. Similarly, we have a pair of SSSD walks with length 4 for the case $j \neq 2$. If $i = j = 2$, then $k \neq 2$. Whence there are a pair of $v_k \xrightarrow{2} v_k$ walks $X_1$ and $X_2$ in $K$ with different signs. Thus we see that $(v_iv_k) + X_1 + (v_kv_j)$ and $(v_iv_k) + X_2 + (v_kv_j)$ are a pair of SSSD walks with length 4. Hence $l(K) \leq 4$.

Let $f_1, f_2 : A \rightarrow \{1, -1\}$,

$$f_1(v_iv_j) = \begin{cases} -1, & i = 1 \text{ and } j = 2 \\ 1, & \text{otherwise} \end{cases}$$

and

$$f_2(v_iv_j) = \begin{cases} -1, & i = 1 \text{ and } j = 2, 3 \\ 1, & \text{otherwise} \end{cases}$$

Then $(V, A, f_1)$ and $(V, A, f_2)$ are examples of signed digraph over complete graphs with loops with bases 3 and 4 respectively. Hence the possible bases of signed digraph over complete graphs with loops on 3 vertices are 3, 4 and 5.

Note that if

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

then

$$A^4 = \begin{pmatrix} 1 & \# & \# \\ \# & 1 & \# \\ \# & \# & 1 \end{pmatrix}.$$

### References


Byung Chul Song
Department of Mathematics
Gangneung-Wonju National University
Gangneung 210-702, Korea
E-mail: bcsong@gwnu.ac.kr

Byeong Moon Kim
Department of Mathematics
Gangneung-Wonju National University
Gangneung 210-702, Korea
E-mail: kbm@gwnu.ac.kr