Finitely \( t \)-Valuative Domains

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Abstract. Let \( D \) be an integral domain with quotient field \( K \). In [1], the authors called \( D \) a finitely valuative domain if, for each \( 0 \neq u \in K \), there is a saturated chain of rings \( D = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n = D[x] \), where \( x = u \) or \( u^{-1} \). They then studied some properties of finitely valuative domains. For example, they showed that the integral closure of a finitely valuative domain is a Prüfer domain. In this paper, we introduce the notion of finitely \( t \)-valuative domains, which is the \( t \)-operation analog of finitely valuative domains, and we then generalize some properties of finitely valuative domains.

1. Introduction

Let \( D \) be an integral domain with quotient field \( K \). Let \( R \) be an overring of \( D \), i.e., a ring between \( D \) and \( K \). As in [1], we say that \( R \) is within \( n \) steps of \( D \) if there is a saturated chain of overrings \( D = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_m = R \) where \( m \leq n \). We say that \( R \) is within finitely many steps of \( D \) if \( R \) is within \( n \) steps of \( D \) for some integer \( n \geq 1 \). An \( x \in K \) is said to be within \( n \) steps of \( D \) if \( D[x] \) is within \( n \) steps of \( D \). An integral domain \( D \) is an \( n \) valuative domain if, for each \( 0 \neq u \in K \), at least one of \( u \) or \( u^{-1} \) is within \( n \) steps of \( D \), while \( D \) is a finitely valuative domain if, for each \( 0 \neq u \in K \), at least one of \( u \) or \( u^{-1} \) is...
within \( n \) steps of \( D \) for some integer \( n = n(u) \geq 1 \). Clearly, an \( n \) valuative domain is a finitely valuative domain. In this paper, we introduce the notion of finitely \( t \)-valuative domains, which is the \( t \)-operation analog of finitely valuative domains, and we then generalize some results of finitely valuative domains.

To facilitate the reading of introduction, we first review the definitions related to the \( t \)-operation. Let \( \overline{D} \) be the integral closure of \( D \) in \( K \), \( X \) be an indeterminate over \( D \), and \( D[X] \) be the polynomial ring over \( D \). For a polynomial \( f \in K[X] \), we denote by \( c_D(f) \) (simply, \( c(f) \)) the fractional ideal of \( D \) generated by the coefficients of \( f \). Let \( F(D) \) (resp., \( f(D) \)) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of \( D \); so \( f(D) \subseteq F(D) \). For \( I \in F(D) \), let \( I^{-1} = \{ u \in K \mid uI \subseteq D \} \), \( I_v = (I^{-1})^{-1} \), and \( I_t = \cup \{ J_v \mid J \in f(D) \text{ and } J \subseteq I \} \). Clearly, if \( I \in f(D) \), then \( I_v = I_t \). We say that \( I \in F(D) \) is a \( t \)-ideal if \( I_t = I \); a \( t \)-ideal is a maximal \( t \)-ideal if it is maximal among proper integral \( t \)-ideals; and \( t \)-Max(\( D \)) is the set of maximal \( t \)-ideals of \( D \). It is well known that each maximal \( t \)-ideal is a prime ideal and \( t \)-Max(\( D \)) \( \neq \emptyset \) when \( D \) is not a field. An \( I \in F(D) \) is said to be \( t \)-invertible if \( (I^{-1})_t = D \). We say that \( D \) is a Prüfer \( v \)-multiplication domain (PvMD) if each nonzero finitely generated ideal of \( D \) is \( t \)-invertible. An upper to zero in \( D[X] \) is a nonzero prime ideal \( Q \) of \( D[X] \) with \( Q \cap D = (0) \). A domain \( D \) is called a UMT-domain if each upper to zero in \( D[X] \) is a maximal \( t \)-ideal. It is known that \( D \) is a UMT-domain if and only if \( \overline{D}_P \) is a Prüfer domain for all \( P \in \text{t-Max}(D) \) [5, Theorem 1.5]. In particular, \( \overline{D} \) is a Prüfer domain if and only if \( D \) is a UMT-domain whose maximal ideals are \( t \)-ideal [4, Theorem 1.1 and Corollary 1.3]. It is also known that \( D \) is a PvMD if and only if \( D \) is an integrally closed UMT-domain [6, Proposition 3.2]. Recall that \( D \) is a GCD-domain if and only if \( I_v \) is principal for all \( I \in f(D) \); so GCD-domains are PvMDs. An overring \( R \) of \( D \) is said to be \( t \)-linked over \( D \) if \( I^{-1} = D \) for \( I \in f(D) \) implies \( (IR)^{-1} = R \). For an overring \( R \) of \( D \), let \( R_w = \{ x \in K \mid xJ \subseteq R \text{ for some } J \in f(D) \text{ with } J^{-1} = D \} \). It is known that \( R_w \) is the smallest \( t \)-linked overring of \( D \) that contains \( R \) [2, Remark 3.3]; hence \( R \) is \( t \)-linked over \( D \) if and only if \( R_w = R \). Also, if we let \( N_v = \{ f \in D[X] \mid c(f)_v = D \} \), then \( R[X]_{N_v} \cap K = R_w \), and hence \( R \) is \( t \)-linked over \( D \) if and only if \( R[X]_{N_v} \cap K = R \) [2, Lemma 3.2].

Let \( R \) be a \( t \)-linked overring of \( D \). We say that \( R \) is within \( t \)-linked \( n \) steps of \( D \) if there is a saturated chain of \( t \)-linked overrings \( D = D_0 \subset D_1 \subset \ldots \subset D_n = R \).
$D_1 \subset D_2 \subset \cdots \subset D_m = R$ where $m \leq n$. We say that $R$ is within $t$-linked finitely many steps of $D$ if $R$ is within $t$-linked $n$ steps of $D$ for some integer $n \geq 1$. We say that a nonzero $u \in K$ is within $t$-linked finitely many steps of $D$ if $(D[u])_w$ is within $t$-linked finitely many steps of $D$. We say that $D$ is a finitely $t$-valuative domain if, for each nonzero $u \in K$, at least one of $u$ or $u^{-1}$ is within $t$-linked finitely many steps of $D$. Our first result of this paper shows that if there is an integer $n \geq 1$ such that for each $0 \neq u \in K$, at least one of $u$ or $u^{-1}$ is within $t$-linked $n$ steps of $D$, then $D$ is an $n$-valuative domain, which shows why we don’t need to define the $t$-operation analog of $n$-valuative domains. We prove that if $D$ is a finitely $t$-valuative domain, then $D$ is a UMT-domain, and hence an integrally closed finitely $t$-valuative domain is a PoMD. It is also shown that (i) Krull domains are finitely $t$-valuative; (ii) if $D$ is a GCD-domain, then $D$ is finitely $t$-valuative if and only if $D[X]$ is finitely $t$-valuative, and if only if $D[X]_{N_u}$ is finitely valuative; and (iii) if $D$ is an integrally closed $n$ valuative domain for an integer $n \geq 1$, then $D[X]$ is a finitely $t$-valuative domain.

2. Finitely $t$-valuative domains

Throughout $D$ is an integral domain with quotient field $K$, $X$ is an indeterminate over $D$, $D[X]$ is the polynomial ring over $D$, and $N_v = \{ f \in D[X] \mid c(f)_v = D \}$.

**Proposition 1.** Let $n$ be a positive integer. If, for each $0 \neq u \in K$, either $u$ or $u^{-1}$ is within $t$-linked $n$ steps of $D$, then $|t$-Max$(D)| \leq 2n + 1$. Hence $t$-Max$(D) = \text{Max}(D)$, the set of maximal ideals of $D$, and thus $D$ is an $n$-valuative domain.

**Proof.** Assume $|t$-Max$(D)| \geq 2n + 2$. Let $\{P_i \mid i = 1, \ldots, 2n + 2\}$ be a set of maximal $t$-ideals of $D$, and set $S = D \setminus \bigcup_{i=1}^{2n+2} P_i$. Then Max$(D_S) = \{P_i D_S \mid i = 1, \ldots, 2n + 2\}$. Let $0 \neq u \in K$, and let $x = u$ or $u^{-1}$. Note that $(D[x]_w)_S = D[x]_S = D_S[x]$; hence if $A$ is a ring such that $D_S \subseteq A \subseteq D[x]_S$, then $A = (A \cap D[x]_w)_S$ and $A \cap D[x]_w$ is $t$-linked over $D$ (note that both $A$ and $D[x]_w$ are $t$-linked over $D$). Hence, either $u$ or $u^{-1}$ is within $n$ steps of $D_S$. Thus, $D_S$ is an $n$-valuative domain, and so by [1, Theorem 2.6], $D_S$ has at most $2n + 1$ maximal ideals, a contradiction. Therefore, $|t$-Max$(D)| \leq 2n + 1$. Moreover, if $M$ is a maximal ideal of $D$, then $M \subseteq \bigcup_{P \in t$-Max$(D)} P$, and since $|t$-Max$(D)| \leq 2n + 1$, we have
\[M \subseteq P \text{ or } M = P \text{ for some } P \in t\text{-Max}(D). \] Thus, each maximal ideal of 
\(D\) is a \(t\)-ideal, which means that \(t\text{-Max}(D) = \text{Max}(D)\) and each overring 
of \(D\) is \(t\)-linked over \(D\).

As we prove in Proposition 1, if there is a positive integer \(n\) such 
that, for each \(0 \neq u \in K\), either \(u\) or \(u^{-1}\) is within \(t\)-linked \(n\) steps 
of \(D\), then \(D\) is an \(n\)-valuative domain. So, in this paper, we focus on 
finitely \(t\)-valuative domains. Our next result shows the relationship be-
tween finitely valuative domains and finitely \(t\)-valuative domains.

**Proposition 2.** \(D\) is finitely valuative if and only if \(D\) is finitely \(t\)-valuative and each maximal ideal of \(D\) is a \(t\)-ideal.

Proof. Assume that \(D\) is finitely valuative. Then the integral closure 
of \(D\) is a Pr"ufer domain [1, Theorem 3.4], and hence \(D\) is a UMT-domain 
in which each maximal ideal of \(D\) is a \(t\)-ideal. Moreover, note that if each 
maximal ideal of \(D\) is a \(t\)-ideal, then every overring of \(D\) is \(t\)-linked over 
\(D\). Thus, \(D\) is finitely \(t\)-valuative. The converse is clear.

We next give the finitely \(t\)-valuative domain analog of [1, Theorem 
3.4] that the integral closure of a finitely valuative domain is a Pr"ufer 
domain.

**Theorem 3.** If \(D\) is a finitely \(t\)-valuative domain, then \(D\) is a UMT-
domain. In particular, an integrally closed finitely \(t\)-valuative domain is a \(PoMD\).

Proof. Let \(P\) be a maximal \(t\)-ideal of \(D\). It suffices to show that the 
integral closure of \(D_P\) is a Pr"ufer domain [5, Theorem 1.5]. To show 
this, let \(0 \neq u \in K\). Then at least one of \(u\) or \(u^{-1}\), for convenience, 
say \(u\), is within \(t\)-linked finitely many steps of \(D\). Hence there exists a 
saturated chain of \(t\)-linked overrings of \(D\), say, \(D = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_n = (D[u])_u\). Clearly, \(D_P = (D_0)_P \subsetneq (D_1)_P \subsetneq \cdots \subsetneq (D_n)_P = ((D[u])_u)_P = (D[u])_P = D_P[u]\) is a chain of overrings of \(D_P\). Let 
\(R\) be a ring such that \((D_i)_P \subsetneq R \subsetneq (D_{i+1})_P\). Note that \(R = (R \cap 
D_{i+1})_P\); \(D_i \subseteq R \cap D_{i+1} \subseteq (R \cap D_{i+1})_w \subseteq (D_{i+1})_w = D_{i+1};\) and \((R \cap 
D_{i+1})_w\) is \(t\)-linked over \(D\). Hence, either \((R \cap D_{i+1})_w = D_i\) or \((R \cap 
D_{i+1})_w = D_{i+1}\), and thus \(R = (R \cap D_{i+1})_P = ((R \cap D_{i+1})_w)_P = 
(D_i)_P\) or \(R = ((R \cap D_{i+1})_w)_P = (D_{i+1})_P\). Therefore, the chain 
\(D_P = (D_0)_P \subsetneq (D_1)_P \subsetneq \cdots \subsetneq (D_n)_P\) is saturated. Hence \(D_P\) is a 
finitely valuative domain, and thus the integral closure of \(D_P\) is a Pr"ufer
domain [1, Theorem 3.4]. The “in particular” part follows because an integrally closed UMT-domain is a PvMD.

By Theorem 3, an integrally closed finitely t-valuative domain is a PvMD. Thus, it is reasonable to study PvMDs that are finitely t-valuative domains. Let \( N_v = \{ f \in D[X] \mid c(f)_v = D \} \). It is well known that \( D \) is a PvMD if and only if \( D[X]_{N_v} \) is a Prüfer domain, and only if each ideal of \( D[X]_{N_v} \) is extended from \( D \) [7, Theorems 3.1 and 3.7]; in this case, \( fD[X]_{N_v} = c(f)D[X]_{N_v} \) for each \( f \in D[X] \).

Lemma 4. Let \( D \) be a PvMD and \( \{ D_\alpha \} \) be the set of t-linked overrings of \( D \).

1. The mapping \( D_\alpha \mapsto D_\alpha[X]_{N_v} \) is a bijection from the set \( \{ D_\alpha \} \) onto the set of overrings of \( D[X]_{N_v} \), where \( N_v = \{ f \in D_\alpha[X] \mid c_{D_\alpha}(f)_v = D_\alpha \} \).
2. If \( 0 \neq u \in K \), then \( u \) is within t-linked \( n \) steps of \( D \) if and only if \( u \) is within \( n \) steps of \( D[X]_{N_v} \).
3. If \( D[X]_{N_v} \) is a finitely valuative domain, then \( D \) is a finitely t-valuative domain.

Proof. (1) This follows directly from [3, Lemma 2 and Corollary 6]. (2) This is an immediate consequence of (1), because \( D[u] = D[u][X]_{N_v} \cap K \) and \( D[u][X]_{N_v} = (D[X]_{N_v})[u] \). (3) This is an immediate consequence of (2).

We say that \( D \) is of finite character (resp., finite t-character) if each nonzero nonunit of \( D \) is contained in a finite number of maximal ideals (resp., maximal t-ideals) of \( D \). The t-dimension of a PvMD \( D \), denoted by \( t\dim(D) \), is \( \sup \{ \text{ht}P \mid P \in t\text{-Max}(D) \} \). It is clear that if \( D \) is a Krull domain, then \( D \) is a PvMD of \( t\dim(D) = 1 \) and finite t-character.

Corollary 5. If \( D \) is a PvMD of \( t\dim(D) < \infty \) and finite t-character, then \( D \) is a finitely t-valuative domain. Hence a Krull domain is finitely t-valuative.

Proof. Clearly, \( D[X]_{N_v} \) is a finite dimensional Prüfer domain of finite character, and hence \( D[X]_{N_v} \) is a finitely valuative domain [1, Corollary 4.15]. Thus, \( D \) is a finitely t-valuative domain by Lemma 4(3).

Let \( I \) be an ideal of \( D \). As in [1], we say that \( I \) is finitely light if \( I \) is contained in finitely many prime ideals of \( D \). Similarly, we say that \( I \) is finitely t-light if the number of prime t-ideals of \( D \) containing \( I \) is finite.
Recall that if $P$ is a nonzero prime ideal of a PrvMD $D$, then $P_t \subseteq D$ if and only if $P$ is a $t$-ideal; so if $I_t \subseteq D$, then $I$ is finitely $t$-light if and only if $ID[X]_{N_v}$ is finitely light.

**Corollary 6.** The following are equivalent for an integrally closed domain $D$.

1. $D$ is a finitely $t$-valuative domain.
2. $D$ is a PrvMD such that for $0 \neq b, c \in D$, letting $I = bD + cD$, at least one of $bI - 1$ or $cI - 1$ is finitely $t$-light.

**Proof.** $(1) \Rightarrow (2)$ First, note that $D$ is a PrvMD by Theorem 3, and hence $D[X]_{N_v}$ is a Prufer domain. Let $u = \frac{b}{c}$. Then either $u$ or $u^{-1}$ is within $t$-linked $n$ steps of $D$ for some integer $n = n(u) \geq 1$, and thus either $u$ or $u^{-1}$ is within $n$ steps of $D[X]_{N_v}$ by Lemma 4(2). Hence, by [1, Corollary 1.15], either $(D[X]_{N_v} : D[X]_{N_v} u) = c \cdot (ID[X]_{N_v})^{-1} = (cI^{-1})D[X]_{N_v}$ or $(D[X]_{N_v} : D[X]_{N_v} u^{-1}) = (bI^{-1})D[X]_{N_v}$ is contained in exactly $n$ primes. Thus, either $bI^{-1}$ or $cI^{-1}$ is contained in exactly $n$ prime $t$-ideals of $D$. Hence at least one of $bI^{-1}$ or $cI^{-1}$ is finitely $t$-light.

$(2) \Rightarrow (1)$ By assumption, $D[X]_{N_v}$ is a Prufer domain and either $(cI^{-1})D[X]_{N_v}$ or $(bI^{-1})D[X]_{N_v}$ is finitely light. Hence if $u = \frac{b}{c}$, then $u$ or $u^{-1}$ is within finitely many steps of $D[X]_{N_v}$ [1, Lemma 4.4], and so by Lemma 4(2), $u$ or $u^{-1}$ is within $t$-linked finitely many steps of $D$. Thus, $D$ is finitely $t$-valuative.

It is known that if $D$ is an integrally closed $n$-valuative domain, then $D$ is a Prufer domain with at most $2n + 1$ maximal ideals [1, Proposition 4.2]. Hence, an integrally closed $n$-valuative domain is a Bezout domain (and so a GCD-domain). This is why we next study GCD-domains that are finitely $t$-valuative domains.

**Corollary 7.** The following are equivalent for a GCD-domain $D$.

1. $D$ is a finitely $t$-valuative domain.
2. $D[X]_{N_v}$ is a finitely valuative domain.
3. $D[X]$ is a finitely $t$-valuative domain.
4. For each pair of $t$-comaximal elements $a, b \in D$, i.e., $(aD + bD)_t = D$, at least one of $a$ or $b$ is finitely $t$-light.
5. For each pair of $t$-comaximal finitely generated ideals $I$ and $J$ of $D$, i.e., $(I + J)_t = D$, at least one of $I$ or $J$ is finitely $t$-light.

**Proof.** $(1) \Rightarrow (4)$ Corollary 6.
(4) $\iff$ (5) This follows because $A_t$ is principal for all nonzero finitely generated ideals $A$ of a GCD-domain and $(I + J)_t = (I_t + J_t)_t$.

(5) $\Rightarrow$ (2) Let $f, g \in D[X]$ be nonzero such that $fD[X]_{N_v} + gD[X]_{N_v} = D[X]_{N_v}$. Then $fD[X]_{N_v} = c(f)D[X]_{N_v}$; $gD[X]_{N_v} = c(g)D[X]_{N_v}$; and $(c(f) + c(g))_t = D$. Hence by (5), at least one of $c(f)$ or $c(g)$ is finitely $t$-light, and thus either $f$ or $g$ is finitely light. Thus, $D[X]_{N_v}$ is finitely valuative [1, Theorem 4.5].

(2) $\Rightarrow$ (1) Lemma 4(3).

(3) $\Rightarrow$ (4) Note that $a, b \in D$ are $t$-comaximal in $D$ if and only if $a, b$ are $t$-comaximal in $D[X]$ and that $P[X]$ is a prime $t$-ideal of $D[X]$ for all prime $t$-ideals $P$ of $D$. Thus, the proof is completed by the equivalence of (1) and (4).

(5) $\Rightarrow$ (3) Let $f, g \in D[X]$ be $t$-comaximal elements of $D[X]$. Then $c(f)$ and $c(g)$ are $t$-comaximal finitely generated ideals of $D$, and hence at least one of $c(f)$ or $c(g)$ is finitely $t$-light. Note that if $Q$ is a prime $t$-ideal of $D[X]$, then $Q \cap D = (0)$ or $Q = (Q \cap D)[X]$ and $Q \cap D$ is a prime $t$-ideal of $D$ (cf. [7, Theorem 3.1] and [6, Theorem 1.4]). Clearly, each nonzero element of $D[X]$ is contained in only finitely many prime $t$-ideals $Q$ of $D[X]$ with $Q \cap D = (0)$, because $D[X]_{D \setminus \{0\}}$ is a principal ideal domain. Thus, either $f$ or $g$ is finitely $t$-light. Therefore, $D[X]$ is a finitely $t$-valuative domain by the equivalence of (1) and (4).

Corollary 8. If $D$ is an integrally closed $n$-valuative domain for some integer $n \geq 1$, then $D[X]$ is a finitely $t$-valuative domain.

Proof. Recall from [1, Proposition 4.2] that $D$ is a Bezout domain (hence GCD-domain). Thus, by Corollary 7, $D[X]$ is a finitely $t$-valuative domain.

References


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