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CONTINUITY OF THE SPECTRUM ON A CLASS $A(k)$

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Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathscr{H}$. An operator T is called class A operator if $|T^2| \geq |T|^2$ and is called class $A(k)$ operator if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$ for a positive number k. In this paper, we show that σ is continuous when restricted to the set of class $A(k)$ operators.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on a complex Hilbert space $\mathscr{H}$. Recall ([1], [3], [5], [7]) that an operator $T \in \mathscr{L}(\mathscr{H})$ is called p-hyponormal if

$$(T^*T)^p \geq (TT^*)^p \quad \text{for } p \in (0, 1].$$

Especially, if $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal. It is well known that q-hyponormal operators are p-hyponormal for $p \leq q$. An operator T is called *paranormal* if $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathscr{H}$, and T is called *normaloid* if $||T^n|| = ||T||^n$ for $n \in \mathbb{N}$ (equivalently, $||T|| = r(T)$, the spectral radius of T). For positive numbers s and t, an operator T belongs to class $A(s,t)$ if $(|T^*|^t|T|^{2s}|T|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$. Especially, we denote class $A(1,1)$ by class

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A, simply. It is well known that for $p \in (0,1]$

$$\begin{aligned}\{\text{hyponormal}\} \subset\ & \{p-\text{hyponormal}\}\\ \subset\ & \{\text{class } A(s,t), s,t \in (0,1]\}\\ \subset\ & \{\text{class } A\}\\ \subset\ & \{\text{paranormal}\}\\ \subset\ & \{\text{normaloid}\}.\end{aligned}$$

Let $T \in \mathscr{L}(\mathscr{H})$ and $T = U|T|$ be a polar decomposition, where U is a partial isometry with initial and final spaces $\overline{\text{ran}T^*}$ and $\overline{\text{ran}T}$, respectively. Note that if $T \in \mathscr{L}(\mathscr{H})$ then $\text{ker}T = \text{ker}|T|^\alpha$ for every $\alpha > 0$. Thus if $T = U|T|$ is a p-hyponormal operator then $\text{ker}(|T|^{2p}) \subseteq \text{ker}(|T^*|^{2p})$, so that $\text{ker}T \subseteq \text{ker}T^*$, which implies $\overline{\text{ran}T} \subseteq \overline{\text{ran}T^*}$. Thus, in the polardecomposition $T = U|T|$, the operator U can be extended to an isometry from $\mathscr{H}$ to $\mathscr{H}$.

Let $\mathfrak{G}$ denote the set, equipped with the Hausdorff metric, of all compact subsets of $\mathbb{C}$. If $\mathfrak{U}$ is a unital Banach algebra then the spectrum can be viewed as a function $\sigma : \mathfrak{U} \to \mathfrak{G}$, mapping each $T \in \mathfrak{U}$ to its spectrum $\sigma(T)$. It is well known that the function σ is upper semicontinuous and that in noncommutative algebras, σ does have points of discontinuity. The work of J. Newburgh ([15]) contains the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel ([4]) have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the C^*-algebra of all operators acting on a complex separable Hilbert space. Of interest is the identification of points of spectral continuity and of classes $\mathfrak{C}$ of operators for which σ becomes continuous when restricted to $\mathfrak{C}$. Recently Farenick and Lee ([8]) and Hwang and Lee ([11]) was considered the spectral continuity when restricted to certain subsets of the entire manifold of Toeplitz operators. The set of normal operators is perhaps the most immediate in the latter direction: σ is continuous on the set of normal operators. As noted in solution 105 of Hilbert space problem book, Newburgh's argument uses the fact that the inverses of normal resolvents are normaloid. This argument can be easily extended to the set of hyponormal operators because the inverses of hyponormal resolvents are also hyponormal and hence normaloid.

Now we consider the generalization of class A operator. For positive number k, an operator $T \in \mathscr{L}(\mathscr{H})$ belongs to *class* $A(k)$ if

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2.$$

It is well known that for $0 < p, k \leq 1$, the following inclusion relation holds

$$\{\text{hyponormal}\} \subset \{p-\text{hyponormal}\} \subset \{\text{class } A(k)\} \subset \{\text{class } A\}.$$

Although class $A(k)$ operators are normaloid for $0 < k \leq 1$, class $A(k)$ operator is not translation-invariant. Thus the arguments of Newburgh cannot apply to show that σ is continuous when restricted to the set of class $A(k)$ operators. In this paper, using the arguments of Cho and Yamazaki ([6]), we show that spectrum is continuous when restricted to the set of class $A(k)$ operators.

2. Results

We begin with the following lemma.

Lemma 2.1. ([11], Theorem) The spectrum σ is continuous on the set of all p-hyponormal operators.

Lemma 2.2. ([6], Theorem A) Let A and B are positive operators. Then for each $p \geq 0$ and $r \geq 0$

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \Rightarrow A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}.$$

Using the above lemmas we can have the following lemma which is used for proof of the main theorem.

Lemma 2.3. If $T \in \mathscr{L}(\mathscr{H})$ belongs to class $A(k)$, then $|T|^kU|T|$ is $\frac{1}{k+1}$-hyponormal, where $T = U|T|$ is the polar decomposition of T.

Proof. Firstly, we claim that

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2 \Leftrightarrow \left(|T^*||T|^{2k}|T^*|\right)^{\frac{1}{k+1}} \geq |T^*|^2.$$

It is well known that $T^* = U^*|T^*|$ is also the polar decomposition of T^* if $T = U|T|$ is the polar decomposition of T. Suppose that

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2.$$

Then we have

$$\begin{aligned}\left(|T^*||T|^{2k}|T^*|\right)^{\frac{1}{k+1}} &= UU^*\left(|T^*||T|^{2k}|T^*|\right)^{\frac{1}{k+1}}UU^* \\ &= U\left(U^*|T^*||T|^{2k}|T^*|U\right)^{\frac{1}{k+1}}U^* \\ &= U\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}}U^* \\ &\geq U|T|^2U^* = |T^*|^2.\end{aligned}$$

Conversely, suppose that

$$\left(|T^*||T|^{2k}|T^*|\right)^{\frac{1}{k+1}} \geq |T^*|^2.$$

Then we have

$$\begin{aligned}\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} &= \left(U^*|T^*||T|^{2k}|T^*|U\right)^{\frac{1}{k+1}} \\ &= U^*\left(|T^*||T|^{2k|}T^*|\right)^{\frac{1}{k+1}}U \\ &\geq U^*|T^*|^2U = |T|^2.\end{aligned}$$

Now let $\tilde{T} = |T|^kU|T|$. Since T belongs to class $A(k)$ by assumption,

$$\left(|T^*||T|^{2k}|T^*|\right)^{\frac{1}{k+1}} \geq |T^*|^2.$$

Thus by Lemma 2, we have

$$\begin{aligned}|T|^2 &\geq \left(|T|^k|T^*|^2|T|^k\right)^{\frac{1}{k+1}} \\ &= \left(|T|^kTT^*|T|^k\right)^{\frac{1}{k+1}} \\ &= \left(|T|^kU|T|^2U^*|T|^k\right)^{\frac{1}{k+1}} \\ &= (\tilde{T}\tilde{T}^*)^{\frac{1}{k+1}}.\end{aligned}$$

Therefore

$$\begin{aligned}(\tilde{T}^*\tilde{T})^{\frac{1}{k+1}} &= \left(|T|U^*|T|^{2k}U|T|\right)^{\frac{1}{k+1}} \\ &= \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \\ &\geq |T|^2 \\ &\geq (\tilde{T}\tilde{T}^*)^{\frac{1}{k+1}}.\end{aligned}$$

Hence $\tilde{T}$ is $\frac{1}{k+1}$-hyponormal. □

Lemma 2.4. ([12], Lemma 5) If $T \in \mathscr{L}(\mathscr{H})$ is an operator such that $T = V|T|$ with partial isometric operator V, then for $s \geq t \geq 0$

$$\sigma\left(|T|^t V|T|^{s-t}\right) = \sigma\left(V|T|^s\right).$$

We are ready for proving the main theorem.

Theorem 2.5. The spectrum σ is continuous when restricted to the set of class $A(k)$ operators.

Proof. Suppose that T and T_n for $n \in \mathbb{N}$ belong to class $A(k)$ and $T = U|T|$ and $T_n = U_n|T_n|$ are polar decompositions of T and T_n, respectively. Suppose that T_n converges to T. By the Lemma 2.1, Lemma 2.3 and Lemma 2.4, it is sufficient to show that

$$\sigma\left(U|T|^{k+1}\right) = \{r^{k+1}e^{i\theta} : re^{i\theta} \in \sigma(T)\}.$$

Let $T(k) = U|T|^{k+1}$. Then since $|T(k)| = |T|^{k+1}$ and $|T(k)^*| = |T^*|^{k+1}$, we have

$$\begin{aligned}
&T \text{ belongs to class } A(k)\\
&\Longleftrightarrow \left(|T^*||T|^{2k}|T^*|\right)^{\frac{1}{k+1}} \geq |T^*|^2\\
&\Longleftrightarrow \left(|T(k)^*|^{\frac{1}{k+1}}|T(k)|^{\frac{2k}{k+1}}|T(k)^*|^{\frac{1}{k+1}}\right)^{\frac{1}{k+1}} \geq |T(k)^*|^{\frac{2}{k+1}}\\
&\Longleftrightarrow T(k) \text{ belongs to class } A(\frac{k}{k+1}, \frac{1}{k+1})\\
&\Longrightarrow T(k) \text{ belongs to class } A.
\end{aligned}$$

Now applying the Cho and Yamazaki's argument (the proof of ([6], Theorem 2.2)): if $T(t) = U|T|^{t+1}$ and $\tau_t\left(re^{i\theta}\right) = r^{t+1}e^{i\theta}$, and if $T(t)$ belongs to the class A, then $\sigma(T(t)) = \tau_t(\sigma(T))$ for all $t \in [0,1]$, we have the result.

$\square$

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