STRUCTURAL AND SPECTRAL PROPERTIES OF
k-QUASI-*-PARANORMAL OPERATORS

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Abstract. For a positive integer \( k \), an operator \( T \) is said to be \( k \)-quasi-*-paranormal if \( ||T^{k+2}x||/||T^k x|| \geq ||T^*T^k x||^2 \) for all \( x \in H \), which is a generalization of *-paranormal operator. In this paper, we give a necessary and sufficient condition for \( T \) to be a \( k \)-quasi-*-paranormal operator. We also prove that the spectrum is continuous on the class of all \( k \)-quasi-*-paranormal operators.

1. Introduction

Let \( B(H) \) denote the \( C^* \)-algebra of all bounded linear operators on an infinite dimensional separable Hilbert space \( H \). In paper [10] authors introduced the class of \( k \)-quasi-*-paranormal operators defined as follows:

**Definition 1.1.** \( T \) is a \( k \)-quasi-*-paranormal operator if
\[
||T^{k+2}x||/||T^k x|| \geq ||T^*T^k x||^2
\]
for every \( x \in H \), where \( k \) is a natural number.
A \( k \)-quasi-\(*\)-paranormal operator for a positive integer \( k \) is an extension of \(*\)-paranormal operator, i.e., \( ||T^2x|| \geq ||T^*x||^2 \) for unit vector \( x \). A 1-quasi-\(*\)-paranormal operator is called a quasi-\(*\)-paranormal operator and it is normaloid [10], i.e., \( ||T^n|| = ||T||^n \) for \( n \in \mathbb{N} \) (equivalently, \( ||T|| = r(T) \), the spectral radius of \( T \)). *-paranormal operator and quasi-*-paranormal operator have been studied by many authors and it is known that they have many interesting properties similar to those of hyponormal operators (see [5, 9, 11, 14]).

It is clear that

\[ *\text{-paranormal} \Rightarrow \text{quasi-} *\text{-paranormal} \Rightarrow \text{normaloid} \]

and

\[ \text{quasi-} *\text{-paranormal} \Rightarrow k\text{-quasi-} *\text{-paranormal} \]

\[ \Rightarrow (k+1)\text{-quasi-} *\text{-paranormal}. \]

In [14], the authors give an example to show that a quasi-*-paranormal operator need not be a *-paranormal operator.

**Example 1.2.** Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) be operators on \( \mathbb{R}^2 \), and let \( H_n = \mathbb{R}^2 \) for all positive integers \( n \). Consider the operator \( T_{A,B} \) on \( \bigoplus_{n=1}^{+\infty} H_n \) defined by

\[
T_{A,B} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
A & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & B & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & B & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & B & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Then \( T_{A,B} \) is a quasi-*-paranormal operator, but not a *-paranormal operator.

We give the following example to show that there also exists a \((k+1)\)-quasi-*-paranormal operator, but not a \( k \)-quasi-*-paranormal operator.

**Example 1.3.** Given a bounded sequence of positive numbers \( \alpha : \alpha_1, \alpha_2, \alpha_3, \ldots \) (called weights), the unilateral weighted shift \( W_\alpha \) associated with \( \alpha \) is the operator on \( l_2 \) defined by \( W_\alpha e_n = \alpha_n e_{n+1} \) for all \( n \geq 1 \), where \( \{e_n\}_{n=1}^{\infty} \) is the canonical orthogonal basis for \( l_2 \). Straightforward
calculations show that $W_\alpha$ is a $k$-quasi-$*$-paranormal operator if and only if

$$W_\alpha = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_1 & 0 & 0 & 0 & 0 & \cdots \\
0 & \alpha_2 & 0 & 0 & 0 & \cdots \\
0 & 0 & \alpha_3 & 0 & 0 & \cdots \\
0 & 0 & 0 & \alpha_4 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix},$$

where

$$\alpha_{i+1}\alpha_{i+2} \geq \alpha_i^2 \ (i = k, k+1, k+2, \cdots).$$

So, if $\alpha_{k+1} \leq \alpha_{k+2} \leq \alpha_{k+3} \leq \cdots$ and $\alpha_k > \alpha_{k+2}$, then $W_\alpha$ is a $(k+1)$-quasi-$*$-paranormal operator, but not a $k$-quasi-$*$-paranormal operator.

Now it is natural to ask whether $k$-quasi-$*$-paranormal operators are normaloid or not. For $k > 1$, an answer has been given: there exists a nilpotent operator which is a $k$-quasi-$*$-paranormal operator. But it need not be normaloid.

In section 2, we give a necessary and sufficient condition for $T$ to be a $k$-quasi-$*$-paranormal operator. In section 3, we prove that the spectrum is continuous on the class of all $k$-quasi-$*$-paranormal operators.

2. $k$-quasi-$*$-paranormal operators

In the sequel, we shall write $N(T)$ and $R(T)$ for the null space and range space of $T$, respectively.

**Lemma 2.1.** [10] $T$ is a $k$-quasi-$*$-paranormal operator $\iff T^*(T^*T^2 - 2\lambda TT^* + \lambda^2)T_3^k \geq 0$ for all $\lambda > 0$.

**Theorem 2.2.** If $T$ does not have a dense range, then the following statements are equivalent:

1. $T$ is a $k$-quasi-$*$-paranormal operator;

2. $T = \begin{pmatrix}
T_1 & T_2 \\
0 & T_3
\end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^*k)$, where $T_1^2T_2^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2 \geq 0$ for all $\lambda > 0$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$. 

Proof. \((1) \Rightarrow (2)\) Consider the matrix representation of \(T\) with respect to the decomposition \(H = \overline{R(T^k)} \oplus N(T^*k)\):
\[
T = \begin{pmatrix}
T_1 & T_2 \\
0 & T_3
\end{pmatrix}.
\]
Let \(P\) be the projection onto \(\overline{R(T^k)}\). Since \(T\) is a \(k\)-quasi-\(*\)-paranormal operator, we have
\[
P(T^*2T^2 - 2\lambda TT^* + \lambda^2)P \geq 0 \text{ for all } \lambda > 0.
\]
Therefore
\[
T^*2T^2 - 2\lambda(T_1T^*_1 + T_2T^*_2) + \lambda^2 \geq 0 \text{ for all } \lambda > 0.
\]
On the other hand, for any \(x = (x_1, x_2) \in H\), we have
\[
(T^*_3 x_2, x_2) = (T^k(I - P)x, (I - P)x) = ((I - P)x, T^*k(I - P)x) = 0,
\]
which implies \(T^*_3 = 0\).
Since \(\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)\), where \(M\) is the union of the holes in \(\sigma(T)\) which happen to be subset of \(\sigma(T_1) \cap \sigma(T_3)\) by Corollary 7 of [8], and \(\sigma(T_1) \cap \sigma(T_3)\) has no interior point and \(T_3\) is nilpotent, we have
\[
\sigma(T) = \sigma(T_1) \cup \{0\}.
\]
\((2) \Rightarrow (1)\) Suppose that \(T = \begin{pmatrix}
T_1 & T_2 \\
0 & T_3
\end{pmatrix}\) on \(H = \overline{R(T^k)} \oplus N(T^*k)\), where \(T^*2T^2 - 2\lambda(T_1T^*_1 + T_2T^*_2) + \lambda^2 \geq 0 \text{ for all } \lambda > 0\) and \(T^*_3 = 0\). Since
\[
T^k = \begin{pmatrix}
T_1^k & \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j} \\
0 & 0
\end{pmatrix},
\]
we have
\[
T^kT^*k = \begin{pmatrix}
T_1^kT^*_1 + \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j} \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}^* & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
A & 0 \\
0 & 0
\end{pmatrix}
\]
where \(A = A^* = T_1^kT^*_1 + \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j} \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}^*\). Hence, for all \(\lambda > 0\),
\[
T^kT^*k(T^*2T^2 - 2\lambda TT^* + \lambda^2)T^kT^*k
\]
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\[
\begin{pmatrix}
  A(T_1^*T_1^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2)A & 0 \\
  0 & 0
\end{pmatrix} \geq 0.
\]

It follows that $T^k(T^{*2}T^{2} - 2\lambda TT^{*} + \lambda^2)T^k \geq 0$ on $H = \overline{R(T^k)} \oplus N(T^k)$. Thus $T$ is a $k$-quasi-∗-paranormal operator. \hfill □

**Corollary 2.3.** [10] Let $T$ be a $k$-quasi-∗-paranormal operator, the range of $T^k$ be not dense and

\[
T = \begin{pmatrix}
  T_1 & T_2 \\
  0 & T_3
\end{pmatrix}
\text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}).
\]

Then $T_1$ is a ∗-paranormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

**Corollary 2.4.** [11] If $T$ is a quasi-∗-paranormal operator and $R(T)$ is not dense, then $T$ has the following matrix representation:

\[
T = \begin{pmatrix}
  T_1 & T_2 \\
  0 & 0
\end{pmatrix}
\text{ on } H = \overline{R(T)} \oplus N(T^{*})
\]

where $T_1$ is a ∗-paranormal operator on $\overline{R(T)}$.

**Corollary 2.5.** Let $T$ be a $k$-quasi-∗-paranormal operator and $0 \neq \mu \in \sigma_p(T)$. If $T$ is of the form $T = \begin{pmatrix}
  \mu & B \\
  0 & C
\end{pmatrix}$ on $H = N(T - \mu) \oplus N(T - \mu)^\perp$, then $B = 0.$

**Proof.** Let $P$ be the projection onto $N(T - \mu)$ and $x \in N(T - \mu)$. Since $T$ is a $k$-quasi-∗-paranormal operator and $x = \frac{1}{\mu^k}T^kx \in R(T^k)$, we have

\[
P(T^{*2}T^{2} - 2\lambda TT^{*} + \lambda^2)P \geq 0 \text{ for all } \lambda > 0,
\]

then

\[
\mu^4 - 2\lambda(\mu^2 + BB^*) + \lambda^2 \geq 0 \text{ for all } \lambda > 0,
\]

which yields that

\[
\mu^4 - 2\lambda\mu^2 + \lambda^2 \geq 2\lambda BB^* \text{ for all } \lambda > 0.
\]

Hence $B = 0.$ \hfill □
3. Spectral properties of $k$-quasi-$*$-paranormal operators

For every $T \in B(H)$, $\sigma(T)$ is a compact subset of $\mathbb{C}$. The function $\sigma$ viewed as a function from $B(H)$ into the set of all compact subsets of $\mathbb{C}$, equipped with the Hausdorff metric, is well known to be upper semi-continuous, but fails to be continuous in general. Conway and Morrel [2] have carried out a detailed study of spectral continuity in $B(H)$. Recently, the continuity of spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators in [6, 12]. It has been proved that is continuous in the set of normal operators and hyponormal operators in [7]. And this result has been extended to quasi-hyponormal operators by Djordjević in [3], to $p$-hyponormal operators by Hwang and Lee in [13], and to $(p, k)$-quasihyponormal, $M$-hyponormal, $*$-paranormal and paranormal operators by Duggal, Jeon and Kim in [4]. In this section we extend this result to $k$-quasi-$*$-paranormal operators.

Lemma 3.1. Let $T$ be a $k$-quasi-$*$-paranormal operator. Then the following assertions hold:

(1) If $T$ is quasinilpotent, then $T^{k+1} = 0$.

(2) For every non-zero $\lambda \in \sigma_p(T)$, the matrix representation of $T$ with respect to the decomposition $H = N(T - \lambda) \oplus (N(T - \lambda))^\perp$ is: $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator $B$ satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.

Proof. (1) Suppose $T$ is a $k$-quasi-$*$-paranormal operator. If the range of $T^k$ is dense, then $T$ is a $*$-paranormal operator, which leads to that $T$ is normaloid, hence $T = 0$. If the range of $T^k$ is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = \overline{R(T^k)} \oplus N(T^k)$

where $T_1$ is a $*$-paranormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Theorem 2.2. Since $\sigma(T_1) = \{0\}$, $T_1 = 0$. Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2 T_3^{k+1} \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

(2) If $\lambda \neq 0$ and $\lambda \in \sigma_p(T)$, we have that $N(T - \lambda)$ reduces $T$ by Corollary 2.5. So we have that $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ on $H = N(T - \lambda) \oplus \overline{R(T^k)} \oplus N(T^k)$.
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$(N(T - \lambda))^\perp$ for some operator $B$ satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.

**LEMMA 3.2.** [1] Let $H$ be a complex Hilbert space. Then there exists a Hilbert space $K$ such that $H \subset K$ and a map $\varphi : B(H) \to B(K)$ such that

1. $\varphi$ is a faithful $\ast$-representation of the algebra $B(H)$ on $K$;
2. $\varphi(A) \geq 0$ for any $A \geq 0$ in $B(H)$;
3. $\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(H)$.

**THEOREM 3.3.** The spectrum $\sigma$ is continuous on the set of $k$-quasi-*$\ast$-paranormal operators.

**Proof.** Suppose $T$ is a $k$-quasi-*$\ast$-paranormal operator. Let $\varphi : B(H) \to B(K)$ be Berberian’s faithful $\ast$-representation of Lemma 3.2. In the following, we shall show that $\varphi(T)$ is also a $k$-quasi-*$\ast$-paranormal operator. In fact, since $T$ is a $k$-quasi-*$\ast$-paranormal operator, we have

$$T^{\ast k}(T^{\ast 2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$$

for all $\lambda > 0$.

Hence we have

$$((\varphi(T))^{\ast k}(\varphi(T))^{\ast 2}(\varphi(T))^2 - 2\lambda \varphi(T)(\varphi(T))^* + \lambda^2)(\varphi(T))^k$$

$$= \varphi(T)^{\ast k}(T^{\ast 2}T^2 - 2\lambda TT^* + \lambda^2)T^k) \text{ by Lemma 3.2}$$

$$\geq 0 \text{ by Lemma 3.2,}$$

so $\varphi(T)$ is also a $k$-quasi-*$\ast$-paranormal operator. By Lemma 3.1, we have $T$ belongs to the set $C(i)$ (see definition in [4]). Therefore, we have that the spectrum $\sigma$ is continuous on the set of $k$-quasi-*$\ast$-paranormal operators by [4, Theorem 1.1].

A complex number $\lambda$ is said to be in the point spectrum $\sigma_p(T)$ of $T$ if there is a nonzero $x \in H$ such that $(T - \lambda)x = 0$. If in addition, $(T^* - \overline{\lambda})x = 0$, then $\lambda$ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of $T$. If $T$ is hyponormal, then $\sigma_{jp}(T) = \sigma_p(T)$. Here we show that if $T$ is a $k$-quasi-*$\ast$-paranormal operator, then $\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.

**LEMMA 3.4.** Let $T$ be a $k$-quasi-*$\ast$-paranormal operator and $\lambda \neq 0$. Then $Tx = \lambda x$ implies $T^*x = \overline{\lambda}x$.

**Proof.** It is obvious from Corollary 2.5.

The following example provides an operator $T$ which is a $k$-quasi-*$\ast$-paranormal operator, however, the relation $N(T) \subseteq N(T^*)$ does not hold.
Example 3.5. [14] Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be operators on $\mathbb{R}^2$, and let $H_n = \mathbb{R}^2$ for all positive integers $n$. Consider the operator $T_{A,B}$ on $\bigoplus_{n=1}^{+\infty} H_n$ defined by

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$ 

Then $T_{A,B}$ is a quasi-*$\ast$-paranormal operator, hence $T_{A,B}$ is a $k$-quasi-*$\ast$-paranormal operator, however for the vector $x = (0, 0, 1, -1, 0, 0, \cdots)$, $T_{A,B}(x) = 0$, but $T_{A,B}^*(x) \neq 0$. Therefore, the relation $N(T_{A,B}) \subseteq N(T_{A,B}^*)$ does not always hold.

**Theorem 3.6.** Let $T$ be a $k$-quasi-*$\ast$-paranormal operator. Then $\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.

**Proof.** It is clearly by Lemma 3.4. \qed

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