ON THE FIELD EQUATIONS IN $g - ESX_n$

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Abstract. This paper is a direct continuation of [1] and [2]. In this paper we investigate some properties of ES-curvature tensor and contracted ES-curvature tensor of $g - ESX_n$. Also, we study the field equations in the $n$-dimensional ES manifold $g - ESX_n$.

1. Preliminaries

This paper is a direct continuation of our previous paper [1] and [2], which will be denoted by I in the present paper. All considerations in this paper are based on our results and symbolism of I([1], [2], [3], [4], [5], [6], [7], [8], [9]). Whenever necessary, these results will be quoted in the text. In this section, we introduce a brief collection of basic concepts, notations, and results of I, which are frequently used in the present paper.

(a) generalized $n$-dimensional Riemannian manifold $X_n$

Let $X_n$ be a generalized $n$-dimensional Riemannian manifold referred to a real coordinate system $x''$, which obeys the coordinate transformations $x'' \to x'''$ for which

$$\det \left( \frac{\partial x'}{\partial x} \right) \neq 0.$$
In $n - g - UFT$ the manifold $X_n$ is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

\begin{equation}
    g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}.
\end{equation}

where

\begin{equation}
    g = \det(g_{\lambda\mu}) \neq 0, \quad h = \det(h_{\lambda\mu}) \neq 0, \quad k = \det(k_{\lambda\mu}).
\end{equation}

In virtue of (1.3) we may define a unique tensor $h^{\lambda\nu}$ by

\begin{equation}
    h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu\mu.
\end{equation}

which together with $h_{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in $X_n$ in the usual manner. There exists a unique tensor $^*g^{\lambda\nu}$ satisfying

\begin{equation}
    g_{\lambda\mu} ^*g^{\lambda\nu} = g_{\mu\lambda} ^*g^{\nu\lambda} = \delta^\nu\mu.
\end{equation}

It may be also decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

\begin{equation}
    ^*g^{\lambda\nu} = ^*h^{\lambda\nu} + ^*k^{\lambda\nu}.
\end{equation}

The manifold $X_n$ is connected by a general real connection $\Gamma^\nu_{\lambda\mu}$ with the following transformation rule:

\begin{equation}
    \Gamma^\nu_{\lambda\mu\nu'} = \frac{\partial x^{\nu'}}{\partial x^\lambda} \left( \frac{\partial x^\alpha}{\partial x^{\lambda'}} \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x^{\lambda'} \partial x^{\mu'}} \right).
\end{equation}

It may also be decomposed into its symmetric part $\Lambda^\nu_{\lambda\mu}$ and its skew-symmetric part $S^\nu_{\lambda\mu}$, called the torsion tensor of $\Gamma^\nu_{\lambda\mu}$:

\begin{equation}
    \Gamma^\nu_{\lambda\mu} = \Lambda^\nu_{\lambda\mu} + S^\nu_{\lambda\mu}; \quad \Lambda^\nu_{\lambda\mu} = \Gamma^{(\nu}_{(\lambda\mu)}; \quad S^\nu_{\lambda\mu} = \Gamma^{[\nu}_{\lambda\mu]}.
\end{equation}

A connection $\Gamma^\nu_{\lambda\mu}$ is said to be Einstein if it satisfies the following system of Einstein’s equations:

\begin{equation}
    \partial_\omega g_{\lambda\mu} - \Gamma^\alpha_{\omega\lambda} g_{\alpha\mu} - \Gamma^\alpha_{\omega\mu} g_{\alpha\lambda} = 0.
\end{equation}

or equivalently

\begin{equation}
    D_\omega g_{\lambda\mu} = 2 S^\alpha_{\omega\mu} g_{\alpha\lambda}.
\end{equation}

where $D_\omega$ is the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$. In order to obtain $g_{\lambda\mu}$ involved in the solution for $\Gamma^\nu_{\lambda\mu}$ in (1.9), certain conditions are imposed. These conditions may be condensed to

\begin{equation}
    S^\lambda_{\lambda\alpha} = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu} Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0.
\end{equation}
where $Y_\lambda$ is an arbitrary vector, and

\begin{equation}
R_{\mu \nu \lambda}^\rho = 2(\partial_{[\mu} \Gamma_{|\lambda|] \omega} + \Gamma_{\alpha [\mu} \Gamma_{|\lambda|] \omega}),
\end{equation}

If the system (1.10) admits a solution $\Gamma_{\lambda \nu \mu}$, it must be of the form (Hlavatý, 1957)

\begin{equation}
\Gamma_{\lambda \nu \mu} = \left\{ \begin{array}{c}
\nu \\
\lambda \mu
\end{array} \right\} + S_{\lambda \mu} \nu + U_{\lambda \mu} \nu.
\end{equation}

where $U_{\nu \lambda \mu} = 2h^{\nu \alpha} S_{\alpha (\beta k_\mu) \beta}$ and $\left\{ \begin{array}{c}
\nu \\
\lambda \mu
\end{array} \right\}$ are Christoffel symbols defined by $h_{\lambda \mu}$.

### (b) Some notations and results

The following quantities are frequently used in our further considerations:

\begin{equation}
g = \frac{g}{\hbar}, \quad k = \frac{\ell}{\hbar}.
\end{equation}

\begin{equation}
K_p = k_{[\alpha_1} \alpha_2 \cdots \alpha_p] \alpha_p, \quad (p = 0, 1, 2, \cdots).
\end{equation}

\begin{equation}
(0)k_{\lambda \nu} = \delta_{\lambda \nu}, \quad (p)k_{\lambda \nu} = k_{\lambda \alpha}^{(p-1)}k_{\alpha \nu} \quad (p = 1, 2, \cdots).
\end{equation}

In $X_n$ it was proved in [5] that

\begin{equation}
K_0 = 1, \quad K_n = k \text{ if } n \text{ is even}, \quad \text{and } K_p = 0 \text{ if } p \text{ is odd}.
\end{equation}

\begin{equation}
g = \hbar(1 + K_1 + K_2 + \cdots + K_n)
\quad \text{or} \quad g = 1 + K_1 + K_2 + \cdots + K_n.
\end{equation}

\begin{equation}
\sum_{s=0}^{n-s} K_s \ (n-s+p)k_{\lambda \nu} = 0 \quad (p = 0, 1, 2, \cdots).
\end{equation}

We also use the following useful abbreviations for an arbitrary vector $Y$, for $p = 1, 2, 3, \cdots$:

\begin{equation}
(p)Y_\lambda = (p-1)k_\lambda \alpha Y_\alpha.
\end{equation}

\begin{equation}
(p)Y^\nu = (p-1)k^\nu \alpha Y^\alpha.
\end{equation}
(c) \textit{n-dimensional ES manifold} \( ESX_n \)

In this subsection, we display an useful representation of the \( ES \) connection in \( n \)-g-UFT.

**Definition 1.1.** A connection \( \Gamma^\nu_{\lambda \mu} \) is said to be \textit{semi-symmetric} if its torsion tensor \( S^\nu_{\lambda \mu} \) is of the form
\[
S^\nu_{\lambda \mu} = 2\delta^\nu_{[\lambda}X_{\mu]} \tag{1.22}
\]
for an arbitrary non-null vector \( X_\mu \).

A connection which is both semi-symmetric and Einstein is called an \( ES \) connection. An \( n \)-dimensional generalized Riemannian manifold \( X_n \), on which the differential geometric structure is imposed by \( g_{\lambda \mu} \) by means of an \( ES \) connection, is called an \( n \)-dimensional \( ES \) manifold. We denote this manifold by \( g - ESX_n \) in our further considerations.

**Theorem 1.2.** Under the condition (1.22), the system of equations (1.10) is equivalent to
\[
\Gamma^\nu_{\lambda \mu} = \left\{ \begin{array}{c} \nu \\ \lambda \mu \end{array} \right\} + 2k^\nu_{(\lambda}X_{\mu)} + 2\delta^\nu_{[\lambda}X_{\mu]} \tag{1.23}
\]

**Proof.** Substituting (1.22) for \( S^\nu_{\lambda \mu} \) into (1.13), we have the representation (1.23). \( \square \)

In \( g - ESX_n \), the following theorem was proved in [1]:

**Theorem 1.3.** In \( g - ESX_n \), the following relations hold for \( p, q = 1, 2, 3, \cdots \) :
\[
S_\lambda = (1 - n)X_\lambda \tag{1.24}
\]
\[
U_\lambda = \frac{1}{2} \partial_\lambda \ln g \tag{1.25}
\]
\[
(p + 1)S_\lambda = (1 - n)(p)U_\lambda \tag{1.26}
\]
\[
(p)^{(p)}U_{\alpha}^{(q)}X^\alpha = 0 \text{ if } p + q - 1 \text{ is odd} \tag{1.27}
\]
\[
D_\lambda X_\mu = \nabla_\lambda X_\mu \tag{1.28}
\]
\[
D_{[\lambda}X_{\mu]} = \nabla_{[\lambda}X_{\mu]} = \partial_{[\lambda}X_{\mu]} \tag{1.29}
\]
\[
\nabla_{[\lambda}U_{\mu]} = 0, \quad D_{[\lambda}U_{\mu]} = 2U_{[\lambda}X_{\mu]} = 2^{(2)}X_{[\lambda}X_{\mu]} \tag{1.30}
\]
where \( \nabla_\omega \) is the symbolic vector of the covariant derivative with respect to the Christoffel symbols defined by \( h_{\lambda \mu} \).

2. The ES curvature tensor and the contracted ES curvature tensor in \( g - ESX_n \)

This chapter is devoted to the study of the ES curvature tensor and the contracted ES curvature tensors in \( g - ESX_n \) and of some useful identities involving them.

**Theorem 2.1.** In \( g - ESX_n \), the ES curvature tensor \( R_{\omega \mu \lambda}^\nu \) may be given by

\[
R_{\omega \mu \lambda}^\nu = L_{\omega \mu \lambda}^\nu + M_{\omega \mu \lambda}^\nu + N_{\omega \mu \lambda}^\nu.
\]

where

\[
L_{\omega \mu \lambda}^\nu = 2 \left( \partial_{[\mu} \left\{ \frac{\nu}{\omega] \lambda} \right\} + \left\{ \frac{\nu}{\alpha[\mu} \right\} \left\{ \frac{\alpha}{\omega] \lambda} \right\} \right).
\]

\[
M_{\omega \mu \lambda}^\nu = 2 \left( \delta^\nu_{\lambda} \partial_{[\mu} X_{\omega]} + \delta^\nu_{\nu} \nabla_\omega \lambda \right)
+ \nabla_{[\mu} U_{\nu] \lambda} + \nabla_{[\mu} U^{\nu}_{\nu] \lambda}.
\]

\[
N_{\omega \mu \lambda}^\nu = 2 \left( \delta^\nu_{\omega} X_{[\mu} \lambda + \frac{1}{2} X_{\lambda} k_{[\mu} \nu X_{\omega]} \right).
\]

**Proof.** Substitute (1.13) into (1.12) and make use of (2.2) to obtain

\[
R_{\omega \mu \lambda}^\nu = 2 \partial_{[\mu} \left\{ \frac{\nu}{\omega] \lambda} \right\} + X_{\omega] \delta^\nu_{\lambda} - \delta^\nu_{\nu} X_{\lambda} + U^{\nu}_{\nu] \lambda}
+ 2 \left( \frac{\nu}{\alpha[\mu} \right\} + \delta^\nu_{\nu} X_{[\mu} \lambda + U^{\nu}_{\nu] \lambda} + U^{\nu}_{[\mu} \lambda \right)
\times \left( \frac{\alpha}{\omega] \lambda} \right)
+ X_{\omega] \delta^\alpha_{\lambda} - \delta^\alpha_{\nu} X_{\lambda} + U^{\alpha}_{\nu] \lambda}
\]

\[
= L_{\omega \mu \lambda}^\nu + 2 \delta^\nu_{\lambda} \partial_{[\mu} X_{\omega]} + 2 \left( \delta^\nu_{[\mu} \partial_{\omega] \lambda} - \delta^\nu_{\nu} \left\{ \frac{\alpha}{\omega] \lambda} \right\} \right) X_{\alpha}
+ 2 \left( \partial_{[\mu} U^{\nu}_{\nu] \lambda} + \left\{ \frac{\alpha}{\lambda \nu} \right\} U^{\nu}_{\nu] \lambda} + \left\{ \frac{\nu}{\alpha[\mu} \right\} U^{\alpha}_{\nu] \lambda} \right)
+ 2 \left( \delta^\nu_{[\omega} X_{\mu]} \lambda - X_{\alpha} \delta^\nu_{[\mu} U^{\alpha}_{\nu] \lambda} + U^{\nu}_{\nu] \lambda} \right)
\]
In virtue of (1.22), the sum of the second, third and fourth terms on the right-hand side of (2.5) is $M_{\omega\mu\lambda}$. On the other hand, using (1.22), (1.25), and (1.27), we have

\begin{equation}
U^\nu_{\lambda\mu} = 2k(\lambda^\nu X_\mu)
\end{equation}

\begin{equation}
-X_\alpha\delta^\nu_{\mu} U^\alpha_{\omega\lambda} = 0
\end{equation}

\begin{equation}
U^\nu_{\alpha\mu} U^\alpha_{\omega\lambda} = (2) X_\lambda k_{\mu} X_\omega
\end{equation}

Substituting (2.7) and (2.8) into the fifth term of (2.5), we find that it is equal to $N_{\omega\mu\lambda}$. Consequently, our proof of the theorem is completed.

The tensors

\begin{equation}
R_{\mu\lambda} = R_{\alpha\mu\lambda}^\alpha, \quad V_{\omega\mu} = R_{\alpha\mu\omega}^\alpha.
\end{equation}

are called the first and second contracted ES curvature tensors of the ES connection $\Gamma_{\lambda\nu\mu}$, respectively. We see in the following two theorems that they appear as functions of the vectors $X_\lambda, S_\lambda, U_\lambda$, and hence also as functions of $g_{\mu\nu}$ and its first two derivatives in virtue of (1.24), (1.25) and (2.1).

**Theorem 2.2.** The first contracted ES curvature tensor $R_{\mu\lambda}$ in $g - ESX_n$ may be given by

\begin{equation}
R_{\mu\lambda} = L_{\mu\lambda} + 2\partial_{[\mu} X_{\lambda]} + \nabla_\mu T_\lambda - \nabla_\alpha U^\alpha_{\mu\lambda} + (n - 1)X_\mu X_\lambda + U_\mu U_\lambda.
\end{equation}

where

\begin{equation}
L_{\mu\lambda} = L_{\alpha\mu\lambda}^\alpha.
\end{equation}

\begin{equation}
T_\lambda^\nu_S = S_{\lambda\nu}^\nu + U^\nu_{\nu\lambda}, \quad T_\lambda = T_\lambda^\alpha_S = S_\lambda + U_\lambda.
\end{equation}

**Proof.** Putting $\omega = \nu = \alpha$ in (2.1) and making use of (2.11), we have

\begin{equation}
R_{\mu\lambda} = L_{\mu\lambda} + M_{\alpha\mu\lambda}^\alpha + N_{\alpha\mu\lambda}^\alpha.
\end{equation}

In virtue of (1.24) and (1.25), it follows from (2.3) that

\begin{equation}
M_{\alpha\mu\lambda}^\alpha = 2\partial_{[\mu} X_{\lambda]} + (1 - n)\nabla_\mu X_\lambda + \nabla_\mu U_\lambda - \nabla_\alpha U^\alpha_{\mu\lambda} = 2\partial_{[\mu} X_{\lambda]} + \nabla_\mu T_\lambda - \nabla_\alpha U^\alpha_{\mu\lambda}.
\end{equation}
On the other hand, in virtue of (1.25) the relation (2.4) gives
\begin{equation}
N_{\alpha\mu\lambda} = (n - 1)X_{\mu}X_{\lambda} + (2)X_{\mu}^{(2)}X_{\lambda} - (2)X_{\lambda}X_{\mu}^{(2)}; \alpha \nonumber
\end{equation}
\begin{equation}
= (n - 1)X_{\mu}X_{\lambda} + U_{\mu}U_{\lambda}. \nonumber
\end{equation}
Our assertion follows immediately from (2.13), (2.14) and (2.15).

**Theorem 2.3.** The second contracted ES curvature tensor $V_{\omega\mu}$ in $g - ESX_n$ is a curl of the vector $S_{\lambda}$. That is,
\begin{equation}
V_{\omega\mu} = 2\partial_{[\omega}S_{\mu]}. \nonumber
\end{equation}

**Proof.** Putting $\lambda = \nu = \alpha$ in (2.1), we have
\begin{equation}
V_{\omega\mu} = L_{\omega\mu\alpha} + M_{\omega\mu\alpha} + N_{\omega\mu\alpha}. \nonumber
\end{equation}
In virtue of (1.11), (1.24), (1.25) and (1.30), the relations (2.2), (2.3) and (2.4) give
\begin{equation}
L_{\omega\mu\alpha} = N_{\omega\mu\alpha} = 0 \nonumber
\end{equation}
\begin{equation}
M_{\omega\mu\alpha} = 2(1 - n)\partial_{[\omega}X_{\mu]} + 2\nabla_{[\mu}U_{\omega]} = 2(1 - n)\partial_{[\omega}X_{\mu]} = 2\partial_{[\omega}S_{\mu]}
\end{equation}
which together with (2.17) proves our assertion.

**Theorem 2.4.** The tensor $R_{\mu\lambda}$ is symmetric when $n = 3$.

**Proof.** The relation (2.10) may be written as
\begin{equation}
R_{\mu\lambda} = L_{\mu\lambda} + (3 - n)\nabla_{\mu}X_{\lambda} - 2\nabla_{[\mu}X_{\lambda]} + \nabla_{\mu}U_{\lambda} - \nabla_{\alpha}U_{\mu\lambda} + (n - 1)X_{\mu}X_{\lambda} + U_{\mu}U_{\lambda}, \nonumber
\end{equation}
where use has been made of (1.24), (1.29) and (2.12). Hence, in virtue of (1.29) and (1.30) we have $R_{[\mu\lambda]} = 0$ if and only if $(3 - n)\nabla_{[\mu}X_{\lambda]} = (3 - n)\partial_{[\mu}X_{\lambda]} = 0$

**Remark 2.5.** In the proof of the Theorem (2.4), we excluded the case that $\partial_{[\mu}X_{\lambda]} = 0$, because we assumed that $X_{\lambda}$ is not a gradient vector in the definition of semi-symmetric connection in (1.22). In fact, the assumption that $X_{\lambda}$ is not a gradient vector is essential in the discussions of the field equations in $g - ESX_n$.

**Theorem 2.6.** The contracted ES curvature tensors in $g - ESX_n$ are related by
\begin{equation}
2R_{[\mu\lambda]} = 4\partial_{[\mu}X_{\lambda]} + V_{\mu\lambda}. \nonumber
\end{equation}
Proof. In virtue of (1.24), (1.29) and (1.30), the relation (2.19) may be proved from (2.18) as in the following way:

\begin{align*}
2R_{[\mu\lambda]} &= 2(3 - n)\partial_{[\mu}X_{\lambda]} \\
&= 2(1 - n)\partial_{[\mu}X_{\lambda]} + 4\partial_{[\mu}X_{\lambda]} \\
&= 2\partial_{[\mu}S_{\lambda]} + 4\partial_{[\mu}X_{\lambda]} \\
&= V_{\mu\lambda} + 4\partial_{[\mu}X_{\lambda]}.
\end{align*}

3. The field equations in \(g - ENX_n\)

By field equations we mean a set of partial equations for \(g_{\lambda\mu}\). In the present section we are concerned with the geometry of field equations in \(g - ENX_n\) and not with their physical applications. We saw in the previous section that ES curvature tensor \(R_{\omega\mu\lambda}^\nu\) together with its contracted curvature tensor \(R_{\alpha\lambda}\) appear as a function of \(g_{\lambda\mu}\). In order to obtain the tensor \(g_{\lambda\mu}\) with which we started in dealing with (1.9), (1.10) and (1.11), we suggest the following conditions for it in terms of \(R_{\mu\lambda}\):

\begin{align*}
    R_{[\mu\lambda]} &= \partial_{[\mu}X_{\lambda]} \\
    R_{(\mu\lambda)} &= 0
\end{align*}

where \(X_{\lambda}\) is an arbitrary vector. The conditions (3.1) and (3.2) represent a system of \(n^2\) differential equations of the second order for \(g_{\lambda\mu}\).

The unified field theory in the \(n\)-dimensional ES manifold \(ENX_n\) is governed by the following set of equations: \(n^3\) equations (1.10) under the conditions (1.22), which determine the unique ES connection \(\Gamma_{\lambda\mu}^\nu\), and \(n^2\) field equations (3.1) and (3.2) for \(n^2\) unknowns \(g_{\lambda\mu}\). In Theorem (3.3), it states that the unknowns \(Y_{\lambda}\) are uniquely determined in \(ENX_n\). The conditions (3.1) and (3.2) are of a purely geometrical nature and physical interpretation is not involved in them a priori. Einstein suggested several different sets of field equations in his four-dimensional unified field theory. It would seem natural to follow the analogy of Einstein’s field equations (1.11) in our manifold \(ENX_n\), too. However, the restriction \(S_{\lambda} = 0\) is too strong in our unified field theory in the ES manifold \(ENX_n\),
since this condition implies $X_\lambda = 0$ and hence $\Gamma_{\lambda \mu \nu} = \left\{ \begin{array}{c} \nu \\ \lambda \mu \end{array} \right\}$ in virtue of (1.23) and (1.24). Therefore, we shall not adopt (1.11) as a starting point, exclude the condition $S_\lambda = 0$, and impose the field equations in $ESX_n$ as given in (3.1) and (3.2).

**Remark 3.1.** In our further considerations we restrict ourselves to the conditions

(3.3) $X_\lambda \neq 0$ and $X_\lambda$ not a gradient vector

This restriction is quite natural in view of (3.1) and (3.2) and Remark (3.1). The first consequence of (3.3) is the following theorem.

**Theorem 3.2.** In $g - ESX_n$ we have

(3.4) $U^\nu_{\lambda \mu} \neq 0$

*Proof.* Assume that $U^\nu_{\lambda \mu} \neq 0$. Then (1.22) implies that

(3.5) $k_{\lambda \nu}X_\mu + k_{\mu \nu}X_\lambda = 0$ for every $\lambda$, $\mu$, $\nu$.

In virtue of the condition (3.3), there exists at least one fixed index $\delta$ such that $X_\delta \neq 0$. Hence

(3.6) $k_{\lambda \nu}X_\delta + k_{\delta \nu}X_\lambda = 0$ for every $\lambda$, $\nu$.

Putting $\lambda = \delta$ in (3.6), we have $k_{\delta \nu} = 0$ for every $\nu$. If $\lambda \neq \delta$, then $k_{\lambda \nu} = 0$ for every $\nu$, since $k_{\delta \nu} = 0$. Hence we have

(3.7) $k_{\lambda \nu} = 0$ for every $\lambda$, $\nu$

which is a contradiction to the non-symmetry of $g_{\lambda \mu}$. \hfill \Box

**Theorem 3.3.** In $g - ESX_n$, the field equation (3.1) is satisfied by a unique vector $Y_\lambda$ given by

(3.8) $Y_\lambda = (3 - n)X_\lambda$

when $n \neq 3$

*Proof.* In virtue of (2.18), we have

$$R_{[\mu \lambda]} = (3 - n)\partial_{[\mu}X_{\lambda]}$$

from which (3.8) follows. \hfill \Box
Theorem 3.4. In $g - ESX_n$, the field equation (3.2) is equivalent to

\[(3.9) \quad L_{\mu\lambda} + \nabla_{(\mu}T_{\lambda)} - \nabla_{\alpha}U^{\alpha}_{\mu\lambda} + (n - 1)X_{\mu}X_{\lambda} + U_{\mu}U_{\lambda} = 0\]

Proof. (3.9) is a immediate consequence of (2.10) and (3.2).

References

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