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ON THE SPECTRAL CONTINUITY

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Abstract. In this paper we show that the spectrum is continuous on the class of $\star$-paranormal operators but the approximate point spectrum generally is not continuous at $\star$-paranormal operators.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on an infinite dimensional separable complex Hilbert space $\mathscr{H}$. Given an operator $T \in \mathscr{L}(\mathscr{H})$, let $\alpha(T) = \dim \ker(T)$ and $\beta(T) = \dim \ker(T^*)$. T is upper semi-Fredholm if $T\mathscr{H}$ is closed and $\alpha(T) < \infty$ and T is lower semi-Fredholm if $T\mathscr{H}$ is closed and $\beta(T) < \infty$. If T is semi-Fredholm, then the index of T, $\text{ind}(T)$, is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. T is said to be Fredholm if $T\mathscr{H}$ is closed and the deficiency indices $\alpha(T)$ and $\beta(T)$ are (both) finite. Let $\mathscr{F}$ denote the set of Fredholm operators and $\mathscr{SF}$ the set of semi-Fredholm operators. Let

$$P_n(T) = \{\lambda \in \sigma(T) : T - \lambda \in \mathscr{SF} \text{ and } \text{ind}(T - \lambda) = n\}$$

for $n \in \mathbb{Z} \cup \{\pm\infty\}$ and let

$$P_\pm(T) = \bigcup\{P_n(T) : n \neq 0\} \text{ and } P_-(T) = \bigcup\{P_n(T) : -\infty \leq n \leq -1\}.$$

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Let σ_p and σ_{ap} denote the point spectrum and the approximate point spectrum, respectively. For an operator $T \in \mathscr{L}(\mathscr{H})$, it is well known that $\sigma_{ap}(T) = \sigma_l(T)$. Recall that if $\mathscr{C}$ is the ideal of compact operators on $\mathscr{H}$ and

$$\pi : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})/\mathscr{C}$$

is the natural projection map, then the essential spectrum of T is defined by $\sigma_e(T) = \sigma(\pi(T))$ and let $\sigma_{le}(T)$ and $\sigma_{re}(T)$ denote the left and right spectrum of $\pi(T)$. So

$$\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T) = [\sigma_{le}(T) \cap \sigma_{re}(T)] \cup P_{+\infty}(T) \cup P_{-\infty}(T).$$

Recall also that a *hole* in $\sigma_e(T)$ is bounded component of $\mathbb{C} \setminus \sigma_e(T)$ and a *pseudohole* in $\sigma_e(T)$ is a component of

$$[\mathrm{int}(\sigma_e(T)) \setminus \sigma_{le}(T)] \text{ or } [\mathrm{int}(\sigma_e(T)) \setminus \sigma_{re}(T)]$$

Hence a hole and a pseudohole in $\sigma_e(T)$ is an open subset of the complex plane $\mathbb{C}$.

Let $\mathcal{K}$ denote the set of all compact subsets of the complex plane $\mathbb{C}$. Equipping $\mathcal{K}$ with the Hausdorff metric, one may consider the spectrum σ as a function $\sigma : \mathscr{L}(\mathscr{H}) \to \mathcal{K}$ mapping operators $T \in \mathscr{L}(\mathscr{H})$ into their spectrum $\sigma(T)$. Newburgh [16] may be the first to have systematically investigated the continuity of the spectrum. he showed that the spectrum of an element of a Banach algebra is upper semicontinuous and that the spectrum is continuous at any element with totally disconnected spectrum. In addition, he showed that the spectrum is continuous on an abelian Banach algebra and on the class of operators satisfying G_1-condition. Studies identifying sets $\mathcal{C}$ of operators for which σ becomes continuous when restricted to $\mathcal{C}$ has been carried out by a number authors (see, for example, [7–9,12]). On the other hand, Conway and Morrel [4, 5] have undertaken a detailed study on the continuity of various spectra in $\mathscr{L}(\mathscr{H})$.

Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be $\star$-*paranormal* if

$$||T^*x||^2 \leq ||T^2x|| \text{ for every unit vector } x \in \mathscr{H}.$$

It is well known [10] that the set of all hyponormal operators is a proper subset of the class of $\star$-paranormal operators and a kernel property

$$\ker(T-\lambda) \subseteq \ker(T-\lambda)^* \ \textit{for all}\ \lambda \in \mathbb{C} \tag{1.1}$$

holds for an $\star$-paranormal operator T.

In this paper we show that the spectrum is continuous on the class of $\star$-paranormal operators but the approximate point spectrum generally is not continuous at $\star$-paranormal operators.

2. Main results

Let $T^\circ \in \mathscr{L}(\mathscr{K})$ denote the Berberian extension of an operator $T \in \mathscr{L}(\mathscr{H})$. Then the Berberian extension theorem [3] says that given an operator $T \in \mathscr{L}(\mathscr{H})$ there exists a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ and an isometric $*$-isomorphism $T \to T^\circ \in \mathscr{L}(\mathscr{K})$ preserving order such that $\sigma(T) = \sigma(T^\circ)$ and $\sigma_p(T^\circ) = \sigma_{ap}(T^\circ) = \sigma_{ap}(T)$. In the following, we shall denote the set of accumulation points (resp. isolated points) of $\sigma(T)$ by $\mathrm{acc}\sigma(T)$(resp. $\mathrm{iso}\sigma(T)$).

The following result is not new and was essentially proved in [9], but we give a proof for the completeness and revising mistakes in the proof of Theorem 1.1 of [14].

Proposition 2.1. *The spectrum σ is continuous on the set of all $\star$-paranormal operators.*

Proof. To prove the theorem it would suffice to prove that if $\{T_n\} \subset \mathscr{L}(\mathscr{H})$ is a sequence of $\star$-paranormal operators such that

$$\lim_{n\to\infty} ||T_n - T|| = 0$$

for some $\star$-paranormal operator T, then

$$\mathrm{acc}\sigma(A) \subseteq \liminf_n \sigma(A_n). \tag{2.1}$$

because the function σ is upper semi-continuous [11] and it is well known that, from an argument of Newburgh [16, Lemma 3],

$$\mathrm{iso}\sigma(T) \subseteq \liminf_n \sigma(T_n). \tag{2.2}$$

Now, we consider corresponding the Berberian extensions to T and the sequence $\{T_n\}$ as mensioned above, and then have that

$$\sigma(T) = \sigma(T^\circ), \sigma(T_n) = \sigma(T_n^\circ) \text{ and } \sigma_{ap}(T) = \sigma_{ap}(T^\circ) = \sigma_p(T^\circ).$$

Since if T is $\star$-paranormal then T° is $\star$-paranormal, we have that

$$\mathrm{acc}\sigma(T) \subseteq \liminf_n \sigma(T_n) \Longleftrightarrow \mathrm{acc}\sigma(T^\circ) \subseteq \liminf_n \sigma(T_n^\circ). \tag{2.3}$$

Now, assume that $\lambda \in \mathrm{acc}\sigma(T^\circ)$. First, we consider the case that

$$\lambda \in \sigma(T^\circ) \setminus \sigma_{ap}(T^\circ). \tag{2.4}$$

So $T^\circ - \lambda$ is upper semi-Fredholm and $\alpha(T^\circ - \lambda) = 0$. Suppose that $\lambda \notin \liminf_n \sigma(T_n^\circ)$, then there exists a $\delta > 0$, a neighbourhood $\mathcal{N}_\delta(\lambda)$ of λ and a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that

$$\sigma(T_{n_k}) \cap \mathcal{N}_\delta(\lambda) = \emptyset \text{ for every } k \geq 1.$$

Evidently, $T_{n_k}^\circ - \lambda$ is Fredholm, with $\mathrm{ind}(T_{n_k}^\circ - \lambda) = 0$, and

$$\lim_{n\to\infty} ||(T_{n_k}^\circ - \lambda) - (T^\circ - \lambda)|| = 0.$$

It follows from the continuity of the index that $\mathrm{ind}(T^\circ - \lambda) = 0$ and $T^\circ - \lambda$ is Fredholm. Since $\alpha(T^\circ - \lambda) = 0$, $\lambda \notin \sigma(T^\circ)$. This contradicts to (2.4). Hence we have that $\lambda \in \liminf_n \sigma(A_n^\circ)$.

Second, we consider the case that

$$\lambda \in \sigma_{ap}(T^\circ).$$

Since T° is $\star$-paranormal and $\sigma_{ap}(T^\circ) = \sigma_p(T^\circ)$, T° is reduced by an eigenspace $\ker(T^\circ - \lambda)$ [10, Lemma 2.2]. So we have a representation of T°,

$$T^\circ = \lambda \oplus B \quad \text{on} \quad \mathscr{K} = \ker(T^\circ - \lambda) \oplus \{\ker(T^\circ - \lambda)\}^\perp$$

Evidently, $B - \lambda$ is upper semi-Fredholm and $\alpha(B - \lambda) = 0$. There exists an $\epsilon > 0$ such that $B - (\lambda - \mu_o)$ is upper semi-Fredholm with $\mathrm{ind}(B - (\lambda - \mu_o)) = \mathrm{ind}(B - \lambda)$ and $\alpha(B - (\lambda - \mu_o)) = 0$ for every μ_o satisfying $0 < |\mu_o| < \epsilon$. Choose $0 < \epsilon < \delta$ and set $\mu = \lambda - \mu_o$ $(0 < |\mu_o| < \epsilon)$. (Here $\delta > 0$ as above mentioned.) Then $B - \mu$ is upper semi-Fredholm, $\mathrm{ind}(B-\mu) = \mathrm{ind}(B-\lambda)$ and $\alpha(B-\mu) = 0$. This implies that

$$T^\circ - \mu = \lambda - \mu \oplus B - \mu$$

is upper semi-Fredholm ,

$$\mathrm{ind}(T^\circ - \mu) = \mathrm{ind}(B - \mu) \text{ and } \alpha(T^\circ - \mu) = 0.$$

Assume to the contrary that $\lambda \notin \liminf_n \sigma(T_n^\circ)$, then evidently, $T_{n_k}^\circ - \mu$ is Fredholm, with $\mathrm{ind}(T_{n_k}^\circ - \mu) = 0$, and

$$\lim_{n\to\infty} ||(T_{n_k}^\circ - \mu) - (T^\circ - \mu)|| = 0.$$

It follows from the continuity of the index that $\mathrm{ind}(T^\circ - \mu) = 0$ and $T^\circ - \mu$ is Fredholm. Since $\alpha(T^\circ - \mu) = 0$, $\mu \notin \sigma(T^\circ)$ for every μ in a deleted

ϵ-neighbourhood of λ. This contradicts to the assumption $\lambda \in \mathrm{acc}\sigma(T^\circ)$. Hence we must have that $\lambda \in \liminf_n \sigma(T_n^\circ)$. □

For the following lemma, we need some concepts. It is well known [2] that an operator is *quasitriangular* if and only if for each $\lambda \in \mathbb{C}$ such that $T-\lambda$ semi-Fredholm, $\mathrm{ind}(T-\lambda) \geq 0$. It follows that an operator T is *biquasitriangular* if and only if for each $\lambda \in \mathbb{C}$ such that $T-\lambda$ semi-Fredholm, $\mathrm{ind}(T-\lambda) = 0$. Consequently, T is biqusitriangular if and only if $P_{\pm}(T) = \emptyset$

LEMMA 2.2. *If an operator $T \in \mathscr{L}(\mathscr{H})$ is $\star$-paranormal, then T is quasitriangular if and only if T is biquasitriangular.*

Furthermore, if T is $\star$-paranormal and quasitriangular, then $\sigma_{ap}(T)$ is continuous at T.

Proof. First, since T is $\star$-paranormal, the equivalence relation immediately follows from that T satisfies the kernel condition (1.1).

Second, suppose that T is quasitriangular. We claim that $\sigma(T) = \sigma_{ap}(T)$. Let $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T)$. Then $T-\lambda$ is semi-Fredholm and $\alpha(T-\lambda) = 0$. Since T is biquasitriangular, $T-\lambda$ has $\mathrm{ind}(T-\lambda) = 0$. Thus $\lambda \in \mathbb{C} \setminus \sigma(T)$. Therefore $\sigma(T) = \sigma_{ap}(T)$. Consequently, from Proposition 2.1, it is immediately follows that $\sigma_{ap}(T)$ is continuous at T □

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0 for short, if for every open neighbourhood U of λ_0, the only analytic function $f : U \to \mathscr{H}$ which satisfies the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. The single valued extension property was introduced by Dunford [6] and plays an important role in local spectral theory. It is well known [10] that a $\star$-paranormal operator has finite ascent and so have SVEP.

For an operator $T \in \mathscr{L}(\mathscr{H})$, the *quasi-nilpotent part of* T is the set

$$H_0(T) = \{x \in \mathscr{H} : \lim_{n\to\infty} ||T^n x||^{\frac{1}{n}} = 0\}, \tag{2.5}$$

which is a linear subspace of $\mathscr{H}$, generally not closed. It is well known [10] that for a $\star$-paranormal operator T,

$$H_0(T-\lambda) = \ker(T-\lambda) \text{ for every } \lambda \in \mathbb{C}.$$

Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *semi-regular* if $T(\mathscr{H})$ is closed and

$$\ker(T) \subset T^{\infty}(\mathscr{H}) = \bigcap_{n \in \mathbb{N}} T^n(\mathscr{H}).$$

Following Mbekhta [15], we shall say that T admits a *generalized Kato decomposition*, GKD for short, if there exists a pair of T-invariant closed subspaces (M, N) such that $\mathscr{H} = M \oplus N$, the restriction $T|_M$ is semi-regular and $T|_N$ is quasinilpotent. We say that T is *Kato type* at a point λ_0 if $(T - \lambda_0)|_N$ is nilpotent in the GKD for $T - \lambda_0$.

In [5, Theorem 5.1], Conway and Morrel characterized the continuity of the approximate point spectrum at an operator $T \in \mathscr{L}(\mathscr{H})$ as the following:

PROPOSITION 2.3. *If $T \in \mathscr{L}(\mathscr{H})$, then σ_{ap} is continuous at T if and only if the following conditions are satisfied:*
(a) σ is continuous at T.
(b) $P_{-}(T) \cap \sigma_p(T) = \emptyset$.
(c) T satisfies $(CC)_k$ for $-\infty \leq k \leq -1$.
(d) $P_{-\infty}(T) = \operatorname{int}(\overline{P_{-\infty}(T)})$.

Where $(CC)_k$ means that for each point $\lambda \in [\operatorname{int}(\overline{P_k(T)})] \setminus P_k(T)$ and for every $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{le}(T) \cap \sigma_{re}(T)$.

If T is $\star$-paranormal, we can see from Lemma 2.2 that T is biquasi-triangular if and only if $P_{-}(T) = \emptyset$ and in that case, σ_{ap} is continuous at T.

Now we consider the case that σ_{ap} is not continuous at a $\star$-paranormal operator T.

THEOREM 2.4. *Let an operator $T \in \mathscr{L}(\mathscr{H})$ be $\star$-paranormal and $P_{-}(T) \neq \emptyset$. Then σ_{ap} is not continuous at T.*

Proof. To prove the theorem, we use the characterization of Proposition 2.3 for continuity of $\sigma_{ap}(T)$. To put it concretely, provided $P_{-}(T) \neq \emptyset$, we will show that the necessary condition

$$P_{-}(T) \cap \sigma_p(T) = \emptyset$$

does not hold for a $\star$-paranormal operator T. We claim that

$$P_{-}(T) \subset \sigma_p(T). \tag{2.6}$$

Let $\lambda_0 \in P_{-}(T)$. Since $T-\lambda_0$ is semi-Fredholm, Kato's classical result [13, Theorem 4] implies that $T-\lambda_0$ is a Kato type. So there exists a pair of T-invariant closed subspaces (M, N) such that $\mathscr{H} = M \oplus N$, the restriction $(T-\lambda_0)|_M$ is semi-regular and $(T-\lambda_0)|_N$ is nilpotent in the GKD for $T-\lambda_0$. Since T is $\star$-paranormal, T has the SVEP at λ_0 ([10]) and

$$N = H_0(T-\lambda_0) = \ker(T-\lambda_0).$$

Assume to the contrary that $\lambda_0 \notin \sigma_p(T)$. So it follows from [1, Theorem 2.6] that $(T-\lambda_0)|_M$ is injective and $M = \mathscr{H}$. This is a contradiction. Thus (2.6) holds, and so $P_{-}(T) \cap \sigma_p(T) \neq \emptyset$. Hence Proposition 2.3 implies that σ_{ap} is not continuous at T. □

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