A STUDY ON THE CATEGORY OF NORMAL FUZZY HYPERGROUPS

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Abstract. Although the category \textit{NFHG} of normal fuzzy hypergroups is not a topos, it forms a pseudo topos. Also we show that there are pseudo power objects in \textit{NFHG}.

1. Introduction

Sun [3] showed that the category \textit{NFHG} of normal fuzzy hypergroups satisfies all the axiom of topos except for the subobject classifier axiom. So we define a pseudo subobject classifier, pseudo topos and pseudo power object. Also Goldblatt [1] showed that any topos has power objects.

In this paper, we show that \textit{NFHG} has a pseudo subobject classifier. So \textit{NFHG} forms a pseudo topos. Also we show that there are pseudo power objects in \textit{NFHG} which is not a topos.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.
Definition 2.1. An *elementary topos* is a category $E$ that satisfies the following:

(T1) $E$ is finitely complete,

(T2) $E$ has exponentiation,

(T3) $E$ has a subobject classifier.

(T2) means that for every object $A$ in $E$, the endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object $A$ in $E$, there exists an object $B^A$, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of $A$, such that for any $Y$ and $f : Y \times A \to B$ in $E$, there exists a unique morphism $g$ such that $ev_A \circ (g \times id) = f$;

$$
\begin{array}{ccc}
Y \times A & \xrightarrow{f} & B \\
g \times id & \downarrow & id \\
B^A \times A & \xrightarrow{ev_A} & B 
\end{array}
$$

And subobject classifier in (T3) is an $E$-object $\Omega$, together with a morphism $\top : 1 \to \Omega$ such that for any monomorphism $h : D \to C$, there is a unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ which makes the following diagram a pull-back:

$$
\begin{array}{ccc}
D & \xrightarrow{!} & 1 \\
\downarrow h & & \downarrow \top \\
C & \xrightarrow{\chi_h} & \Omega
\end{array}
$$

Example 2.2. Category $\text{Set}$ is a topos. $\{\ast\}$ is a terminal object. $\Omega = \{0, 1\}$ and $\top : \{\ast\} \to \Omega$ with $\top(\ast) = 1$ is a subobject classifier. If we define

$\chi_h = 1$ if $c = h(d)$ for some $d \in D$,

$\chi_h = 0$ otherwise

then $\chi_h$ is a characteristic function of $D$.

Let $H$ be a nonempty set and $F(H) = [0, 1]^H$ be the set of all fuzzy subset of $H$ and $F^*(H) = F(H) - \{\phi\}$. A fuzzy hyperoperation on $H$ is a mapping $*: H^2 \to F(H)$ and the couple $(H, *)$ is called a partial fuzzy hypergroupoid. If the fuzzy hyperoperation $*$ maps $H^2$ into $F^*(H)$, then $(H, *)$ is called a fuzzy hypergroupoid.
**Definition 2.3.**

1. A **fuzzy semihypergroup** is a fuzzy hypergroupoid \((H, \ast)\) which satisfies the associative law.
2. A **fuzzy quasihypergroup** is a fuzzy hypergroupoid \((H, \ast)\) which satisfies the reproductive law.
3. A **fuzzy hypergroup** is a fuzzy semihypergroup which is also a fuzzy quasihypergroup.
4. A **fuzzy subhypergroup** \((A, \bullet)\) of a fuzzy hypergroup \((B, \cdot)\) is a nonempty subset \(A \subseteq B\) such that for any \(a \in A\), \(a \bullet A = A = A \ast a\).

**Definition 2.4.** A fuzzy hypergroup \((H, \ast)\) is said to be **normal** if it satisfies the following three conditions:

1. \((x \ast x)(x) = 1\) for all \(x \in H\);
2. \(x \ast y = x \ast x \cup y \ast y\) for all \(x, y \in H\);
3. \((x \ast x)(z) \geq (x \ast x)(y) \land (y \ast y)(z)\) for all \(x, y, z \in H\).

Let \(NFHG\) be a category, where objects are normal fuzzy hypergroups and a morphism from \((H, \circ)\) to \((K, \ast)\) is a mapping \(f : H \rightarrow K\) such that \(f(a \circ b) \subseteq f(a) \ast f(b)\).

**Definition 2.5.** A **pseudo subobject classifier** in a category \(E\) is an object \(\Omega\), together with a morphism \(\top : 1 \rightarrow \Omega\) such that for any \((A, \ast) \subseteq (B, \ast)\) and any inclusion \(k : A \rightarrow B\), there is a unique morphism \(\chi_k : B \rightarrow \Omega\) which makes the following diagram a pull-back:

\[
\begin{array}{ccc}
A & \xrightarrow{1} & 1 \\
\downarrow{k} & & \downarrow{\top} \\
B & \xrightarrow{\chi_k} & \Omega
\end{array}
\]

**Definition 2.6.** A **pseudo topos** is a category \(E\) that satisfies the following:

(T1) \(E\) is finitely complete,
(T2) \(E\) has exponentiation,
(T3) \(E\) has a pseudo subobject classifier.

**Definition 2.7.** A category \(E\) is said to have **pseudo power objects** if to each object \(A\), there are objects \(P(A)\) and \(E(A)\), and inclusion \(e : E(A) \rightarrow P(A) \times A\), such that for any object \(B\), and ”relation”,
$r : R \to B \times A$ there is exactly one morphism $f_r : B \to P(A)$ for which there is a pullback of the form

\[
\begin{array}{ccc}
R & \longrightarrow & E(A) \\
\downarrow r & & \downarrow e \\
B \times A & \longrightarrow & P(A) \times A
\end{array}
\]

\section{Pseudo Topos \textit{NFHG} and Pseudo Power Object}

\textbf{Theorem 3.1.} \textit{NFHG} has a pseudo subobject classifier.

\textit{Proof.} Let $\Omega = \{\top, \bot\}$ and $\odot : \Omega \times \Omega \to [0, 1]^\Omega$ defined by

\[
(\top \odot \top)(\top) = 1 = (\top \odot \bot)(\bot),
\]

\[
(\bot \odot \bot)(\top) = 1 = (\bot \odot \bot)(\bot)
\]

\[
(\top \odot \bot) = (\top \odot \top) \cup (\bot \odot \bot).
\]

Then $(\Omega, \odot)$ is a normal fuzzy hypergroup.

For any normal fuzzy subhypergroup $(K, \star) \subseteq (H, \star)$ and inclusion $f : K \to H$ defined by $f(k) = k$ for any $k \in K$, we construct a morphism $\chi_f : H \to \Omega$ defined by

\[
\chi_f(h) = \top \text{ if } x \in K
\]

\[
\chi_f(h) = \bot \text{ otherwise.}
\]

For any $z \in \Omega$, $\chi_f(u \star v)(z) \leq (\chi_f(u) \odot \chi_f(v))(z) = 1$. So $\chi_f(u \star v) \subseteq \chi_f(u) \odot \chi_f(v)$. Thus $\chi_f : H \to \Omega$ is a morphism. For any $h : (M, \oplus) \to (H, \star)$ and $! : (M, \oplus) \to \{\ast\}, \odot$ with $\chi_f \circ h = \top \circ !$, we have that $\chi_f \circ h = \top \circ !$ implies $h(m) \in \text{Im}(f)$. That is, $h(m) = f(k)$ for some $k \in K$. So there exists a morphism $g : (M, \oplus) \to (K, \star)$ such that $g(m) = k$ with $h(m) = f(k)$ for all $m \in M$. Clearly, $f \circ g = h$ and such a morphism is unique.

\[
\begin{array}{ccc}
K & \longrightarrow & \{\ast\} \\
\downarrow f & & \downarrow \top \\
H & \longrightarrow & \Omega \end{array}
\]

\textbf{Corollary 3.2.} \textit{NFHG} is a pseudo topos.
THEOREM 3.3. In category $NFHG$, for each object $(A, \odot)$ there are objects $(P(A), \star)$, $(E(A), \triangle)$ and inclusion $g : (E(A), \triangle) \to (P(A), \star) \times (A, \odot)$ such that for any object $(B, \oplus)$ and relation $(R, \triangledown)$ from $(A, \odot)$ to $(B, \oplus)$, there is exactly one morphism $f_r : (B, \oplus) \to (P(A), \star)$ for which there is a pullback of the form

$$
\begin{array}{c}
(R, \triangledown) \\
\downarrow r
\end{array}
\xrightarrow{\mathcal{T}}
\begin{array}{c}
(E(A), \triangle) \\
\downarrow g
\end{array}

(B, \oplus) \times (A, \odot)
\xrightarrow{f_r \times i_A}
(P(A), \star) \times (A, \odot)
$$

where $((b_1, a_1) \triangledown (b_2, a_2))(r_1, r_2) = ((b_1 \oplus b_1)(r_1) \land (a_1 \odot a_1)(r_2)) \lor ((b_2 \oplus b_2)(r_1) \land (a_2 \odot a_2)(r_2))$ and $r(b, a) = (b, a)$.

Proof. Let $P(A) = (\Omega, \circ)^{(A, \odot)} = \{f : A \to \Omega\}$ where $\star : P(A) \times P(A) \to [0, 1]^{P(A)}$ defined by $(f \star f)(h) = \land(f(x) \circ f(x))h(x)$ and $E(A) = \{< f, a > | f \in P(A), a \in A, f(a) = \top\}$ where $\triangle : E(A) \times E(A) \to [0, 1]^{E(A)}$ defined by $((f, a) \triangle (g, b))(h, c) = ((f \star f)(h) \land (a \odot a)(c)) \lor ((g \star g)(h) \land (b \odot b)(c))$. Then we obtain objects $(P(A), \star)$ and $(E(A), \triangle)$. Consider

$$
\begin{array}{c}
E(A) \\
\downarrow g
\end{array}
\xrightarrow{!}
\begin{array}{c}
\{\star\} \\
\downarrow \top
\end{array}

P(A) \times A
\xrightarrow{\chi_g}
\Omega
$$

Let $\chi_g < f, a > = f(a)$, then $\chi_g$ is a morphism and $\chi_g \circ g = \top \circ !$. By the property of $(P(A), \star)$ and $(E(A), \triangle)$, $\Omega$ is a pseudo subobject classifier of the inclusion $g : E(A) \to P(A) \times A$. So the previous square is a pullback.

Consider

$$
\begin{array}{c}
R \\
\downarrow r
\end{array}
\xrightarrow{\mathcal{T}}
\begin{array}{c}
E(A) \\
\downarrow g
\end{array}
\xrightarrow{!}
\begin{array}{c}
\{\star\} \\
\downarrow \top
\end{array}

B \times A
\xrightarrow{f_r \times i_r}
P(A) \times A
\xrightarrow{\chi_g}
\Omega
$$

Let $f_r : B \to P(A)$ defined by

$(f_r(b))(a) = (\top \circ !) < b, a >$, if $< b, a > \in R$

$(f_r(b))(a) = \bot$, otherwise
Then $f_r : B \to P(A)$ is a morphism. And $\Omega$ is a pseudo subobject classifier of the inclusion $r : R \to B \times A$ with $! : R \to \{\ast\}$. So the outer square is a pullback. By definition of pullback, there is exactly one morphism $\overline{f} : R \to E(A)$ such that $g \circ \overline{f} = (f_r \times i_r) \circ r$. By pullback Lemma, the left square is a pullback. 

**Corollary 3.4.** $NFGH$ has pseudo power objects.

**References**


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