ON SEMI-IFP RINGS

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Abstract. We in this note introduce the concept of semi-IFP rings which is a generalization of IFP rings. We study the basic structure of semi-IFP rings, and construct suitable examples to the situations raised naturally in the process. We also show that the semi-IFP does not go up to polynomial rings.

1. Semi-IFP rings

Throughout this paper all rings are associative with identity unless otherwise stated. Let $R$ be a ring. $N_s(R)$, $N^*(R)$, and $N(R)$ denote the lower nilradical (i.e., the prime radical), the upper nilradical (i.e., the sum of nil ideals), and the set of all nilpotent elements in $R$, respectively. Note that $N_s(R) \subseteq N^*(R) \subseteq N(R)$. The polynomial ring with an indeterminate $x$ over a ring $R$ is denoted by $R[x]$. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo $n$. Denote the $n$ by $n$ (resp., upper triangular) matrix ring over $R$ by $\text{Mat}_n(R)$ (resp., $U_n(R)$). Use $e_{ij}$ for the matrix with $(i,j)$-entry 1 and elsewhere 0. $\mathbb{Z}$ (resp., $\mathbb{Z}_n$) denotes the ring of integers (resp., modulo $n$).

It is well-known that the set of all nilpotent elements in a commutative ring coincides with the prime radical. This fact is also possessed by certain sorts of noncommutative rings, and such rings are called $2$-primal
by Birkenmeier et al. [3]. Shin [13, Proposition 1.11] proved that given a ring $R$, $N_*(R) = N(R)$ if and only if every minimal prime ideal $P$ of $R$ is completely prime (i.e., $R/P$ is a domain).

A well-known property between “commutative” and “2-primal” is the insertion-of-factors-property (simply, IFP), introduced by Bell [2]. A right (or left) ideal $I$ of a ring $R$ is said to have the IFP if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$, and we will call a ring IFP if the zero ideal has the IFP. Narbonne [12] and Shin [13] used the terms semicommutative and SI for the IFP, respectively; while, IFP rings were also studied under the name zero insertive by Habeb [7]. IFP rings are 2-primal [13, Theorem 1.5].

A ring is called reduced if it has no nonzero nilpotent elements. It is trivial to check that reduced rings are IFP, whence the IFP condition is also between “reduced” and “2-primal”. It is trivial that subrings of IFP rings are also IFP, so we use this fact freely in this note. A ring is called Abelian if every idempotent is central. IFP rings are Abelian by a simple computation.

Following the literature, the index (of nilpotency) of a nilpotent element $a$ in a ring $R$ is the least positive integer $n$ such that $a^n = 0$, write $i(a)$ for $n$; the index (of nilpotency) of a subset $S$ of $R$ is the supremum of the indices (of nilpotency) of all nilpotent elements in $S$, write $i(S)$; and if such a supremum is finite, then $S$ is said to be of bounded index (of nilpotency).

We now introduce the concept of semi-IFP rings as a generalization of IFP, and study relationships between semi-IFP rings and near related ring theoretic properties.

**Definition 1.1.** A ring $R$ is called semi-IFP if $a^2 = 0$ for $a \in R$ implies $aRa = 0$.

It is obvious that a ring $R$ is semi-IFP if and only if $a^2 = 0$ for $a \in R$ implies $(RaR)^2 = 0$. Clearly, the class of semi-IFP rings is closed under subrings. We will use this fact freely.

Following [10] a ring $R$ is said to be near-IFP if $\sum_{i=0}^n Ra_i R$ contains a nonzero nilpotent ideal whenever a nonzero polynomial $\sum_{i=0}^n a_i x^i$ over a ring $R$ is nilpotent.

$U_2(\mathbb{Z}_4)$ is near-IFP by [10, Proposition 1.10(1)], but $U_2(\mathbb{Z}_4)$ is not semi-IFP by Example 1.6 to follow. Let $R$ be a semi-IFP ring and $R$ is of
bounded index 2 (of nilpotency). Then \( R \) is near-IFP by [10, Proposition 1.2]. However we do not know whether semi-IFP rings are near-IFP when given rings are of bounded index (of nilpotency) \( \geq 3 \).

We see in the following that \( \text{Mat}_n(R) \) cannot be semi-IFP for any ring \( R \) and \( n \geq 2 \), and that the class of semi-IFP rings is not closed under homomorphic images.

**Example 1.2.** Consider the ring \( \text{Mat}_2(R) \) over any ring \( R \). For \( e_{12} \in \text{Mat}_2(R) \), we have \( e_{12}^2 = 0 \) but

\[
0 \neq e_{12} = e_{12}e_{21} \in e_{12}\text{Mat}_2(R)e_{12},
\]

showing that \( \text{Mat}_2(R) \) is not semi-IFP. Consequently, \( \text{Mat}_n(R) \) for \( n \geq 2 \) cannot be semi-IFP.

This result also illuminates that the class of semi-IFP rings is not closed under homomorphic images. In fact, let \( R \) be the ring of quaternions with integer coefficients. Then \( R \) is a domain, and so semi-IFP. However, for any odd prime integer \( q \), the ring \( R/qR \cong \text{Mat}_n(Z_q) \), by the argument in [6, Exercise 2A]. Since \( \text{Mat}_n(Z_q) \) is not semi-IFP by above, and thus \( R/qR \) cannot be semi-IFP.

However, we have the following.

**Proposition 1.3.** Let \( R \) be a reduced ring. Then \( U_n(R) \) is a semi-IFP ring for \( n = 2, 3 \).

**Proof.** It is enough to show that \( U_3(R) \) over a reduced ring \( R \) is semi-IFP. Let \( M^2 = 0 \) for

\[
M = \begin{pmatrix}
a & b & c \\
0 & d & e \\
0 & 0 & f
\end{pmatrix} \in U_3(R).
\]

Then \( a = d = f = 0 \) and \( be = 0 \). Here \( be = 0 \) implies \( bRe = 0 \) since reduced rings are IFP. So for any

\[
\begin{pmatrix}
x & y & z \\
0 & u & v \\
0 & 0 & w
\end{pmatrix} \in U_3(R),
\]

we have

\[
\begin{pmatrix}
0 & b & c \\
0 & 0 & e \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
x & y & z \\
0 & u & v \\
0 & 0 & w
\end{pmatrix} \begin{pmatrix}
0 & b & c \\
0 & 0 & e \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & bue \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = 0,
\]

On semi-IFP rings 39
entailing that $MU_3(R)M = 0$. Therefore $U_3(R)$ is semi-IFP.

Recall that for a ring $R$ and an $(R, R)$-bimodule $M$, the trivial extension of $R$ by $M$ is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

By Proposition 1.3, we have the following since $T(R, R)$ is a subring of $U_2(R)$.

**Corollary 1.4.** If $R$ is a reduced ring, then $T(R, R)$ is a semi-IFP ring.

Based on Proposition 1.3, one may suspect that $U_n(R)$ over a reduced ring $R$ may be also a semi-IFP ring for $n \geq 4$. But the following example erases the possibility.

**Example 1.5.** Let $R$ be any ring and consider $U_4(R)$. Let

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in U_4(R).$$

Then $M^2 = 0$, but

$$0 \neq \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = M \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} M \in MU_4(R)M,$$

showing that $U_4(R)$ is not semi-IFP. Therefore $U_n(R)$ cannot be semi-IFP for $n \geq 5$.

$U_n(R)$ is 2-primal over a 2-primal ring $R$ by [3]. So Example 1.5 shows that 2-primal rings need not be semi-IFP.

IFP rings are clearly semi-IFP but the converse need not hold. For example, the ring $U_2(R)$ is non-Abelian over any ring $R$, so $U_2(R)$ cannot be IFP. But $U_2(A)$ is semi-IFP over a reduced ring $A$ by Proposition 1.3, and thus this also says that semi-IFP rings need not be Abelian.

The condition “$A$ is a reduced ring” in Proposition 1.3 cannot be weakened by the condition “$A$ is a semi-IFP ring” by the following.
Example 1.6. Consider the ring $U_2(\mathbb{Z}_4)$. Note that $\mathbb{Z}_4$ is a semi-IFP ring. For

$$M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \in U_2(\mathbb{Z}_4),$$

$M^2 = 0$, but

$$0 \neq \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = M \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M \in MU_2(\mathbb{Z}_4).$$

So $U_2(\mathbb{Z}_4)$ is not semi-IFP.

Let $R$ be a ring and $I$ be a nilpotent ideal of $R$ with $I^3 = 0$. Then $I$ is a semi-IFP ring without identity. So it is natural to ask whether $R$ is a semi-IFP ring if both $R/I$ and $I$ are semi-IFP rings. However we have a negative answer to this situation by Example 1.6. Indeed, let

$$I = \begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}.$$ 

Then $I$ is a semi-IFP ring with $I^2 = 0$, and $R/I$ is a commutative ring. But $R$ is not semi-IFP.

However if we take another condition “$I$ is reduced” then we can have an affirmative answer as follows. The following is a similar result to [9, Theorem 6] for IFP rings.

**Theorem 1.7.** For a ring $R$ suppose that $R/I$ is a semi-IFP ring for a proper ideal $I$ of $R$. If $I$ is a reduced ring without identity then $R$ is semi-IFP.

**Proof.** Let $a^2 = 0$ for $a \in R$. Then $aRa \subseteq I$ since $R/I$ is semi-IFP. Moreover $a^2 = 0$ implies $(aRa)^2 = 0$. But since $I$ is reduced, $aRa = 0$. Therefore $R$ is semi-IFP.

We let $N_0(R)$ be the Wedderburn radical (i.e., the sum of all nilpotent ideals) of a ring $R$. Note $N_0(R) \subseteq N_s(R)$.

**Proposition 1.8.** Let $R$ be a semi-IFP ring with $i(R) = 2$.

1. $N_0(R) = N_s(R) = N^s(R) = N(R)$.

2. If $f(x)^2 = 0$ for $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$, then $(R[x]f(x)R[x])^{n+2} = 0$.

**Proof.** (1) Let $R$ be a semi-IFP ring with $i(R) = 2$. Take $a \in N(R)$. Then $a^2 = 0$, and since $R$ is semi-IFP we have $(RaR)^2 = 0$. This implies $a \in N_0(R)$.
(2) Let \( f(x)^2 = 0 \) for \( f(x) = \sum_{i=0}^{n} a_i x^i \in R[x] \). By (1), \( N_0(R) = N(R) \) and so \( R/N_0(R) \) is a reduced ring. It then follows from \( f(x)^2 = 0 \) that \( a_i \in N_0(R) \) for all \( i \). Since \( R \) is semi-IFP, we get \( (Ra_iR)^2 = 0 \). Consider any sum-factor of any coefficient of polynomials in \( (R[x]f(x)R[x])^{n+2} \), \( c \) say. Then two or more \( a_j \)'s occur in \( c \) for some \( j \), so \( c \) is contained in \( (Ra_jR)^2 \). Thus \( c = 0 \) and this yields \( (R[x]f(x)R[x])^{n+2} = 0 \).

It is well-known that a ring \( R \) is reduced if and only if \( R \) is 2-primal and semiprime. Moreover, we have the following.

**Proposition 1.9.** For a semiprime ring \( R \), then the following conditions are equivalent:

1. \( R \) is reduced;
2. \( R \) is IFP;
3. \( R \) is 2-primal;
4. \( R \) is semi-IFP;
5. \( R \) is near-IFP.

**Proof.** It suffices to show (4) \( \Rightarrow \) (1) and (5) \( \Rightarrow \) (1). Assume that the condition (4) holds. If \( a^2 = 0 \) for \( a \in R \), then \( aRa = 0 \) by assumption. Since \( R \) is semiprime, \( a = 0 \) and so \( R \) is reduced.

Let \( R \) be a near-IFP and assume that \( a^2 = 0 \) for \( a \in R \). Here assume \( a \neq 0 \). Then by [10, Proposition 1.2], \( RaR(\neq 0) \) contains a nonzero nilpotent ideal of \( R \), \( I \) say. But \( R \) is semiprime, and so \( I = 0 \), a contradiction. so \( a \) must be zero, entailing that \( R \) is reduced. \( \Box \)

Following [5], a ring \( R \) is called (von Neumann) regular if for each \( a \in R \) there exists \( b \in R \) such that \( a = aba \). Regular rings are semiprimitive (hence semiprime) by [5]. So we get the following from Proposition 1.9.

**Corollary 1.10.** For a regular ring \( R \), then the following conditions are equivalent:

1. \( R \) is reduced;
2. \( R \) is IFP;
3. \( R \) is 2-primal;
4. \( R \) is semi-IFP;
5. \( R \) is near-IFP.

A ring \( R \) is called right Ore if for \( a, b \in R \) with \( b \) regular there exist \( a_1, b_1 \in R \) with \( b_1 \) regular such that \( ab_1 = ba_1 \). It is a well-known fact that \( R \) is a right Ore ring if and only if there exists the classical right quotient ring of \( R \), and that \( R \) is a semiprime right Goldie ring if and only
if there exists the classical right quotient ring of $R$ which is semisimple Artinian.

Combing this fact with Proposition 1.9, we have the following which is an extension of [9, Corollary 13].

**Proposition 1.11.** Let $R$ be a semiprime right Goldie ring and $Q$ be the classical right quotient ring of $R$. Then the following conditions are equivalent:

1. $R$ is a reduced ring;
2. $R$ is an IFP ring;
3. $R$ is a semi-IFP ring;
4. $R$ is near-IFP;
5. $Q$ is a reduced ring;
6. $Q$ is an IFP ring;
7. $Q$ is a semi-IFP ring;
8. $Q$ is a finite direct product of division rings.

**Proof.** By Proposition 1.9 and the proof of [9, Corollary 13]. \qed

2. Polynomial rings over semi-IFP rings

Concerning polynomial rings over reduced rings and 2-primal rings, we have the following useful facts:

1. A ring $R$ is reduced if and only if $R[x]$ is reduced obviously.
2. A ring $R$ is 2-primal if and only if $R[x]$ is 2-primal by [3].

Based on these results one may naturally conjecture that a ring $R$ is semi-IFP if and only if $R[x]$ is semi-IFP. However the following example erases the possibility.

**Example 2.1.** The construction and computation are similar to [9, Example 2] for IFP rings. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, c \rangle$ be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, c$ over $\mathbb{Z}_2$. Let $B$ be the set of all polynomials in $A$ with zero constant term. Let $I$ be the ideal of $A$ generated by

\[ a_0a_0, a_1a_2 + a_2a_1, a_0a_1 + a_1a_0, a_0a_2 + a_1a_1 + a_2a_0, a_2a_2, \]

and

\[ a_0ra_0, a_2ra_2, (a_0 + a_1 + a_2)r(a_0 + a_1 + a_2), r_1r_2r_3r_4 \]
for \( r, r_1, r_2, r_3, r_4 \in B \). Then clearly \( B^4 \subseteq I \). Set \( R = A/I \). Notice that 
\[
(a_0 + a_1x + a_2x^2)(a_0 + a_1x + a_2x^2) \in I[x], \quad \text{but} \quad (a_0 + a_1x + a_2x^2)c(a_0 + a_1x + a_2x^2) \notin I[x] \text{ because } a_0c_1 + a_1c_0 a_2 \notin I; \quad \text{hence } R[x] \text{ is not semi-IFP.}
\]

Next we claim that \( R \) is semi-IFP. Each product of indeterminates \( a_0, a_1, a_2, c \) is called a monomial and we say that \( \alpha \) is a monomial of degree \( n \) if it is a product of exactly \( n \) generators. Let \( H_n \) be the set of all linear combinations of monomials of degree \( n \) over \( \mathbb{Z}_2 \). Observe that
\( H_n \) is finite for any \( n \) and that the ideal \( I \) of \( R \) is homogeneous (i.e., if \( \sum_{i=1}^n r_i \in I \) with \( r_i \in H_i \) then every \( r_i \) is in \( I \)).

**Claim 1.** If \( f_1^2 \in I \) with \( f_1 \in H_1 \) then \( f_1r_1f_1 \in I \) for any \( r \in A \).

**Proof.** By the definition of \( I \) we obtain the following cases:
\[
(f_1 = a_0), (f_1 = a_2), \text{ or } (f_1 = a_0 + a_1 + a_2).
\]
So we complete the proof, using the definition of \( I \) again. \(\square\)

**Claim 2.** If \( f^2 \in I \) with \( f \in B \) then \( frf \in I \) for all \( r \in B \).

**Proof.** Observe that \( f = f_1 + f_2 + f_3 + f_4 \) and \( r = r_1 + r_2 + r_3 + r_4 \) for some \( f_1, r_1 \in H_1, f_2, r_2 \in H_2, f_3, r_3 \in H_3 \) and some \( f_4, r_4 \in I \). Note that \( H_i \subseteq I \) for \( i \geq 4 \). So \( frf = f_1r_1f_1 + h \) for some \( h \in I \). \( f^2 \in I \) implies \( f_1f_1 \in I \) since \( I \) is homogeneous; hence \( f_1r_1f_1 \in I \) by Claim 1. Consequently \( frf \in I \). \(\square\)

To see that \( R \) is semi-IFP, it suffices to show that \( yry \in I \) for all \( r \in A \) if \( y^2 \in I \) with \( y \in A \). By help of Claim 2, we can obtain the following computations. First write \( y = \alpha + z \) for some \( \alpha \in \mathbb{Z}_2 \) and some \( z \in B \). So \( \alpha^2 + \alpha z + z \alpha + z^2 = y^2 \in I \); hence \( \alpha = 0 \). Then \( z^2 \in I \); hence \( yry = zrz \in I \) for all \( r \in A \). Therefore \( R \) is a semi-IFP ring.

**Proposition 2.2.** Suppose that a ring \( R \) is semiprime. Then the following conditions are equivalent:

1. \( R \) is semi-IFP;
2. \( R[x] \) is semi-IFP.

**Proof.** If \( R \) is semiprime and semi-IFP then \( R \) is reduced by Proposition 1.9, entailing that \( R[x] \) is reduced. \(\square\)

Given a ring \( R \), an endomorphism \( \sigma \) of \( R \), and a \( \sigma \)-derivation \( \delta \) of \( R \), the *Ore extension* \( R[x; \sigma, \delta] \) of \( R \) is the ring obtained by giving the polynomial ring \( R[x] \) with the new multiplication \( xr = \sigma(r)x + \delta(r) \) for
all \( r \in R \). If \( \delta = 0 \) then we write \( R[x; \sigma] \) and call it a skew polynomial ring. If \( \sigma = 1 \) then we write \( R[x; \delta] \) and call it a differential polynomial ring. It is also natural to ask whether the class of semi-IFP rings is closed under these two kinds of extensions. But the following provides negative answers.

**Example 2.3.** There exists a semi-IFP ring over which the skew polynomial ring is not semi-IFP. The argument is essentially due to [4, Example 3.1(1)]. For a division ring \( D \) let \( R = D \oplus D \), then \( R \) is semi-IFP obviously. Define \( \sigma : R \to R \) by \( \sigma(s, t) = (t, s) \). Then \( \sigma \) is an automorphism of \( R \). Let \( S = R[x; \sigma] \) be the skew polynomial ring over \( R \) by \( \sigma \). We claim that \( S \) is semiprime. Let \( I \) be a nonzero ideal of \( S \). Then we pick a nonzero \( f(x) \) in \( I \) which is of the smallest degree in \( I \). Say \( f(x) = a + bx + \cdots + cx^n \) with \( a, b, \ldots, c \in R \) and \( c \neq 0 \). If \( n \) is even, then \( f(x)^2 = a^2 + \cdots + c^n(c)x^{2n} = a^2 + \cdots + c^2x^{2n} \in I^2 \subseteq I \) is nonzero because \( c \) is nonzero and \( \sigma \) is of order 2. Next if \( n \) is odd then \( f(x)x = ax + bx^2 + \cdots + cx^{n+1} \in I \); hence \( [f(x)x]^2 \in I^2 \subseteq I \) is also nonzero by the same computation. Thus \( I^2 \) is nonzero and so \( S \) is semiprime. But \( N(S) \neq 0 \) as can be seen by \((1, 0)x)((1, 0)x) = 0\); hence \( S \) is not reduced. By Proposition 1.9, \( S \) is not semi-IFP.

**Example 2.4.** There exists a semi-IFP ring over which the differential polynomial ring is not semi-IFP. The argument is essentially due to [1, Example 11], [6, Proposition 1.14], and [8, Example 2.1]. Let \( R = F[x]/(x^2) \) and define \( \delta : R \to R \) by \( \delta(x + (x^2)) = 1 + (x^2) \), where \( F \) is a field of characteristic 2 and \( (x^2) = F[x]x^2 \). Then \( R \) is semi-IFP since \( R \) is commutative and \( \frac{R}{F[x]x} \cong F \). Next let \( S = R[x; \delta] \). Then \( [x + (x^2)]^2 = 0 \) but \( [x + (x^2)]S[x + (x^2)] \neq 0 \) so \( R \) is not semi-IFP.

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