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INSERTION-OF-FACTORS-PROPERTY WITH FACTORS NILPOTENTS

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Abstract. We in this note study a ring theoretic property which unifies Armendariz and IFP. We call this new concept *INFP*. We first show that idempotents and nilpotents are connected by the Abelian ring property. Next the structure of INFP rings is studied in relation to several sorts of algebraic systems.

1. INFP rings

Throughout this note every ring is an associative ring with identity unless otherwise stated. Given a ring R, let $I(R)$ and $N(R)$ denote the set of all idempotents and the set of all nilpotent elements in R, respectively. A nilpotent elements is also called a *nilpotent* simply. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. $R[x]$ denotes the polynomial ring with an indeterminate x over R. For $f(x) \in R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$.

Following the literature, the *index of nilpotency* of $a \in N(R)$ is the least positive integer n such that $a^n = 0$, write $i(a)$ for n. The *index of nilpotency* of a subset S of R is the supremum of the indices of nilpotency

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of all nilpotent elements in S, write $i(S)$; and if such a supremum is finite, then S is said to be *of bounded index of nilpotency*. Define $N_t(R) = \{a \in R \mid i(a) \leq t\}$. It is easily shown that $N(R) = \cup_{t=1}^{\infty} N_t(R)$, and so if R is of bounded index of nilpotency then $N(R) = \cup_{t=1}^{n} N_t(R)$ for some $n \geq 1$.

Lemma 1.1. *For a ring R and $e \in I(R)$, the following conditions are equivalent:*
(1) *e is central;*
(2) *$ef = fe$ for every $f \in I(R)$;*
(3) *$ea = ae$ for every $a \in N(R)$;*
(4) *$eb = be$ for every $b \in N_2(R)$.*

Proof. (1)⇒(2), (1)⇒(3), and (3)⇒(4) are obvious.

(2)⇒(1): Suppose that the condition (2) holds. Let $r \in R$. Consider the element $f = e + er(1-e)$. Then $f^2 = f$, so $ef = fe$ and this yields $er(1-e) = 0$. Thus $er = ere$. Next we get $re = ere$ similarly, using $(1-e) + (1-e)re \in I(R)$. These yield $er = re$.

(4)⇒(1): Suppose that the condition (4) holds. Let $r \in R$. Since $(er(1-e))^2 = 0$ and $((1-e)re)^2 = 0$, we have $er(1-e) = 0$ and $(1-e)re = 0$. Thus $er = re$. □

A ring (possibly without identity) is usually called *Abelian* if every idempotent is central. The following is obtained by Lemma 1.1.

Proposition 1.2. *For a ring R, the following conditions are equivalent:*
(1) *R is Abelian;*
(2) *$ef = fe$ for all $e, f \in I(R)$;*
(3) *$ea = ae$ for all $e \in I(R)$ and $a \in N(R)$;*
(4) *$eb = be$ for all $e \in I(R)$ and $b \in N_2(R)$.*

A ring (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. Suppose that $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$ over a reduced ring R (possibly without identity). In this situation, Armendariz [2, Lemma 1] proved that $ab = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$. Rege and Chhawchharia [15] called a ring (possibly without identity) *Armendariz* if it satisfies such property. This shows that reduced rings are Armendariz.

Recall that Armendariz rings are also Abelian by [8, Corollary 8]. We can obtain this fact independently by help of Proposition 1.2 as follows.

Corollary 1.3. *Armendariz rings are Abelian.*

Proof. Let R be an Armendariz ring and consider $e \in I(R)$, $b \in N_2(R)$ (i.e., $b^2 = 0$). Take polynomials

$$f(x) = eb + ex, g(x) = b(1-e) - (1-e)x$$

and

$$f_1(x) = (1-e)b + (1-e)x, g_1(x) = be - ex$$

in $R[x]$. Then $f(x)g(x) = 0$ and $f_1(x)g_1(x) = 0$. Since R is Armendariz, we have $eb(1-e) = 0, (1-e)be = 0$. This yields $eb = be$ and so R is Abelian by Proposition 1.2. □

We now concentrate on a condition, called *Insertion-of-Factors-Property* (simply *IFP*) by Bell [3], between reduced rings and commutative rings. A ring R (possibly without identity) is usually called *IFP* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Shin [16] used the term *SI* for the IFP, while Narbonne [14] used *semicommutative* in place of the IFP. Reduced rings are simply shown to be IFP. It is also easily checked that every IFP ring is Abelian.

Now we consider a condition that

$$acb = 0 \text{ for } c \in N(R) \text{ whenever } ab = 0 \text{ for } a, b \in R.$$

This new concept is clearly a generalization of IFP rings and the following example shows that this generalization is proper. Based on these arguments, a ring R (possibly without identity) will be called *Insertion-of-Nilpotent-Factors-Property* (simply *INFP*) if $acb = 0$ for all $c \in N(R)$ whenever $ab = 0$ for $a, b \in R$.

Example 1.4. We use the ring in [1, Example 4.8]. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2 and set $R = A/I$. Identify a and b with their images in R for simplicity. $b^2 = 0$ but $bab \neq 0$, so R is not IFP.

We will show that R is INFP. Let $\alpha, \beta \in R\backslash\{0\}$ with $\alpha\beta = 0$. We apply the computation in the proof of [10, Theorem 1]. α and β can be rewritten by

$$\alpha = \alpha_0 + \alpha_1 b \text{ and } \beta = \beta_0 + b\beta_1,$$

where $\alpha_i, \beta_j \in R$ for $i, j \in \{0, 1\}$ and every sum-factor of α_i's does not end by b and every sum-factor of β_j's does not start by b. Note $\alpha_1 bb\beta_1 = 0$.

From $0 = \alpha\beta = \alpha_0\beta_0 + \alpha_0 b\beta_1 + \alpha_1 b\beta_0$, we have

$$\alpha_0\beta_0 = 0 \text{ and } \alpha_0 b\beta_1 + \alpha_1 b\beta_0 = 0$$

since $\alpha_0\beta_0$ is not a term of same kind to each of $\alpha_0 b\beta_1$ and $\alpha_1 b\beta_0$. Thus we have $\alpha_0 = 0$ or $\beta_0 = 0$, entailing

$$\text{“}\alpha = \alpha_1 b, \beta = \beta_0 + b\beta_1\text{” or “}\alpha = \alpha_0 + \alpha_1 b, \beta = b\beta_1\text{”}.$$

Here assume $\alpha_0 \neq 0$. Then $\beta_0 = 0$, so $\beta = b\beta_1$. But $\alpha_0 b\beta_1 = 0$ implies $b\beta_1 = 0$ (hence $\beta = 0$) since every sum-factor of α_0 does not end by b and every sum-factor of β_1 does not start by b. Similarly we get $\alpha = 0$ when $\beta_0 \neq 0$. Consequently we have $\alpha = \alpha_1 b$ and $\beta = b\beta_1$.

Applying this result, we also get that every nilpotent element is of the form brb with $r \in R$. This yields $\alpha brb\beta = \alpha_1 bbrbb\beta_1 = 0$, concluding that R is INFP.

In the following we see connections among INFP, Abelian, and Armendariz.

Proposition 1.5. (1) *Every INFP ring is Abelian.*

(2) *Armendariz rings are INFP.*

Proof. (1) Let R be an INFP ring and $e \in I(R), r \in R$. Then $e(1-e) = 0 = (1-e)e$ and $er(1-e), (1-e)re \in N(R)$. Since R is INFP, we have $er(1-e) = eer(1-e)(1-e) = 0$ and $(1-e)re = (1-e)(1-e)ree = 0$. These yield $er = re$.

Another proof of (1): Let $e \in I(R)$. Then $e(1-e) = 0 = (1-e)e$. Since R is INFP, we have $ea(1-e) = 0$ and $(1-e)ae = 0$ for all $a \in N(R)$. These yield $ea = ae$, so e is central by Lemma 1.1.

(2) is shown by [8, Lemma 7(1)]. □

Armendariz rings are Abelian by Corollary 1.3. The classes of Armendariz rings and IFP rings are independent of each other by [8, Examples 2 and 14]. So INFP rings need not be Armendariz, entailing that the converse of Proposition 1.5(2) need not hold.

The converse of Proposition 1.5(1) also need not hold as follows. Following the literature, let

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in U_n(R) \mid a, a_{ij} \in R \right\}$$

over a given ring R. By [7, Lemma 2], every $D_n(R)$ is Abelian if and only if R is Abelian. However $D_n(R)$ (over any ring R) cannot be INFP when $n \geq 4$ by [12, Example 1.3]. Indeed, $(e_{12} - e_{13})(e_{24} + e_{34}) = 0$ but $(e_{12} - e_{13})e_{23}(e_{24} + e_{34}) = e_{14} \neq 0$ in spite of $e_{23} \in N(D_n(R))$.

Recall that $D_3(R)$ is a non-reduced Armendariz ring by [11, Proposition 2].

Proposition 1.6. *For a ring R the following conditions are equivalent:*

(1) *R is a reduced ring;*
(2) *$D_3(R)$ is Armendariz;*
(3) *$D_3(R)$ is IFP;*
(4) *$D_3(R)$ is INFP.*

Proof. The equivalences of the conditions (1), (2), and (3) are proved by [9, Proposition 2.8], and (3)⇒(4) is obvious.

(4)⇒(1): Let $D_3(R)$ be INFP, and assume on the contrary that there is a nonzero $a \in R$ with $a^2 = 0$. We apply the computation in [9, Proposition 2.8]. Take $A = \begin{pmatrix} a & a & 1 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$, $B = \begin{pmatrix} a & 0 & a \\ 0 & a & -1 \\ 0 & 0 & a \end{pmatrix}$ in $D_3(R)$. Then $AB = 0$ but

$$\begin{pmatrix} a & a & 1 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & a \\ 0 & a & -1 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$$

noting $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N(D_3(R))$. This induces a contradiction to $D_3(R)$ being INFP. Thus R is reduced.

□

If $D_3(R)$ is INFP then $D_2(R)$ is clearly INFP since $D_2(R)$ is isomorphic to a subring of $D_3(R)$. But $D_2(R)$ being INFP need not be a sufficient condition of R being reduced. In fact, $D_2(R)$ is commutative (hence IFP) for any non-reduced commutative ring R (e.g., $\mathbb{Z}_{k^l}$ for $k, l \geq 2$).

A ring R is usually called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. We obtain the following equivalences by help of [2, Lemma 1], Proposition 1.5, and [6, Theorem 3.2].

Proposition 1.7. *Let R be a regular ring. Then R is IFP if and only if R is INFP if and only if R is Abelian if and only if R is reduced if and only if R is Armendariz.*

2. On nilpotents of INFP rings

In this section we focus our eyes on the basic property of nilpotents in INFP rings.

Proposition 2.1. *For an INFP ring we have the following.*

(1) *$N(R)$ forms a subring (without identity) of R.*

(2) *Let S be a subring (possibly without identity) of R. Then $N(S)$ is a nil subring of R.*

(3) *$eN(R)e$ is a nil subring of R for all $e \in I(R)$.*

Proof. Note that the class of INFP rings is closed under subrings.

(1) Let $a, b \in N(R)$ and say that $a^m = 0$, $b^n = 0$ for some $m, n \geq 1$. Then since R is INFP, $\underbrace{(ab)(ab)\cdots(ab)}_{m-\text{times}} = 0$ and so $ab, ba \in N(R)$. Similarly we have $ar_1ar_2\cdots r_{m-1}a = 0$ and $bs_1bs_2b\cdots s_{n-1}b = 0$ for all $r_i, s_j \in \{1, a^h, b^k \mid h, k \geq 1\}$.

Next consider the expansion of $(a+b)^{m+n}$, $\mathcal{S}$ say. Then every term in $\mathcal{S}$ contains $ar_1ar_2\cdots r_{m-1}a$ or $bs_1bs_2b\cdots s_{n-1}b$, and hence it is zero by the preceding result. This yields $(a+b)^{m+n} = 0$, entailing $a+b \in N(R)$.

(2) is an immediate consequence of (1), and (3) is shown by (1) and Proposition 1.5(1). □

For an Armendariz ring R, $N(R)$ also forms a subring of R by [1, Corollary 3.3].

Let K be a field and R_1, R_2 be K-algebras. $R_1 *_K R_2$ denotes the ring coproduct of R_1 and R_2 (see Antoine [1] and Bergman [4, 5] for details.)

The following theorem can be shown by Proposition 1.5(2), Lemma 2.1, [1, Theorem 4.7], and [10, Theorem 1]. But we here provide another proof.

THEOREM 2.2. *Let K be a field and A be a K-algebra. Let $C = K[b]$ be the polynomial ring with an indeterminate b over K, and I be the ideal of C generated by b^n for $n \geq 2$. Set $B = C/I$ and $R = A *_K B$. Then the following conditions are equivalent:*

(1) *R is Armendariz;*
(2) *A is a domain and $U(A) = K\backslash\{0\}$;*
(3) *R is INFP;*
(4) *$N(R)$ forms a subring of R.*

Proof. The equivalence of the conditions (1) and (2) is obtained from [1, Theorem 4.7]. (4) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are proved by [10, Theorem 1] and Lemma 2.1 respectively. So it remains to show (2) $\Rightarrow$ (3).

Assume the condition (2) and $st = 0$ for $0 \neq s, t \in R$. Then there exist $v, w \in R$ and $h, k \geq 1$ $s = vb^h$ and $t = b^k w$ with $h + k \geq n$ by the computation in [10, page 5].

Let $r \in N(R)$ and say $r^g = 0$ for some $g \geq 1$. Then we obtain $r = b^l r' b^m$ for some $r' \in R$ and $l, m \geq 1$ with $l + m \geq n$, from the equalities $rr^{g-1} = 0 = r^{g-1}r$. Now we have

$$srt = (vb^h)(b^l r' b^m)(b^k w) = vb^{h+l} r' b^{m+k} w = 0$$

since $h + k, l + m \geq n$ implies that $h + l \geq n$ or $m + k \geq n$. Therefore R is INFP.

□

Observing the contents of Theorem 2.2, one may ask whether R is also INFP if A is a domain and $U(A) \supsetneq K\backslash\{0\}$. But the answer is negative as follows. We use the ring in [10, Example 2(1)]. Let K be a field and $A = K[[a]]$, the power series ring with an indeterminate a over K, and B the ring in Theorem 2.2. Let $n \geq 3$ and set $R = A *_K B$. Then $(1-a)(1+a+a^2+\cdots) = 1$ and $b^{n-1}(1-a)(1+a+a^2+\cdots)b = b^n = 0$. But

$$\begin{aligned} &b^{n-1}(1-a)b(1+a+a^2+\cdots)b \\ &\qquad =(-b^{n-1}ab)(b+ab+a^2b+\cdots) \\ &\qquad =-b^{n-1}ab^2 - b^{n-1}abab - b^{n-1}aba^2b - \cdots \neq 0. \end{aligned}$$

This concludes that R is not INFP.

Following Marks [13], a ring R is called *NI* if $N(R) = N^*(R)$. Every IFP ring is NI through a simple computation, so one may ask whether INFP rings are also NI. But Theorem 2.2 answers this negatively. Let $A = K\langle a, b\rangle$ be the free algebra over a field K and I be the ideal of A generated by b^2. Set $R = A/I$. Then $R = K[a] *_K B$ with $B = K[b]/b^2K[b]$. Then R satisfies the condition (2) in Theorem 2.2, and so R is INFP. But R is not NI as can be seen by $\bar{b} \in N(R)$ and $\bar{a}\bar{b} \notin N(R)$.

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