Korean J. Math. **23** (2015), No. 2, pp. 293–312
http://dx.doi.org/10.11568/kjm.2015.23.2.293

A BOUNDARY CONTROL PROBLEM FOR VORTICITY MINIMIZATION IN TIME-DEPENDENT 2D NAVIER-STOKES EQUATIONS

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Abstract. We deal with a boundary control problem for the vorticity minimization, in which the flow is governed by the time-dependent two dimensional incompressible Navier-Stokes equations. We derive a mathematical formulation and a process for an appropriate control along the portion of the boundary to minimize the vorticity motion due to the flow in the fluid domain. After showing the existence of an optimal solution, we derive the optimality system for which optimal solutions may be determined. The differentiability of the state solution in regard to the control parameter shall be conjunct with the necessary conditions for the optimal solutions.

1. Introduction

In this paper, we study a class of optimal flow control problem for which the vorticity of the flow is controlled by the velocity forcing along the portion of the boundary in the fluid domain. We are concerned with a boundary control problem for the vorticity minimization in a flow governed by the time-dependent two dimensional incompressible Navier-Stokes equations. Let us describe the boundary control problem for the Navier-Stokes equations that represents the minimization of the vorticity in a fluid flow. Let Ω be a bounded domain in $\mathbb{R}^2$ with the

Received January 21, 2015. Revised May 31, 2015. Accepted June 9, 2015.
2010 Mathematics Subject Classification: 35Q30, 49J20, 49J50, 76F70.
Key words and phrases: Navier-Stokes equations, boundary control, vorticity minimization.

smooth boundary. For practical purposes, we assume that the boundary $\partial\Omega \equiv \Gamma$ is composed of two disjoint parts with positive measures; the homogeneous part Γ_0 and the control part Γ_c such that $\Gamma = \Gamma_0 \cup \Gamma_c$. We consider two dimensional flow over the time interval $[0,\ T]$ in the phisical flow domain Ω with the control applied to the boundary Γ_c:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } (0,T)\times\Omega\,, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } (0,T)\times\Omega \tag{1.2}$$

along with the Dirichlet boundary condition

$$\boldsymbol{u} = \begin{cases} \boldsymbol{g} & \text{on } (0,T)\times\Gamma_c\,, \\ \boldsymbol{0} & \text{on } (0,T)\times\Gamma_0\,, \end{cases} \tag{1.3}$$

and an initial condition

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x}\in\Omega\,. \tag{1.4}$$

Here, the vector field $\boldsymbol{u}(t,\boldsymbol{x}) = (u_1(t,\boldsymbol{x}), u_2(t,\boldsymbol{x}))$ denotes the velocity vector field of the two dimensional flow, p the pressure, and $\nu > 0$ the kinematic viscosity of the fluid. We will use the time variable by t, the state variable by $\boldsymbol{x}$ in the flow domain Ω, and the boundary variable by $\boldsymbol{s}$ for consistency. In our problem, the control parameter is the forcing velocity $\boldsymbol{g}$ of the fluid flow such as injection or suction along the boundary portion Γ_c. For balance among initial and boundary data in (1.1)–(1.4), we assume the following threshold conditions. For the compatibility and regularity of the solution, the control parameter $\boldsymbol{g}$ should satisfy

$$\text{support } \boldsymbol{g}(t,\ \cdot) \subset \Gamma_c \quad \text{and} \quad \int_{\Gamma_c} \boldsymbol{g}(t,\boldsymbol{s})\cdot\boldsymbol{n}\,d\boldsymbol{s} = 0\,,\ \forall\, 0 \le t \le T\,, \tag{1.5}$$

where $\boldsymbol{n}$ is the unit normal vector along the boundary Γ_c, and

$$\boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{g}(0,\boldsymbol{x}) \text{ for } \boldsymbol{x}\in\Gamma_c\,. \tag{1.6}$$

Also we need to keep the balance between the initial and boundary data by assuming

$$\nabla\cdot\boldsymbol{u}_0 = 0 \text{ in } \Omega,\ \boldsymbol{u}_0 = \boldsymbol{0} \text{ on } \Gamma_0, \text{ and } \int_{\Gamma_c} \boldsymbol{u}_0\cdot\boldsymbol{n}\,d\boldsymbol{s} = 0\,. \tag{1.7}$$

The conditions (1.5) and (1.7) are necessary in view of the incompressibility condition and (1.6) in order to attain the appropriate regularity for the solution of the Navier-Stokes system.

Typical model for the boundary control problem related to the reduction of the vorticity $\nabla \times \boldsymbol{u}$ in turbulent flows may be formulated as follows: Find the optimal boundary control $\boldsymbol{g}$ along Γ_c minimizing the objective functional

$$(\mathrm{Q}): \quad \mathcal{J}(\boldsymbol{u}, \boldsymbol{g}) = \frac{\nu}{2} \int_0^T \int_\Omega |\nabla \times \boldsymbol{u}|^2 \, d\boldsymbol{x} dt + \frac{\epsilon}{2} \int_0^T \int_{\Gamma_c} |\boldsymbol{g}|^2 \, d\boldsymbol{s} dt \,,$$

where $\boldsymbol{u}$ is subject to the two dimensional Navier-Stokes system (1.1)–(1.4). In (Q), $\nabla \times \boldsymbol{u}$ stands for the curl operator in the two dimensional domain Ω. $\mathcal{J}(\boldsymbol{u}, \boldsymbol{g})$ is the objective functional related to a distributed observation of the vorticity stemmed from the fluid flow during the time interval $(0, T)$ and the driven control along the boundary Γ_c. The second integral appearing in the objective functional often plays the role of a regularization term, where $\epsilon > 0$ is a suitable regularization factor. It is often demanded for a concession of mathematical rigor for the control. The positive penalty parameter ϵ in (Q) may be used to switch the relative importance between terms in the objective functional as in [12]. It is also necessary to keep the uniform boundedness for the control terms. One could examine several physically meaningful objective functionals for the boundary control in practices such as seeking the desired velocity tracking over the special region of the flow body Ω as in [7], or pursuing an optimal drag reduction profile as in [5]. The vorticity introduced by the fluid flow is an important factor dealt with the fluid dynamics and mechanics. It is recognized as the force generating the turbulence. It has been regarded as a major source of the disturbance in the fluid flow, and is closely connected with a variety of technical applications in science and engineering such as aerodynamics and the crystal growth process. Abergel and Temam [1] have considered several turbulence control problems by taking the distributed controls. However, the boundary control for the turbulence minimization raises some significant difficulties in constructing rigorous formulation for the control as remarked at the forward section.

In [8], we have studied the vorticity minimization through the boundary control for the stationary two dimensional Stokes equations. Also in [9], a boundary control problem for the drag minimization in the two dimensional Navier-Stokes equations have been dealt with. The purpose of this paper is to extend the result of [8] to the time-dependent two dimensional case with the aid of [9]. The plan of the study is as follows.

In section 2, we will introduce function spaces and presents some preliminary results that will be useful in what follows. The existence of an optimal solution will be shown in section 3, and the first order necessary conditions shall be derived through a direct sensitivity analysis in section 4.

2. Preliminaries

To denote vectors and spaces of vector-valued functions, we will use boldface notations. For example, $\boldsymbol{H}^s(\Omega) = [H^s(\Omega)]^2$ denotes the space of $\mathbb{R}^2$-valued functions such that each components of an element in $\boldsymbol{H}^s(\Omega)$ belongs to $H^s(\Omega)$. A particular interest for our purpose is the space

$$\boldsymbol{H}^1(\Omega) = \left\{ \boldsymbol{v} = (v_1\,, v_2) \in \boldsymbol{L}^2(\Omega) \,\Big|\, \frac{\partial v_j}{\partial x_i} \in L^2(\Omega)\,,\ 1 \le i,\, j \le 2 \right\}.$$

Whenever $\Gamma_0 \subset \Gamma$ has a positive measure, the space with the homogeneous boundary condition imposed along the boundary portion Γ_0 is defined by $\boldsymbol{H}^1_{\Gamma_0}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \,|\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}$, and we let $\boldsymbol{H}^1_{\Gamma}(\Omega) = \boldsymbol{H}^1_0(\Omega)$.

Of special use, we define the infinitely differentiable divergence free space by

$$\boldsymbol{\mathcal{V}}(\Omega) = \{\, \boldsymbol{u} \in \boldsymbol{\mathcal{C}}^\infty(\bar{\Omega}) \mid \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega\,, \quad \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_0 \}\,,$$

and its completion in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^1(\Omega)$ by

$$\boldsymbol{H} = \left\{ \boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \,\Big|\, \nabla \cdot \boldsymbol{u} = 0 \ \text{ in } \Omega\,,\ \boldsymbol{u} = \boldsymbol{0} \ \text{ on } \Gamma_0 \right\}\,,$$

$$\boldsymbol{V} = \{ \boldsymbol{v} \in \boldsymbol{H}^1_{\Gamma_0}(\Omega) \mid \nabla \cdot \boldsymbol{v} = 0 \ \text{ in } \Omega \}\,,$$

and $\boldsymbol{V}_0 = \boldsymbol{V} \cap \boldsymbol{H}^1_0(\Omega)$ respectively. The norm on $\boldsymbol{H}$ shall be defined by $|\boldsymbol{u}|$. We also denote the seminorm on $\boldsymbol{V}$ by $\|\boldsymbol{v}\| = |\nabla \boldsymbol{v}|$. According to Poincare's inequality([3], [4]), this is equivalent to the norm of $\boldsymbol{H}^1(\Omega)$. Let us denote the dual space of $\mathbf{V}$ by $\boldsymbol{V}^*$ and the duality between $\boldsymbol{V}^*$ and $\boldsymbol{V}$ by $< \cdot, \cdot >_{\boldsymbol{V}^*}$. Since $\boldsymbol{V}$ is compactly embedded in $\boldsymbol{H}$ and $\boldsymbol{H}$ may be identified with its dual $\boldsymbol{H}^*$ by Riesz's theorem, we have the canonical framework for the variational formulation in the sense that the following inclusions imply dense embeddings:

$$\boldsymbol{V} \subset \boldsymbol{H} \subset \boldsymbol{V}^*\,.$$

For the concerned boundary Γ_c, $\boldsymbol{H}^s_0(\Gamma_c)$ denotes the space of functions in $\boldsymbol{H}^s(\Gamma_c)$ with compact support in Γ_c. Restricting the domain of integration, we represent the norm $\|\cdot\|_{s,\Gamma_c}$ for $\boldsymbol{H}^s(\Gamma_c)$. We also define the traces of the velocity to the control part Γ_c by

$$\gamma_c^0 \,:\, \boldsymbol{H}^1_{\Gamma_0}(\Omega) \to \boldsymbol{H}^{1/2}(\Gamma_c)\,;\, \left(\boldsymbol{u} \mapsto \boldsymbol{u}\Big|_{\Gamma_c}\right),$$

and

$$\gamma_c^1 \,:\, \boldsymbol{H}^1_{\Gamma_0}(\Omega) \to \boldsymbol{H}^{-1/2}(\Gamma_c)\,;\, \left(\boldsymbol{u} \mapsto \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\Big|_{\Gamma_c}\right).$$

In order to define a variational form for the Navier-Stokes equations, we introduce the continuous bilinear form

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu \int_\Omega \mathcal{D}(\boldsymbol{u}) \,:\, \mathcal{D}(\boldsymbol{v})\, d\boldsymbol{x}\,, \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\,,$$

and the trilinear form on $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$

$$b(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\, d\boldsymbol{x} = \sum_{i,j=1}^{2} \int_\Omega u_i \left(\frac{\partial v_j}{\partial x_i}\right) w_j\, d\boldsymbol{x}\,.$$

Here $\mathcal{D}(\boldsymbol{u}) \,:\, \mathcal{D}(\boldsymbol{v})$ denotes the tensor product $\sum_{i,j=1}^{2} \mathcal{D}_{ij}(\boldsymbol{u})\mathcal{D}_{ij}(\boldsymbol{v})$, where

$$\mathcal{D}_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Obviously, $a(\cdot,\cdot)$ is a continuous bilinear form on $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$. It is worthwhile to notice that

$$2\nabla\cdot\mathcal{D}(\boldsymbol{u}) = \Delta\boldsymbol{u} + \nabla(\nabla\cdot\boldsymbol{u})\,.$$

If we take a dot product with $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and the integration, by Green's formula we obtain

$$2\int_\Gamma \boldsymbol{v}\cdot\mathcal{D}(\boldsymbol{u})\boldsymbol{n}\, d\boldsymbol{s} = \int_\Omega \Delta\boldsymbol{u}\cdot\boldsymbol{v}\, d\boldsymbol{x} + 2\int_\Omega \mathcal{D}(\boldsymbol{u}) \,:\, \nabla\boldsymbol{v}\, d\boldsymbol{x}\,.$$

Since $\mathcal{D}(\boldsymbol{u})$ is a symmetric tensor, we have

$$\int_\Omega \mathcal{D}(\boldsymbol{u}) \,:\, \nabla\boldsymbol{v}\, dx = \int_\Omega \mathcal{D}(\boldsymbol{u}) \,:\, \mathcal{D}(\boldsymbol{v})\, d\boldsymbol{x}\,.$$

Hence for $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$, it follows that

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu \int_\Omega \mathcal{D}(\boldsymbol{u}) \,:\, \mathcal{D}(\boldsymbol{v})\, d\boldsymbol{x} = -\nu \int_\Omega \Delta\boldsymbol{u}\cdot\boldsymbol{v}\, d\boldsymbol{x}\,.$$

Related to the duality pairing $<\cdot,\cdot>_{\boldsymbol{V}^*}$, we will make use of the following operators:

$$\mathcal{A}:\boldsymbol{V}\longrightarrow\boldsymbol{V}^*,$$

which is defined by

$$<\mathcal{A}\boldsymbol{u},\boldsymbol{v}>_{\boldsymbol{V}^*}=a(\boldsymbol{u},\boldsymbol{v}),\quad\forall\boldsymbol{u}\in\boldsymbol{V},\forall\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega),\tag{2.1}$$

and

$$\mathcal{B}:\boldsymbol{V}\times\boldsymbol{V}\longrightarrow\boldsymbol{V}^*$$

defined by

$$\langle\mathcal{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle_{\boldsymbol{V}^*}=b(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}),\quad\forall\boldsymbol{u},\,\boldsymbol{v}\in\boldsymbol{V},\forall\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega).\tag{2.2}$$

Without any confusion, $\mathcal{B}(\boldsymbol{u},\boldsymbol{u})$ is denoted by $\mathcal{B}(\boldsymbol{u})$ for the sake of brevity, and $\boldsymbol{V}^*$ will be dropped out in the duality between $\boldsymbol{V}^*$ and $\boldsymbol{V}$ so that $<\cdot,\cdot>_{\boldsymbol{V}^*}=<\cdot,\cdot>$.

For the operator $\mathcal{B}$ and its associated trilinear form $b(\cdot;\cdot,\cdot)$, the following results will be useful in the sequel.

Lemma 2.1. ([1], [9], [13])

(1) *The trilinear form $b(\cdot\,;\cdot\,,\cdot)$ has the following orthogonality properties:*

$$\begin{cases} b(\boldsymbol{u};\boldsymbol{v},\boldsymbol{v})=0\,,\quad\forall\boldsymbol{u}\in\boldsymbol{V},\forall\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\,,\\ b(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})=-b(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})\,,\quad\forall\boldsymbol{u}\in\boldsymbol{V}\,,\forall\boldsymbol{v}\,,\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega)\,.\end{cases}\tag{2.3}$$

$$\begin{cases} |\,b(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})|\le C\|\boldsymbol{u}\|\|\boldsymbol{v}\|\|\boldsymbol{w}\|\,,\quad\forall\boldsymbol{u},\,\boldsymbol{v},\,\boldsymbol{w}\in\mathbf{V}\,,\\ |\,b(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})|\le C\|\boldsymbol{u}\|\|\boldsymbol{v}\||\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}\,,\quad\forall\boldsymbol{u},\,\boldsymbol{v},\,\boldsymbol{w}\in\boldsymbol{V}\,.\end{cases}\tag{2.4}$$

(2) *The map $\mathcal{B}:\boldsymbol{V}\to\boldsymbol{V}^*\,;\,(\boldsymbol{u}\mapsto\mathcal{B}(\boldsymbol{u}))$ is differentiable, and we have*

$$\mathcal{B}'(\boldsymbol{u};\boldsymbol{v})=\frac{d}{d\lambda}\mathcal{B}(\boldsymbol{u}+\lambda\boldsymbol{v})\Big|_{\lambda=0}=\mathcal{B}(\boldsymbol{u},\boldsymbol{v})+\mathcal{B}(\boldsymbol{v},\boldsymbol{u})\,.\tag{2.5}$$

Furthermore, if we represent the corresponding adjoint of $\mathcal{B}'(\cdot\,;\cdot)$ by $\mathcal{B}'(\cdot\,;\cdot)^*$ so that

$$\langle\mathcal{B}'(\boldsymbol{u};\boldsymbol{v})^*,\boldsymbol{w}\rangle=\langle\mathcal{B}'(\boldsymbol{u};\boldsymbol{w}),\boldsymbol{v}\rangle\ ,$$

then it follows for all $\boldsymbol{w}\in\boldsymbol{V}_0$ that

$$<\mathcal{B}'(\boldsymbol{u};\boldsymbol{v})^*,\boldsymbol{w}>=\int_\Omega\sum_{i,j=1}^2 w_j\left(\frac{\partial u_i}{\partial x_j}v_i-u_i\frac{\partial v_j}{\partial x_i}\right)d\boldsymbol{x}\,.\tag{2.6}$$

From (2.5), $\mathcal{B}'(\boldsymbol{u};\boldsymbol{v})$ corresponds to the linearized form for the nonlinear convective term $\mathcal{B}(\boldsymbol{u}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ in the $\boldsymbol{v}$-direction, so that

$$\mathcal{B}'(\boldsymbol{u};\boldsymbol{v}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\,.$$

On the while, one can see from (2.6) that its adjoint $\mathcal{B}'(\boldsymbol{u};\boldsymbol{v})^*$ is represented by

$$\mathcal{B}'(\boldsymbol{u};\boldsymbol{v})^* = (\nabla\boldsymbol{u})^t\boldsymbol{v} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\,, \tag{2.7}$$

where $(\nabla\boldsymbol{u})^t$ denotes the transpose of the tensor. Also, concerned with the orthogonality relations (2.3), it is noticeable in [7] that if $\boldsymbol{u}\in\boldsymbol{V}$ satisfies $\displaystyle\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{n}\,d\boldsymbol{s} = 0$, then

$$b(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) \;=\; -\,b(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v})\,,\quad \forall\,\boldsymbol{v}\in\boldsymbol{H}^1(\Omega),\,\forall\,\boldsymbol{w}\in\boldsymbol{H}^1_0(\Omega)\,.$$

Also, we have

$$\langle\mathcal{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle \;=\; -\,\langle\mathcal{B}(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{v}\rangle\;,\quad \forall\boldsymbol{u},\,\boldsymbol{v}\in\boldsymbol{V},\,\forall\,\boldsymbol{w}\in\boldsymbol{V}_0\,. \tag{2.8}$$

We will project the system (1.1)–(1.7) into the dual space of the divergence free vector fields as in [3] and [13]. Let us denote by $\mathcal{P}$ the orthogonal projector $\mathcal{P}\,:\,\boldsymbol{L}^2(\Omega)\to\boldsymbol{H}$. It is obvious that the operator in (2.1) corresponds to $\mathcal{A} = \mathcal{P}(-\Delta)$, and the operator in (2.2) to $\mathcal{B}(\boldsymbol{u},\boldsymbol{v}) = \mathcal{P}((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})$. In perspective points of view, the major advantage we can get by applying the projector $\mathcal{P}$ to the Navier-Stokes system is that the pressure term can be excluded, so that it is reduced to the system only the velocity concerned. After finding the velocity, the pressure then can be retrieved by applying de Rham's lemma. For details, one may consult [3], [6] and [10].

According to this formulation, the Navier-Stokes system (1.1)–(1.4) can be written by

$$\begin{cases} \dfrac{\partial\boldsymbol{u}}{\partial t} + \nu\mathcal{A}\boldsymbol{u} + \mathcal{B}(\boldsymbol{u}) = \mathcal{P}\boldsymbol{f} \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\boldsymbol{u} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \boldsymbol{u} = \boldsymbol{g} \quad \text{on } (0,T)\times\Gamma_c\,, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } (0,T)\times\Gamma_0\,, \\ \boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})\,,\ \forall\,\boldsymbol{x}\in\Omega\,. \end{cases} \tag{2.9}$$

For consistency in the mathematical formulation, we assume the body force satisfies $\nabla\cdot\boldsymbol{f} = 0$ in $(0,T)\times\Omega$, so that $\mathcal{P}\boldsymbol{f} = \boldsymbol{f}$ in (2.9) as in [3].

From (1.5)–(1.7), the initial velocity $\boldsymbol{u}_0$ must satisfy the compatibility conditions

$$\boldsymbol{u}_0 \in \boldsymbol{V}\,,\ \ \boldsymbol{u}_0(\boldsymbol{s}) = \boldsymbol{g}(0,\boldsymbol{s})\ \forall\, \boldsymbol{s} \in \Gamma_c\,,\ \ \int_{\Gamma_c} \boldsymbol{u}_0 \cdot \boldsymbol{n}\, d\boldsymbol{s} = 0\,. \tag{2.10}$$

In the following theorem, we present some classical results concerning the well-posedness as well as the regularity for the time-dependent two dimensional incompressible Navier-Stokes system.

Theorem 2.2. *Let Ω be a bounded domain with the smooth boundary. Suppose $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H})$, and let $\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ and $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}_0^{1/2}(\Gamma_c))$ satisfy the threshold conditions (1.5)–(1.7).*

Then, there exists a unique admissible weak solution $\boldsymbol{u}$ of (2.9) for which the system is well posed in a sense that

$$\|\boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{V})} + \|\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{H})} \leq C\left(\|\boldsymbol{g}\|^2_{\Gamma_c} + \|\boldsymbol{u}_0\|^2 + \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{H})}\right)\,.$$

Moreover, $\dfrac{\partial \boldsymbol{u}}{\partial t}$ belongs to $L^2(0,T;\boldsymbol{H})$.

For proof, one may follow the compactness argument by employing the Galerkin approximation method as in [3], [6], [11] and [13]. It is worthwhile that if $\|\boldsymbol{g}\|^2_{\Gamma_c}$ is uniformly bounded, then $\boldsymbol{u}$ is uniformly bounded in $L^2(0,T;\boldsymbol{V})$.

In the remainder sections, we will refer to the two dimensional curl operators. Let $\mathfrak{D}(\Omega)$ denote the space of distributions in Ω. In the sense of distributions, two kinds of curl operators are introduced:

$$\vec{\nabla} \times \varphi = \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}\right) \quad \text{for } \varphi \in \mathfrak{D}(\Omega)\,,$$

and

$$\nabla \times\ \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \quad \text{for } \boldsymbol{v} = (v_1, v_2) \in (\mathfrak{D}(\Omega))^2\,.$$

One can easily check the following identities hold:

$$\nabla \times (\vec{\nabla} \times\ \varphi) = -\Delta\varphi\,,$$

and

$$\vec{\nabla} \times (\nabla \times \boldsymbol{v}) = -\Delta\boldsymbol{v} + \nabla(\nabla \cdot \boldsymbol{v})\,.$$

Hence, it is immediately followed that

$$\vec{\nabla} \times (\nabla \times \boldsymbol{v}) = -\Delta\boldsymbol{v}\,, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,. \tag{2.11}$$

3. Existence of an optimal solution

For the boundary vector fields in our need, we use the space

$$\boldsymbol{W} = \left\{ \boldsymbol{g}(t,\cdot) \in \boldsymbol{H}_0^{1/2}(\Gamma_c) \,\Big|\, \int_{\Gamma_c} \boldsymbol{g}(t,\cdot)\cdot \boldsymbol{n}\, d\boldsymbol{s} = 0\,,\ \forall\, 0 \le t \le T \right\}.$$

Then, $\boldsymbol{W}$ is a closed subspace of $\boldsymbol{H}^{1/2}(\Gamma_c)$, and the boundary condition $\boldsymbol{g}$, which is comprised with the control parameter in our case, belongs to the space $L^2(0,T;\boldsymbol{W})$. For the norm of $\boldsymbol{g}$ in $L^2(0,T;\boldsymbol{W})$, one may take

$$\|\boldsymbol{g}\|_{\Gamma_c} = \left(\int_0^T |\boldsymbol{g}|^2_{\Gamma_c}\, dt \right)^{1/2},$$

where $|\cdot|_{\Gamma_c}$ denotes the $\boldsymbol{H}^{1/2}$-norm on Γ_c. We let $\boldsymbol{H}^{-1/2}(\Gamma_c)$ denote the dual space of $\boldsymbol{H}_0^{1/2}(\Gamma_c)$, and $< \cdot,\cdot >_{\Gamma_c}$ denote the duality between $\boldsymbol{H}^{-1/2}(\Gamma_c)$ and $\boldsymbol{H}_0^{1/2}(\Gamma_c)$. We note that the duality between $L^2(0,T;\boldsymbol{W})$ and its dual space $L^2(0,T;\boldsymbol{W}^*)$ can be given by

$$< \boldsymbol{s}^*, \boldsymbol{h} >_{L^2(0,T;\boldsymbol{W}^*)} = \int_0^T < \boldsymbol{s}^*, \gamma_c^0(\boldsymbol{v}) >_{\Gamma_c} dt\,,$$

for $\boldsymbol{v} \in L^2(0,T;\boldsymbol{H}^1_{\Gamma_0}(\Omega))$ with $\gamma_c^0(\boldsymbol{v}) = \boldsymbol{h}$.

We provide a precise formulation for the control problem (Q) and prove the existence of an optimal solution. To comply with our previous discussions, we set the admissible family of sets by

$$\begin{aligned} \mathcal{U}_{ad} = \ & \{(\boldsymbol{u},\boldsymbol{g}) \in L^2(0,T;\boldsymbol{V}) \times L^2(0,T;\boldsymbol{W}) \mid \mathcal{J}(\boldsymbol{u},\boldsymbol{g}) < \infty, \\ & \qquad (\boldsymbol{u},\boldsymbol{g}) \text{ correponds to the system (2.9)}\}\,. \end{aligned}$$

Then the boundary control problem concerned with can be formulated as follows:

Given $\boldsymbol{u}_0 \in \boldsymbol{V}$, find the boundary control $\boldsymbol{g}$ and a velocity field $\boldsymbol{u}$ such that the objective functional

$$\mathcal{J}(\boldsymbol{u},\boldsymbol{g}) = \frac{\nu}{2}\int_0^T \int_\Omega |\nabla \times \boldsymbol{u}|^2 d\boldsymbol{x}dt + \frac{\epsilon}{2}\int_0^T \int_{\Gamma_c} |\boldsymbol{g}|^2\, d\boldsymbol{s}dt\,. \tag{3.1}$$

is minimized subject to $(\boldsymbol{u},\boldsymbol{g}) \in \mathcal{U}_{ad}$ satisfying (1.1)–(1.7).

Let us prove that the optimal control problem (3.1) is well posed and has at least one solution. By Theorem 2.2, the solution $\boldsymbol{u}$ for the system (2.9) can be described as a function of the control parameter as $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{g})$. Since the mapping $\boldsymbol{g} \mapsto \boldsymbol{u}(\boldsymbol{g})$ is nonlinear, the functional

$\mathcal{J}$ is nonconvex and the optimal solutions may not be unique. We will show the existence of its solution by using the built-in coercivity of the functional $\mathcal{J}$.

THEOREM 3.1. *Let $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H})$ and $\boldsymbol{u}_0 \in \boldsymbol{V}$ be given. Suppose $\boldsymbol{u}_0$ satisfies the compatibility conditions (2.10). Then, there exists at least one optimal solution $(\boldsymbol{u},\boldsymbol{g}) \in \mathcal{U}_{ad}$ which minimizes the functional (3.1), and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{g})$ satisfies $\gamma_c^0(\boldsymbol{u}) = \boldsymbol{g}$ and $\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$.*

Proof. An admissible solution can be found by first setting $\widehat{\boldsymbol{g}}(t,\cdot) = \gamma_c^0(\boldsymbol{u}_0)$ for $0 \le t \le T$, and then by solving the system

$$\begin{cases} \dfrac{\partial \widehat{\boldsymbol{u}}}{\partial t} + \nu \mathcal{A}\widehat{\boldsymbol{u}} + \mathcal{B}(\widehat{\boldsymbol{u}}) = \boldsymbol{f} \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\widehat{\boldsymbol{u}} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \widehat{\boldsymbol{u}} = \widehat{\boldsymbol{g}} \quad \text{on } (0,T)\times\Gamma_c\,, \\ \widehat{\boldsymbol{u}} = \boldsymbol{0} \quad \text{on } (0,T)\times\Gamma_0\,, \\ \widehat{\boldsymbol{u}}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})\,,\ \forall\,\boldsymbol{x}\in\Omega\,. \end{cases}$$

According to Theorem 2.2, the solution $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{g}})$ exists and belongs to $\mathcal{U}_{ad}$.

Since the set of admissible solutions $\mathcal{U}_{ad}$ is not empty and the set of the values assumed by the functional is bounded from below, there exists a minimizing sequence $\boldsymbol{g}_m \in L^2(0,T;\boldsymbol{W})$, and the corresponding sequence for the velocity $\boldsymbol{u}_m = \boldsymbol{u}(\boldsymbol{g}_m)$, where $\boldsymbol{u} = \boldsymbol{u}_m$ is a solution of the system (2.9) with $\boldsymbol{g} = \boldsymbol{g}_m$. Then since the sequence $\{\boldsymbol{g}_m\}$ is uniformly bounded in $L^2(0,T;\boldsymbol{W})$, the corresponding sequence $\{\boldsymbol{u}_m\}$ is also uniformly bounded in $L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$. Thus one can extract from the sequence $\{(\boldsymbol{u}_m,\boldsymbol{g}_m)\}$ a subsequence (denoted again by the same notation) in $L^2(0,T;\boldsymbol{V})\times L^2(0,T;\boldsymbol{W})$ which converges weakly to $(\boldsymbol{u},\boldsymbol{g})$.

Hence one can write

$$\begin{array}{lllll} \boldsymbol{g}_m & \to & \boldsymbol{g} & \text{weakly} & \text{in } L^2(0,T;\boldsymbol{H}^{1/2}(\Gamma_c))\,, \\ \boldsymbol{u}_m & \to & \boldsymbol{u} & \text{weakly} & \text{in } L^2(0,T;\boldsymbol{V})\,, \\ \boldsymbol{u}_m & \to & \boldsymbol{u} & \text{strongly} & \text{in } L^2(0,T;\boldsymbol{H})\,, \\ \gamma_c^0(\boldsymbol{u}_m) & \to & \gamma_c^0(\boldsymbol{u}) & \text{weakly} & \text{in } L^2(0,T;\boldsymbol{H}^{1/2}(\Gamma_c))\,, \\ \boldsymbol{u}_m & \to & \boldsymbol{u} & \text{weakstarly} & \text{in } L^\infty(0,T;\boldsymbol{H})\,. \end{array} \tag{3.2}$$

Concerned with the vorticity term in the objective functional, especially we have

$$\int_0^T \int_\Omega (\nabla \times \boldsymbol{u}_m) \cdot \varphi \, d\boldsymbol{x}\, dt - \int_0^T \int_\Omega \boldsymbol{u}_m \cdot (\vec{\nabla} \times \varphi)\, d\boldsymbol{x}\, dt \\ = \int_0^T \int_{\Gamma_c} (\boldsymbol{u}_m \cdot \boldsymbol{\tau})\varphi \, d\boldsymbol{s}\, dt\,, \ \forall \varphi \in L^2(0,T;H^1(\Omega))\,,$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2)$ denotes the unit tangent vector along the control boundary Γ_c. This can be derived by taking integration by parts. Hence, for every $\boldsymbol{v} \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$, relations (3.2) allow us to pass to the limit with the aid of (2.11) that

$$\begin{aligned} &\int_0^T \int_\Omega \nabla \times \boldsymbol{u}_m \cdot \nabla \times \boldsymbol{v}\, d\boldsymbol{x}\, dt \\ &= \int_0^T \int_\Omega \boldsymbol{u}_m \cdot \vec{\nabla} \times (\nabla \times \boldsymbol{v})\, d\boldsymbol{x}\, dt + \int_0^T \int_{\Gamma_c} (\boldsymbol{u}_m \cdot \boldsymbol{\tau})(\nabla \times \boldsymbol{v})\, d\boldsymbol{s}\, dt \\ &= \int_0^T \int_\Omega \boldsymbol{u}_m \cdot (-\Delta \boldsymbol{v})\, d\boldsymbol{x}\, dt + \int_0^T \int_{\Gamma_c} (\boldsymbol{u}_m \cdot \boldsymbol{\tau})(\nabla \times \boldsymbol{v})\, d\boldsymbol{s}\, dt \\ &\longrightarrow \int_0^T \int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{v})\, d\boldsymbol{x}\, dt + \int_0^T \int_{\Gamma_c} (\boldsymbol{u} \cdot \boldsymbol{\tau})(\nabla \times \boldsymbol{v})\, d\boldsymbol{s}\, dt \\ &\qquad = \int_0^T \int_\Omega \nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}\, d\boldsymbol{x}\, dt\,, \end{aligned}$$

which yields

$$\nabla \times \boldsymbol{u}_m \to \nabla \times \boldsymbol{u} \quad \text{weakly in } L^2(0,T;\boldsymbol{H})\,.$$

We also note that the Young's inequality produces

$$\begin{aligned} \int_0^T \int_\Omega |\nabla \times \boldsymbol{u}|^2\, d\boldsymbol{x}\, dt &= \int_0^T \int_\Omega \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)^2 d\boldsymbol{x}\, dt \\ &\le C \int_0^T \|\boldsymbol{u}\|^2\, dt\,. \end{aligned}$$

This implies that the cost functional $\mathcal{J}$ is strongly continuous and lower semicontinuous. Hence passing to the limit in $\mathcal{U}_{ad}$, we have

$$\mathcal{J}(\boldsymbol{u}, \boldsymbol{g}) \le \liminf_{n \to \infty} \mathcal{J}(\boldsymbol{u}_m, \boldsymbol{g}_m)\,,$$

so that the functional is minimized at $(\boldsymbol{u}, \boldsymbol{g})$.

To complete the proof, it remains to show that $(\boldsymbol{u}, \boldsymbol{g})$ belongs to the admissible set $\mathcal{U}_{ad}$. First of all, we need to note: a priori estimate by using the compactness argument in the fractional time order Sobolev space yields the strong convergence of $\boldsymbol{u}_m$ to $\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{H})$ as stated above, see [3] and [13] for details.

Note that for $\boldsymbol{w} \in L^2(0,T;\boldsymbol{V}_0)$, we have

$$\begin{aligned} & b(\boldsymbol{u}_m;\boldsymbol{w},\boldsymbol{u}_m) - b(\boldsymbol{u};\boldsymbol{w},\boldsymbol{u}) \\ & \quad = b(\boldsymbol{u}_m - \boldsymbol{u};\boldsymbol{w},\boldsymbol{u}_m - \boldsymbol{u}) + b(\boldsymbol{u}_m - \boldsymbol{u};\boldsymbol{w},\boldsymbol{u}) + b(\boldsymbol{u};\boldsymbol{w},\boldsymbol{u}_m - \boldsymbol{u})\,. \end{aligned}$$

Using (2.4), (2.8), the strong convergence of $\boldsymbol{u}_m$ to $\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{H})$, and the Sobolev embedding $\boldsymbol{H}^1(\Omega) \subset \boldsymbol{L}^4(\Omega)$, we have for every $\boldsymbol{w} \in L^2(0,T;\boldsymbol{V}_0)$

$$\begin{aligned} \langle \mathcal{B}(\boldsymbol{u}_m,\boldsymbol{u}_m),\boldsymbol{w}\rangle \; &= -\,\langle \mathcal{B}(\boldsymbol{u}_m,\boldsymbol{w}),\boldsymbol{u}_m\rangle \\ & \qquad \longrightarrow \; -\,\langle \mathcal{B}(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{u}\rangle \; = \; \langle \mathcal{B}(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{w}\rangle \;. \end{aligned}$$

Hence $\boldsymbol{u}$ satisfies the equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nu\,\mathcal{A}\boldsymbol{u} + \mathcal{B}(\boldsymbol{u}) = \boldsymbol{f}\,.$$

Since $\boldsymbol{g}_m = \gamma_c^0(\boldsymbol{u}_m)$ weakly converges to $\boldsymbol{g}$ and the lifting $\boldsymbol{u}_m$ of $\boldsymbol{g}_m$ strongly converges to $\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{V})$, by the continuity of the trace([4]) it follows that $\gamma_c^0(\boldsymbol{u}) = \boldsymbol{g}$, which implies that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{g})$.

Finally, we need to show that $\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0$. Note that $\boldsymbol{u}_m \in L^2(0,T;\boldsymbol{V})$ is a solution of an initial problem for the parabolic system

$$\left\{ \begin{array}{ll} \dfrac{\partial \boldsymbol{u}_m}{\partial t} & = \boldsymbol{f} - \nu\,\mathcal{A}\boldsymbol{u}_m - \mathcal{B}(\boldsymbol{u}_m) \quad \text{on } L^2(0,T;\boldsymbol{V}^*)\,, \\ \boldsymbol{u}_m(0) & = \boldsymbol{u}_0 \quad \text{for every } m\,. \end{array} \right. \tag{3.3}$$

Since $\mathcal{C}^1(0,T) \times \boldsymbol{\mathcal{V}}_0(\Omega)$ is dense in $L^2(0,T;\boldsymbol{V}_0)$, multiplying (3.3) by a trial function $\varphi(t)\boldsymbol{v}$ such that $\varphi \in \mathcal{C}^1(0,T)$ with $\varphi(T) = 0$ and $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}_0$, and taking integration by parts, we have

$$\begin{aligned} -\int_0^T < \boldsymbol{u}_m,\ \ \varphi'(t)\boldsymbol{v} > \,dt = \langle \boldsymbol{u}_0, \varphi(0)\boldsymbol{v}\rangle + \int_0^T \langle \boldsymbol{f}, \varphi(t)\boldsymbol{v}\rangle \,dt \\ -\,\nu \int_0^T a(\boldsymbol{u}_m, \varphi(t)\boldsymbol{v})\,dt - \int_0^T b(\boldsymbol{u}_m;\boldsymbol{u}_m,\varphi(t)\boldsymbol{v})\,dt\,. \end{aligned}$$

After passing to the limit and integrating of the first term by parts, this yields

$$\langle (\boldsymbol{u}(0) - \boldsymbol{u}_0), \boldsymbol{v}\rangle\,\varphi(0) = 0\,, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0\,.$$

Hence, if we choose φ with $\varphi(0) \neq 0$, it follows $\boldsymbol{u}(0) = \boldsymbol{u}_0$. $\square$

4. The first order derivation and the optimality system

Let us examine the question of what relations characterizes an optimal solution. For this purpose, we need to investigate the differentiability of the objective functional as well as the corresponding velocity vector field with respect to the control parameter in order to derive the first order necessary conditions for an optimal solution. Since the velocity $\boldsymbol{u}$ can be described as a function of the control parameter $\boldsymbol{g}$, we recast the functional $\mathcal{J}(\cdot,\cdot)$ equivalently into

$$\mathcal{J}(\boldsymbol{g}) = \mathcal{J}(\boldsymbol{u}(\boldsymbol{g}),\boldsymbol{g}) \text{ for } \boldsymbol{g} \in L^2(0,T;\boldsymbol{W}) .$$

Let us investigate the rate of variation of $\mathcal{J}(\boldsymbol{g})$ with respect to the control parameter $\boldsymbol{g}$. The rate of variation at $\boldsymbol{g}$ in the direction of $\boldsymbol{h}$ can be measured as a directional semi-derivative

$$\mathcal{D}\mathcal{J}(\boldsymbol{g};\boldsymbol{h}) = \frac{d}{d\lambda}\mathcal{J}(\boldsymbol{g}+\lambda\boldsymbol{h})\Big|_{\lambda=0} .$$

Whenever $\boldsymbol{h} \mapsto \mathcal{D}\mathcal{J}(\boldsymbol{g};\boldsymbol{h})$ is linear and continuous, the rate of variation $\mathcal{D}(\boldsymbol{g};\boldsymbol{h})$ is called the Gateaux-derivative at $\boldsymbol{g}$ in the $\boldsymbol{h}$-direction. For all $\boldsymbol{g} \in L^2(0,T;\boldsymbol{W})$, the first order necessary condition is available if the map

$$\boldsymbol{u} \,:\, L^2(0,T;\boldsymbol{W}) \to L^2(0,T;\boldsymbol{V}) ; \quad \Big(\boldsymbol{g} \mapsto \boldsymbol{u}(\boldsymbol{g})\Big)$$

is Gateaux differentiable. In the following theorem, we prove the state solution $\boldsymbol{u}$ is strictly differentiable with respect to the control parameter.

THEOREM 4.1. *Let $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H})$ and $\boldsymbol{u}_0 \in \boldsymbol{V}$ be given. Suppose $(\boldsymbol{u}_0,\boldsymbol{g})$ satisfies the threshold conditions. Then, the mapping*

$$\boldsymbol{u} \,:\, L^2(0,T;\boldsymbol{W}) \to L^2(0,T;\boldsymbol{V}) ; \quad \Big(\boldsymbol{g} \mapsto \boldsymbol{u}(\boldsymbol{g})\Big)$$

is differentiable. Furthermore, if we represent the Gateaux-derivative of $\boldsymbol{u}$ at $\boldsymbol{g}$ in the $\boldsymbol{h}$-direction by $\boldsymbol{u}'(\boldsymbol{h}) \equiv \mathcal{D}\boldsymbol{u}(\boldsymbol{g};\boldsymbol{h})$, then $\boldsymbol{u}' = \boldsymbol{u}'(\boldsymbol{h})$ is the *solution of the linearized problem*

$$
(4.1)\quad \begin{cases} \dfrac{\partial \boldsymbol{u}'}{\partial t} + \nu \mathcal{A}\boldsymbol{u}' + \mathcal{B}'(\boldsymbol{u}(\boldsymbol{g}); \boldsymbol{u}') = \boldsymbol{0} & \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\boldsymbol{u}' = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \boldsymbol{u}' = \boldsymbol{h} \quad \text{on } (0,T)\times\Gamma_c\,, \\ \boldsymbol{u}' = \boldsymbol{0} \quad \text{on } (0,T)\times\Gamma_0\,, \\ \boldsymbol{u}'(0,\boldsymbol{x}) = \boldsymbol{0}\,,\ \forall\,\boldsymbol{x}\in\Omega\,, \end{cases}
$$

and $\boldsymbol{u}'$ *belongs to* $L^2(0,T;\boldsymbol{V})\cap L^\infty(0,T;\boldsymbol{H})$*, where* $\mathcal{B}'(\boldsymbol{u};\boldsymbol{v})$ *corresponds to the linearized form for the nonlinear convective term* $\mathcal{B}(\boldsymbol{u}) = \mathcal{P}((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})$ *in the* $\boldsymbol{v}$*-direction, so that* $\mathcal{B}'(\boldsymbol{u};\boldsymbol{v}) = \mathcal{P}((\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u})$*.*

Proof. In [9], it has been shown that

$$
\|\boldsymbol{u}(\boldsymbol{g}+\lambda\boldsymbol{h}) - \boldsymbol{u}(\boldsymbol{g}) - \lambda\boldsymbol{u}'(\boldsymbol{h})\|_{L^2(0,T;\boldsymbol{V})} \le C\,|\lambda|^k \quad \text{for some } k>1\,,
$$

so that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{g})$ is strictly differentiable. □

We now turn to the differentiability of the functional $\mathcal{J}$. As a result, we will get a first order necessary conditions for the optimal solution to the problem. For this purpose, we need the following preliminary result.

LEMMA 4.2. *Let* $\boldsymbol{u}_0 \in \boldsymbol{V}$ *and* $\boldsymbol{h} \in L^2(0,T;\boldsymbol{W})$ *be given. Suppose* $\boldsymbol{u}' = \boldsymbol{u}'(\boldsymbol{h})$ *is a solution of the system (4.1), then for every* $\boldsymbol{e}\in L^2(0,T;\boldsymbol{H})$*, we have*

$$
\int_0^T\!\!\int_\Omega \boldsymbol{e}\cdot\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x}dt = \int_0^T\!\!\int_{\Gamma_c} -\nu\frac{\partial\widetilde{\boldsymbol{w}}}{\partial\boldsymbol{n}}\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\,,
$$

where $\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{w}}(\boldsymbol{e})$ *is the solution of the adjoint system*

$$
(4.2)\quad \begin{cases} -\dfrac{\partial\widetilde{\boldsymbol{w}}}{\partial t} + \nu\,\mathcal{A}\widetilde{\boldsymbol{w}} + \mathcal{B}'(\boldsymbol{u}(\boldsymbol{g});\widetilde{\boldsymbol{w}})^* = \boldsymbol{e} \ \ \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\widetilde{\boldsymbol{w}} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \widetilde{\boldsymbol{w}} = \boldsymbol{0} \quad \text{on } (0,T)\times\Gamma\,, \\ \widetilde{\boldsymbol{w}}(T,\boldsymbol{x}) = \boldsymbol{0}\,, \quad \forall\,\boldsymbol{x}\in\Omega\,. \end{cases}
$$

Here, $\mathcal{B}'(\boldsymbol{u}(\boldsymbol{g});\widetilde{\boldsymbol{w}})^*$ *denotes the adjoint of* $\mathcal{B}'$ *in a sense that*

$$
\langle\mathcal{B}'(\boldsymbol{u};\widetilde{\boldsymbol{w}})^*,\boldsymbol{v}\rangle = \langle\mathcal{B}'(\boldsymbol{u};\boldsymbol{v}),\widetilde{\boldsymbol{w}}\rangle\,,\ \forall\,\widetilde{\boldsymbol{w}}\in\boldsymbol{V}_0\,,
$$

so that $\mathcal{B}'(\boldsymbol{u};\widetilde{\boldsymbol{w}})^* = \mathcal{P}((\nabla\boldsymbol{u})^t\widetilde{\boldsymbol{w}} - (\boldsymbol{u}\cdot\nabla)\widetilde{\boldsymbol{w}})$*.*

Proof. Since $\boldsymbol{e} \in L(0,T;\boldsymbol{H})$, the adjoint system (4.2) has a solution in $L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$ (c.f. [3], [13]). Also, since the traces $\gamma_c^0(\boldsymbol{u}') = \boldsymbol{h}$ and $\gamma_c^1(\widetilde{\boldsymbol{w}}) = \boldsymbol{0}$, the Green's second identity yields that

$$
\begin{aligned}
&< \mathcal{A}\widehat{\boldsymbol{w}}, \boldsymbol{u}' > - < \mathcal{A}\boldsymbol{u}', \widehat{\boldsymbol{w}} > \\
&\qquad\qquad = - < \gamma_c^1(\widehat{\boldsymbol{w}}), \gamma_c^0(\boldsymbol{u}') >_{\Gamma_c} + < \gamma_c^1(\boldsymbol{u}'), \gamma_c^0(\widehat{\boldsymbol{w}}) >_{\Gamma_c} \\
&\qquad\qquad = - < \gamma_c^1(\widehat{\boldsymbol{w}}), \boldsymbol{h} >_{\Gamma_c} .
\end{aligned}
$$

Hence using the facts that $\boldsymbol{u}'(0,\boldsymbol{x}) = 0 = \widetilde{\boldsymbol{w}}(T,\boldsymbol{x})$, one can derive the following estimations:

$$
\begin{aligned}
&\int_0^T \int_\Omega \boldsymbol{e} \cdot \boldsymbol{u}'(\boldsymbol{h})\, d\boldsymbol{x} dt \\
&= \int_0^T \left\langle \Big(-\frac{\partial \widetilde{\boldsymbol{w}}}{\partial t} + \nu \mathcal{A}\widetilde{\boldsymbol{w}} + \mathcal{B}'(\boldsymbol{u};\widetilde{\boldsymbol{w}})^* \Big), \boldsymbol{u}' \right\rangle dt \\
&= \int_0^T \left\langle -\frac{\partial \widetilde{\boldsymbol{w}}}{\partial t}, \boldsymbol{u}' \right\rangle dt + \int_0^T \langle \nu \mathcal{A}\widetilde{\boldsymbol{w}}, \boldsymbol{u}' \rangle\, dt + \int_0^T \langle \mathcal{B}'(\boldsymbol{u},\widetilde{\boldsymbol{w}})^*, \boldsymbol{u}' \rangle\, dt \\
&= \int_0^T \left\langle \frac{\partial \boldsymbol{u}'}{\partial t}, \widetilde{\boldsymbol{w}} \right\rangle dt + \int_0^T \langle \nu \mathcal{A}\boldsymbol{u}', \widetilde{\boldsymbol{w}} \rangle\, dt \\
&\qquad\qquad - \int_0^T \left\langle \nu \gamma_c^1(\widetilde{\boldsymbol{w}}), \boldsymbol{h} \right\rangle_{\Gamma_c} + \int_0^T \langle \mathcal{B}'(\boldsymbol{u};\boldsymbol{u}'), \widetilde{\boldsymbol{w}} \rangle\, dt \\
&= \int_0^T \left\langle \Big(\frac{\partial \boldsymbol{u}'}{\partial t} + \nu \mathcal{A}\boldsymbol{u}' + \mathcal{B}'(\boldsymbol{u};\boldsymbol{u}') \Big), \widetilde{\boldsymbol{w}} \right\rangle dt - \int_0^T \left\langle \nu \gamma_c^1(\widetilde{\boldsymbol{w}}), \boldsymbol{h} \right\rangle_{\Gamma_c} dt \\
&= \int_0^T - \left\langle \nu \gamma_c^1(\widetilde{\boldsymbol{w}}), \boldsymbol{h} \right\rangle_{\Gamma_c} dt \\
&= \int_0^T \int_{\Gamma_c} - \nu \frac{\partial \widetilde{\boldsymbol{w}}}{\partial \boldsymbol{n}} \cdot \boldsymbol{h}\, d\boldsymbol{s} dt .
\end{aligned}
$$

The transition of $-\dfrac{d\widetilde{\boldsymbol{w}}}{dt}$ into $\dfrac{d\boldsymbol{u}'}{dt}$ is followed by applying integration by parts and $\displaystyle\int_\Omega \widetilde{\boldsymbol{w}}(t,\boldsymbol{x}) \cdot \boldsymbol{u}'(t,\boldsymbol{x}) \Big|_{t=0}^{T} d\boldsymbol{x} = 0$. □

We are now ready to establish the differential framework for the objective functional $\mathcal{J}$.

THEOREM 4.3. *Let $(\boldsymbol{u},\boldsymbol{g})$ be an optimal solution for the problem (3.1) with $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{g})$. Then, the Gateaux derivative for the functional $\mathcal{J}$*

at $\boldsymbol{g}$ *is given by*

$$\text{(4.3)} \quad \mathcal{DJ}(\boldsymbol{g};\boldsymbol{h}) = \int_0^T \Big\langle \Big(-\nu\gamma_c^1(\boldsymbol{w}) + \gamma_c^0(\nabla\times\boldsymbol{u})\,\boldsymbol{\tau} + \epsilon\boldsymbol{g}\Big),\,\boldsymbol{h}\Big\rangle_{\Gamma_c}\,dt,$$

where $\boldsymbol{w}$ *is the solution of the adjoint system*

$$\text{(4.4)} \quad \begin{cases} -\dfrac{\partial\boldsymbol{w}}{\partial t} + \nu\,\mathcal{A}\boldsymbol{w} + \mathcal{B}'(\boldsymbol{u};\boldsymbol{w})^* = \mathcal{A}\boldsymbol{u} \quad \textit{in } (0,T)\times\Omega\,, \\ \nabla\cdot\boldsymbol{w} = 0 \quad \textit{in } (0,T)\times\Omega\,, \\ \boldsymbol{w} = \boldsymbol{0} \quad \textit{on } (0,T)\times\Gamma\,, \\ \boldsymbol{w}(T,\boldsymbol{x}) = \boldsymbol{0}\,,\ \forall\,\boldsymbol{x}\in\Omega\,. \end{cases}$$

Proof. In the two dimensional domain, the following relation holds true between the unit normal vector $\boldsymbol{n} = (n_1, n_2)$ and the unit tangent vector $\boldsymbol{\tau} = (\tau_1, \tau_2)$:

$$\boldsymbol{n} = (n_1, n_2) = (\tau_2, -\tau_1)\,.$$

Hence, if $\varphi \in H^1(\Omega)$ and $\boldsymbol{u}' = \boldsymbol{u}'(\boldsymbol{h})$ is the solution of the system (4.1), the Green's formula yields

$$\int_\Omega \varphi\cdot\nabla\times\boldsymbol{u}'\,d\boldsymbol{x} = \int_\Omega \vec{\nabla}\times\varphi\cdot\boldsymbol{u}'\,d\boldsymbol{x} + \int_{\Gamma_c}(\varphi\,\boldsymbol{\tau})\cdot\boldsymbol{h}\,d\boldsymbol{s}\,.$$

It is also noteworthy as an analogous result in [2] that the curl operator can be generalized into tangential vector fields on the boundary of a three dimensinal Lipschitz domain.

If we now evaluate the Gateaux derivative at $\boldsymbol{g}$ in the $\boldsymbol{h}$-direction, from the above considerations it follows that

$$\begin{aligned}
&\frac{d}{d\lambda}\mathcal{J}(\boldsymbol{g}+\lambda\boldsymbol{h})\Big|_{\lambda=0}\\
&=\nu\int_0^T\int_\Omega \nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x}dt+\epsilon\int_0^T\int_{\Gamma_c}\boldsymbol{g}\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\\
&=\nu\int_0^T\int_\Omega \vec{\nabla}\times(\nabla\times\boldsymbol{u})\cdot\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x}dt+\nu\int_0^T\int_{\Gamma_c}(\nabla\times\boldsymbol{u})\,\boldsymbol{\tau}\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\\
&\qquad\qquad+\epsilon\int_0^T\int_{\Gamma_c}\boldsymbol{g}\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\\
&=\nu\int_0^T\int_\Omega -\Delta\boldsymbol{u}\cdot\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x}dt+\nu\int_0^T\int_{\Gamma_c}(\nabla\times\boldsymbol{u})\,\boldsymbol{\tau}\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\\
&\qquad\qquad+\epsilon\int_0^T\int_{\Gamma_c}\boldsymbol{g}\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\,.
\end{aligned}$$

The relation (2.11) for the curl operator in $\mathbb{R}^2$ has been used. If we replace $\boldsymbol{e}$ by $-\Delta\boldsymbol{u}$ in (4.2), by Lemma 4.2 the term $\int_\Omega -\Delta\boldsymbol{u}\cdot\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x}$ can be written in the form

$$\int_\Omega -\Delta\boldsymbol{u}\cdot\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x}=\int_{\Gamma_c}-\nu\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\cdot\boldsymbol{h}\,d\boldsymbol{s}\,,$$

where $\boldsymbol{w}$ is the solution of the system (4.4). Therefore, the Gateaux derivative for the functional $\mathcal{J}$ can be expressed by the simplified form of the force effected along the control boundary Γ_c as

$$\begin{aligned}
\mathcal{DJ}(\boldsymbol{g};\boldsymbol{h}) &=\int_0^T\int_\Omega -\nu\Delta\boldsymbol{u}\cdot\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x}dt+\int_0^T\int_{\Gamma_c}\nu(\nabla\times\boldsymbol{u})\,\boldsymbol{\tau}\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\\
&\qquad+\epsilon\int_0^T\int_{\Gamma_c}\boldsymbol{g}\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\\
&=\int_0^T\int_{\Gamma_c}\Big(-\nu\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}+\nu(\nabla\times\boldsymbol{u})\,\boldsymbol{\tau}+\epsilon\boldsymbol{g}\Big)\cdot\boldsymbol{h}\,d\boldsymbol{s}dt\,.
\end{aligned}$$

This completes the proof, for (4.3) corresponds to its variational formulation. □

It is remarkable that in the three dimensional space we have some difficulties in evaluating the Gateaux derivative of the functional involving the vorticity term. The variation of $\mathcal{J}$ at $\boldsymbol{g}$ in the direction of $\boldsymbol{h}$ is followed by

$$
\begin{aligned}
&\mathcal{D}\mathcal{J}(\boldsymbol{g};\boldsymbol{h})\\
&= \int_0^T \int_\Omega \vec{\nabla}\times\boldsymbol{u}\cdot\vec{\nabla}\times\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x} + \epsilon\int_0^T\int_{\Gamma_c}\boldsymbol{g}\cdot\boldsymbol{h}\,d\boldsymbol{s}\\
&= \int_0^T \int_\Omega \vec{\nabla}\times(\vec{\nabla}\times\boldsymbol{u})\cdot\boldsymbol{u}'(\boldsymbol{h})\,d\boldsymbol{x} + \int_0^T\int_{\Gamma_c}\vec{\nabla}\times\boldsymbol{u}\cdot\boldsymbol{u}'(\boldsymbol{h})\times\boldsymbol{n}\,d\boldsymbol{s}\\
&\qquad\qquad + \epsilon\int_0^T\int_{\Gamma_c}\boldsymbol{g}\cdot\boldsymbol{h}\,d\boldsymbol{s}\,.
\end{aligned}
$$

Unlike the two dimensional case, we have no other way to detach the $\boldsymbol{h}$ from

$$\int_{\Gamma_c}\vec{\nabla}\times\boldsymbol{u}\cdot\boldsymbol{u}'(\boldsymbol{h})\times\boldsymbol{n}\,d\boldsymbol{s}\,.$$

Hence, we are faced with difficulties getting the differential framework dealt with the boundary control problem for the vorticity minimization.

By (2.7), the adjoint system (4.4) constitutes a formulation of the equations

$$
\text{(4.5)}\quad
\begin{cases}
-\dfrac{\partial\boldsymbol{w}}{\partial t} - \nu\,\Delta\boldsymbol{w} + (\nabla\boldsymbol{u})^t\boldsymbol{w} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\\
\qquad\qquad\qquad + \nabla q = -\Delta\boldsymbol{u} \;\text{ in } (0,T)\times\Omega\,,\\
\nabla\cdot\boldsymbol{w} = 0 \;\text{ in } (0,T)\times\Omega\,,\\
\boldsymbol{w} = \boldsymbol{0} \;\text{ on } (0,T)\times\Gamma\,,\\
\boldsymbol{w}(T,\boldsymbol{x}) = \boldsymbol{0}\,,\;\forall\,\boldsymbol{x}\in\Omega\,.
\end{cases}
$$

In (4.5), q corresponds to the adjoint variable for the pressure p, which can be also identified by de Rham's Lemma. It should be remarked that the adjoint system has to be solved by the backward time steps.

According to Theorem 4.3, the differential framework for the gradient can be represented by

$$
\begin{aligned}
&\langle\nabla\mathcal{J}(\boldsymbol{g}),\boldsymbol{h}\rangle_{L^2(0,T;\boldsymbol{W}^*)}\\
&\quad = \int_0^T \Big\langle\Big(-\nu\gamma_c^1(\boldsymbol{w}) + \nu\gamma_c^0(\nabla\times\boldsymbol{u})\,\boldsymbol{\tau} + \epsilon\boldsymbol{g}\Big),\boldsymbol{h}\Big\rangle_{\Gamma_c}\,dt\,,
\end{aligned}
$$

so that the gradient of $\mathcal{J}$ can be written by

$$\nabla \mathcal{J}(\boldsymbol{g}) = -\nu \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} + \nu(\nabla \times \boldsymbol{u})\,\boldsymbol{\tau} + \epsilon \boldsymbol{g} \in L^2(0,T;\boldsymbol{W}^*)$$

along $(0,T)\times\Gamma_c$. This is the key factor for the first order necessary conditions to find an optimal solution for our boundary control problem to minimize the vorticity due to the flow. The candidate for the minimizer necessarily comes from the critical points of the objective functional $\mathcal{J}$, so that the relation

$$-\nu \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} + \nu(\nabla \times \boldsymbol{u})\,\boldsymbol{\tau} + \epsilon \boldsymbol{g} = \boldsymbol{0} \quad \text{along } (0,T)\times\Gamma_c$$

provides the control actuator along the control boundary Γ_c.

References

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