BOUNDEDNESS IN PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS VIA $t_\infty$-SIMILARITY

Sang Il Choi and Yoon Hoe Goo

Abstract. In this paper, we investigate bounds for solutions of perturbed functional differential systems using the notion of $t_\infty$-similarity.

1. Introduction and preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean $n$-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed functional differential systems of (1.1)

$$y' = f(t, y) + \int_{t_0}^t g(s, y(s)) ds + r(t, y(t), Ty(t)), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $r \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $r(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator.

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix $A$, define the norm $|A|$ of $A$ by $|A| = \sup_{|x| \leq 1} |Ax|$.
Let \( x(t, t_0, x_0) \) denote the unique solution of (1.1) with \( x(t_0, t_0, x_0) = x_0 \), existing on \([t_0, \infty)\). Then, we can consider the associated variational systems around the zero solution of (1.1) and around \( x(t) \), respectively,

\begin{equation}
(1.3) \quad v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0
\end{equation}

and

\begin{equation}
(1.4) \quad z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0.
\end{equation}

The fundamental matrix \( \Phi(t, t_0, x_0) \) of (1.4) is given by

\[ \Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0), \]

and \( \Phi(t, t_0, 0) \) is the fundamental matrix of (1.3).

We recall some notions of \( h \)-stability [14].

**Definition 1.1.** The system (1.1) (the zero solution \( x = 0 \) of (1.1)) is called an \( h \)-system if there exist a constant \( c \geq 1 \), and a positive continuous function \( h \) on \( \mathbb{R}^+ \) such that

\[ |x(t)| \leq c |x_0| h(t) h(t_0)^{-1} \]

for \( t \geq t_0 \geq 0 \) and \( |x_0| \) small enough (here \( h(t)^{-1} = \frac{1}{h(t)} \)).

**Definition 1.2.** The system (1.1) (the zero solution \( x = 0 \) of (1.1)) is called \( (hS)h \)-stable if there exists \( \delta > 0 \) such that (1.1) is an \( h \)-system for \( |x_0| \leq \delta \) and \( h \) is bounded.

The notion of \( h \)-stability \((hS)\) was introduced by Pinto [13, 14] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called \( h \)-systems. Choi and Koo [2] and Choi et al. [3,4] investigated \( h \)-stability and bounds of solutions for the perturbed differential systems. Also, Goo [6,7,8] and Goo et al. [9] studied the boundedness of solutions for the perturbed differential systems.

The main conclusion to be drawn from this paper is that the use of inequalities provides a powerful tool for obtaining bounds for solutions.
Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class $C^1$ with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of $t_\infty$-similarity in $\mathcal{M}$ was introduced by Conti [5].

**Definition 1.3.** A matrix $A(t) \in \mathcal{M}$ is $t_\infty$-similar to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t)$$

for some $S(t) \in \mathcal{N}$.

The notion of $t_\infty$-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [5, 10].

In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of $t_\infty$-similarity.

We give some related properties that we need in the sequel.

**Lemma 1.4.** [14] The linear system

$$x' = A(t)x, \quad x(t_0) = x_0,$$

where $A(t)$ is an $n \times n$ continuous matrix, is an $h$-system (respectively $h$-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function $h$ defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point $(t_0, y_0)$ in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].
Lemma 1.5. If $y_0 \in \mathbb{R}^n$, then for all $t$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$,
\[
y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) g(s, y(s)) ds.
\]

Theorem 1.6. [3] If the zero solution of (1.1) is $hS$, then the zero solution of (1.3) is $hS$.

Theorem 1.7. [4] Suppose that $f_x(t, 0)$ is $t_\infty$-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is $hS$, then the solution $z = 0$ of (1.4) is $hS$.

Lemma 1.8. (Bihari – type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in $u$. Suppose that, for some $c > 0$,
\[
u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \quad t \geq t_0 \geq 0.
\]
Then
\[
u(t) \leq W^{-1} \left[ W(c) + \int_{t_0}^{t} \lambda(s) ds \right], \quad t_0 \leq t < b_1,
\]
where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and
\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \text{dom} W^{-1} \right\}.
\]

Lemma 1.9. Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in $u$, $u \leq w(u)$. Suppose that, for some $c > 0$ and $0 \leq t_0 \leq t$,
\[
u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) u(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) w(u(r)) dr) d\tau ds + \int_{t_0}^{t} \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau) w(u(\tau)) d\tau ds.
\]
Then
\[
u(t) \leq W^{-1} \left[ W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau + \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau) d\tau) ds \right],
\]
Boundedness in functional differential systems via $t_\infty$-similarity

$t_0 \leq t < b_1$, where $W$, $W^{-1}$ are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)) s (\lambda_3(\tau) + \lambda_4(\tau)) d\tau + \lambda_6(s) \int_{t_0}^{t} \lambda_7(\tau)d\tau ds \in \text{dom}W^{-1} \right\}.$$

**Proof.** Define a function $z(t)$ by the right member of (1.9). Then, we have $z(t_0) = c$ and

$$z'(t) = \lambda_1(t)w(u(t)) + \lambda_2(t) \int_{t_0}^{t} (\lambda_3(s)u(s) + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)w(u(\tau))d\tau)ds + \lambda_6(t) \int_{t_0}^{t} \lambda_7(s)w(u(s))ds$$

$$\leq (\lambda_1(t) + \lambda_2(t) \int_{t_0}^{t} (\lambda_3(s) + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau)d\tau)ds + \lambda_6(t) \int_{t_0}^{t} \lambda_7(s)ds)w(z(t)).$$

$t \geq t_0$, since $z(t)$ and $w(u)$ are nondecreasing, $u \leq w(u)$, and $u(t) \leq z(t)$. Therefore, by integrating on $[t_0, t]$, the function $z$ satisfies

$$z(t) \leq c + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(\tau)d\tau) d\tau + \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)d\tau w(z(s)))ds.$$

(1.11)

It follows from Lemma 1.8 that (1.11) yields the estimate (1.10). \hfill \square

**Corollary 1.10.** Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in $u$, $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} \lambda_2(\tau)w(u(\tau)) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1}\left[ W(c) + \int_{t_0}^{t} (\lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(\tau)d\tau)ds \right].$$
$t_0 \leq t < b_1$, where $W, W^{-1}$ are the same functions as in Lemma 1.8, and
\[ b_1 = \sup\left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)dr)d\tau)ds \in \text{dom} W^{-1} \right\}. \]

**Corollary 1.11.** Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+), w \in C((0, \infty))$, and $w(u)$ be nondecreasing in $u$, $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,
\[ u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} \lambda_2(\tau)u(\tau)d\tau + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)w(u(\tau))d\tau ds. \]
Then
\[ u(t) \leq W^{-1}\left[ W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} \lambda_2(\tau)d\tau + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau ds \right], \]
$0 \leq t < b_1$, where $W, W^{-1}$ are the same functions as in Lemma 1.8, and
\[ b_1 = \sup\left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau)ds \in \text{dom} W^{-1} \right\}. \]

2. **Main Results**

In this section, we investigate boundedness for solutions of the nonlinear perturbed differential systems via $t_\infty$-similarity.

To obtain the bounded property, the following assumptions are needed:
1. (H1) $w(u)$ is nondecreasing in $u$ such that $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$.
2. (H2) $f_x(t, 0)$ is $t_\infty$-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$.
3. (H3) The solution $x = 0$ of (1.1) is $hS$ with the increasing function $h$.

**Theorem 2.1.** Let $a, b, c, k, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and $g$ in (1.2) satisfies
\[ |g(t, y(t))| \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)w(|y(s)|)ds \]
and

\[(2.2) \quad |r(t, y(t), Ty(t))| \leq \int_{t_0}^{t} c(s)w(|y(s)|)ds,\]

where \(a, b, c, k \in L_1(\mathbb{R}^+)\). Then, any solution \(y(t) = y(t, t_0, y_0)\) of (1.2) is bounded on \([t_0, \infty)\) and it satisfies

\[|y(t)| \leq h(t)W^{-1}\left[W(c) + c_2 \int_{t_0}^{t} \int_{t_0}^{s} (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau ds\right],\]

where \(W, W^{-1}\) are the same functions as in Lemma 1.8, and

\[b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} \int_{t_0}^{s} (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau ds \in \text{dom}W^{-1} \right\}.\]

**Proof.** Using the nonlinear variation of constants formula of Alekseev [1], any solution \(y(t) = y(t, t_0, y_0)\) of (1.2) passing through \((t_0, y_0)\) is given by

\[(2.3) \quad y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) \left( \int_{t_0}^{s} g(\tau, y(\tau))d\tau + r(s, y(s), Ty(s))ds \right).\]

By Theorem 1.6, since the solution \(x = 0\) of (1.1) is hS, the solution \(v = 0\) of (1.3) is hS. Therefore, by Theorem 1.7, the solution \(z = 0\) of (1.4) is hS. Using the nonlinear variation of constants formula (2.3), Lemma 1.4, the hS condition of \(x = 0\) of (1.1), (2.1), and (2.2), we have

\[|y(t)| \leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1} \left( \int_{t_0}^{s} (a(\tau) + c(\tau))w(|y(\tau)|)d\tau ds \right.\]

\[+ b(\tau) \int_{t_0}^{\tau} k(r)w(|y(r)|)dr)d\tau ds + \left. \int_{t_0}^{s} c(\tau)w(|y(\tau)|)d\tau ds \right)\]

\[\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) \left( \int_{t_0}^{s} ((a(\tau) + c(\tau))w\left(\frac{|y(\tau)|}{h(\tau)}\right)d\tau \right)\]

\[+ b(\tau) \int_{t_0}^{\tau} k(r)w\left(\frac{|y(r)|}{h(r)}\right)dr)d\tau ds.\]
Set \( u(t) = |y(t)||h(t)|^{-1} \). Then, by Corollary 1.10, we have

\[
|y(t)| \leq h(t)W^{-1}\left[ W(c) + c_2 \int_{t_0}^t \int_{t_0}^s (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^\tau k(r)dr) d\tau ds \right],
\]

where \( c = c_1|y_0|h(t_0)^{-1} \). From the above estimation, we obtain the desired result. Thus, the proof is complete.

**Remark 2.2.** Letting \( c(s) = 0 \) in Theorem 2.1, we obtain the similar result as that of Theorem 3.6 in [8].

We need the following lemma for the proof of Theorem 2.4.

**Lemma 2.3.** Let \( u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+) \), \( w \in C((0, \infty)) \) and \( w(u) \) be nondecreasing in \( u \). Suppose that, for some \( c \geq 0 \), we have

\[
(2.4)\quad u(t) \leq c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s) \left( \int_{t_0}^s (\lambda_3(\tau)w(u(\tau)) \right. \\
+ \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(s)w(u(r))dr)d\tau + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)w(u(\tau))d\tau \Big)ds, \quad t \geq t_0.
\]

Then

\[
(2.5)\quad u(t) \leq W^{-1}\left[ W(c) + \int_{t_0}^t [\lambda_1(s) + \lambda_2(s) \left( \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)dr)d\tau \\
+ \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau \right)ds \right] \right], \quad t \geq t_0,
\]

where \( W, W^{-1} \) are the same functions as in Lemma 1.8, and

\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t [\lambda_1(s) + \lambda_2(s) \left( \int_{t_0}^s (\lambda_3(\tau) \\
+ \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r)dr)d\tau + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau \right)ds \in \text{dom}W^{-1} \right\}.
\]
Proof. Define a function $v(t)$ by the right member of (2.4). Then, we have $v(t_0) = c$ and

$$v'(t) = \lambda_1(t)w(u(t)) + \lambda_2(t)\left(\int_{t_0}^{t} (\lambda_3(s)w(u(s)) + \lambda_4(t)\int_{t_0}^{t} \lambda_5(s)w(u(s))ds\right)$$

$$+ \lambda_6(t)\int_{t_0}^{t} \lambda_7(s)ds)w(v(t)),$$

$t \geq t_0$, since $v(t)$ is nondecreasing and $u(t) \leq v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$v(t) \leq c + \int_{t_0}^{t} \left(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(s)\int_{t_0}^{\tau} \lambda_5(r)dr)\right)ds$$

$$+ \lambda_6(t)\int_{t_0}^{t} \lambda_7(s)ds)w(v(s))ds.$$

Thus, by Lemma 1.8, (2.6) yields the estimate (2.5).

**Theorem 2.4.** Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and $g$ in (1.2) satisfies

$$|g(t, y(t))| \leq a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds$$

and

$$|r(t, y(t), Ty(t))| \leq c(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)|$$

$$\leq \int_{t_0}^{t} q(s)w(|y(s)|)ds, \quad t \geq t_0 \geq 0,$$

where $a, b, c, k, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and

$$|y(t)| \leq h(t)W^{-1}\left[W(c) + c_2\int_{t_0}^{t} (c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau)\int_{t_0}^{\tau} k(\tau)dr) d\tau)ight.$$

$$+ c(s)\int_{t_0}^{s} q(\tau)dr)ds.$$
where \( W, W^{-1} \) are the same functions as in Lemma 1.8, and
\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} (c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr) d\tau
\right. \\
\left. + c(s) \int_{t_0}^{s} q(\tau)d\tau)ds \in \text{dom} W^{-1} \right\}.
\]

**Proof.** Let \( x(t) = x(t, t_0, y_0) \) and \( y(t) = y(t, t_0, y_0) \) be solutions of (1.1) and (1.2), respectively. By the same argument as the proof in Theorem 2.1, the solution \( z = 0 \) of (1.4) is hS. Applying Lemma 1.4, the hS condition of \( x = 0 \) of (1.1), (2.3), (2.7), (2.8), and the given conditions, we have
\[
|y(t)| \leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) \left(c(s)w\left(\frac{|y(s)|}{h(s)}\right) + \int_{t_0}^{s} (a(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right)
\right. \\
\left. + b(\tau) \int_{t_0}^{\tau} k(r)w\left(\frac{|y(r)|}{h(r)}\right)dr\right) d\tau + c(s) \int_{t_0}^{s} q(\tau)d\tau\left. \right)ds.
\]

Set \( u(t) = |y(t)||h(t)|^{-1} \). Then, it follows from Lemma 2.3 that we have
\[
|y(t)| \leq h(t) W^{-1}\left[ W(c) + c_2 \int_{t_0}^{t} (c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr) d\tau
\right.
\left. + c(s) \int_{t_0}^{s} q(\tau)d\tau)ds \right],
\]
where \( c = c_1 |y_0| h(t_0)^{-1} \). From the above estimation, we obtain the desired result. Thus, the theorem is proved.

**Remark 2.5.** Letting \( c(t) = 0 \) in Theorem 2.4, we obtain the similar result as that of Theorem 3.6 in [8].

We obtain the following corollary from Lemma 2.3 to prove Theorem 2.7.

**Corollary 2.6.** Let \( u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+) \), \( w \in C((0, \infty)) \) and \( w(u) \) be nondecreasing in \( u \). Suppose that, for some \( c \geq 0 \), we have
\[
u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_2(s) \left( \int_{t_0}^{s} \lambda_3(\tau) w(u(\tau)) d\tau
\right. \\
\left. + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau) w(u(\tau)) d\tau \right) ds, \ t \geq t_0.
\]
Then
\[ u(t) \leq W^{-1}\left[W(c) + \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s) \left( \int_{t_0}^{s} \lambda_3(\tau)d\tau + \int_{t_0}^{s} \lambda_5(\tau)d\tau \right)]ds\right], \]
t \geq t_0, where \( W, W^{-1} \) are the same functions as in Lemma 1.8, and
\[ b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s) \left( \int_{t_0}^{s} \lambda_3(\tau)d\tau + \int_{t_0}^{s} \lambda_5(\tau)d\tau \right)]ds \in \text{dom}W^{-1} \right\}. \]

**Theorem 2.7.** Let \( a, b, c, k, u, w \in C(\mathbb{R}^+) \). Suppose that (H1), (H2), (H3), and \( g \) in (1.2) satisfies
\[ \int_{t_0}^{t} |g(s, y(s))|ds \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)w(|y(s)|)ds, \quad t \geq t_0 \geq 0, \]
and
\[ |r(t, y(t), Ty(t))| \leq \int_{t_0}^{t} c(s)w(|y(s)|)ds, \]
where \( a, b, c, k \in L_1(\mathbb{R}^+) \). Then, any solution \( y(t) = y(t, t_0, y_0) \) of (1.2) is bounded on \( [t_0, \infty) \) and it satisfies
\[ |y(t)| \leq h(t)W^{-1}\left[W(c) + c_2 \int_{t_0}^{t} (a(s) + \int_{t_0}^{s} c(\tau)d\tau + b(s) \int_{t_0}^{s} k(\tau)d\tau)ds\right], \]
t \geq t \geq b_1, \]
where \( W, W^{-1} \) are the same functions as in Lemma 1.8 and
\[ b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} (a(s) + \int_{t_0}^{s} c(\tau)d\tau + b(s) \int_{t_0}^{s} k(\tau)d\tau)ds \in \text{dom}W^{-1} \right\}. \]

**Proof.** It is well known that the solution of (1.2) is represented by the integral equation (2.3). By the same argument as the proof in Theorem 2.1, the solution \( z = 0 \) of (1.4) is hS. Using the nonlinear variation of constants formula (2.3), the hS condition of \( x = 0 \) of (1.1), (2.9), and
(2.10), we have

\[ |y(t)| \leq c_1|y_0|h(t) h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) a(s) w\left(\frac{|y(s)|}{h(s)}\right) ds \\
+ \int_{t_0}^{t} c_2 h(t) (\int_{t_0}^{s} c(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau + b(s) \int_{t_0}^{s} k(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau) ds. \]

Set \( u(t) = |y(t)||h(t)|^{-1} \). Then, an application of Corollary 2.6 yields

\[ |y(t)| \leq h(t) W^{-1} \left[ W(c) + c_2 \int_{t_0}^{t} (a(s) + \int_{t_0}^{s} c(\tau) d\tau + b(s) \int_{t_0}^{s} k(\tau) d\tau) ds \right], \]

where \( c = c_1|y_0|h(t_0)^{-1} \). Thus, any solution \( y(t) = y(t, t_0, y_0) \) of (1.2) is bounded on \([t_0, \infty)\), and so the proof is complete.

**Remark 2.8.** Letting \( c(s) = 0 \) in Theorem 2.7, we obtain the same result as that of Theorem 3.2 in [6].

**Theorem 2.9.** Let \( a, b, c, u, w \in C(\mathbb{R}^+), w(u) \) be nondecreasing in \( u \) such that \( u \leq w(u) \) and \( \frac{1}{v}w(u) \leq w\left(\frac{2}{v}\right) \) for some \( v > 0 \). Suppose that (H2), (H3), and \( g \) in (1.2) satisfies

\[ |g(t, y(t))| \leq a(t) w(|y(t)|), \quad |r(t, y(t), Ty(t))| \leq b(t) \int_{t_0}^{t} c(s) |y(s)| ds, \]

where \( a, b, c \in L_1(\mathbb{R}^+) \). Then, any solution \( y(t) = y(t, t_0, y_0) \) of (1.2) is bounded on \([t_0, \infty)\) and

\[ |y(t)| \leq h(t) W^{-1} \left[ W(c) + c_2 \int_{t_0}^{t} (b(s) \int_{t_0}^{s} c(\tau) d\tau + \int_{t_0}^{s} a(\tau) d\tau) ds \right] \]

where \( W, W^{-1} \) are the same functions as in Lemma 1.8 and

\[ b_1 = \sup \left\{ t \geq t_0 : W(c)+c_2 \int_{t_0}^{t} (b(s) \int_{t_0}^{s} c(\tau) d\tau + \int_{t_0}^{s} a(\tau) d\tau) ds \in \text{dom} W^{-1} \right\}. \]

**Proof.** Let \( x(t) = x(t, t_0, y_0) \) and \( y(t) = y(t, t_0, y_0) \) be solutions of (1.1) and (1.2), respectively. By the same argument as the proof in Theorem 2.1, the solution \( z = 0 \) of (1.4) is hS. By the hS condition of \( x = 0 \) of
(1.1), (2.3), and (2.11), it follows that
\[
|y(t)| \leq c_1|y_0|h(t) h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) (b(s) \int_{t_0}^{s} c(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau)
\]
\[
+ \int_{t_0}^{s} a(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau) ds.
\]
Set \( u(t) = |y(t)||h(t)|^{-1} \). Then, an application of Corollary 1.11 yields
\[
|y(t)| \leq h(t) W^{-1}\left[W(c) + c_2 \int_{t_0}^{t} (b(s) \int_{t_0}^{s} c(\tau)d\tau + \int_{t_0}^{s} a(\tau) d\tau) ds\right],
\]
where \( c = c_1|y_0|h(t_0)^{-1} \). Thus, any solution \( y(t) = y(t,t_0,y_0) \) of (1.2) is bounded on \([t_0, \infty)\). This completes the proof.

**Remark 2.10.** Letting \( b(t) = 0 \) in Theorem 2.9, we obtain the similar result as that of Theorem 3.5 in [9].

**Acknowledgement.** The authors are very grateful for the referee’s valuable comments.

**References**


Sang Il Choi  
Department of Mathematics  
Hanseo University  
Seosan 356-706, Republic of Korea  
E-mail: schoi@hanseo.ac.kr

Yoon Hoe Goo  
Department of Mathematics  
Hanseo University  
Seosan 356-706, Republic of Korea  
E-mail: yhgoo@hanseo.ac.kr