ON THE ANTICYCLOTOMIC $\mathbb{Z}_p$-EXTENSION OF AN IMAGINARY QUADRATIC FIELD

JANGHEON OH

ABSTRACT. We prove that if a subfield of the Hilbert class field of an imaginary quadratic field $k$ meets the anticyclotomic $\mathbb{Z}_p$-extension $k^a_\infty$ of $k$, then it is contained in $k^a_\infty$. And we give an example of an imaginary quadratic field $k$ with $\lambda_3(k^a_\infty) \geq 8$.

1. Introduction

An abelian extension $L$ of $k$ is called an anti-cyclotomic extension of $k$ if it is Galois over $\mathbb{Q}$, and $\text{Gal}(k/\mathbb{Q})$ acts on $\text{Gal}(L/k)$ by $-1$. For each prime number $p$, the compositum $K$ of all $\mathbb{Z}_p$-extensions over $k$ becomes a $\mathbb{Z}_p^2$-extension, and $K$ is the compositum of the cyclotomic $\mathbb{Z}_p$-extension $k^c_\infty$ and the anti-cyclotomic $\mathbb{Z}_p$-extension $k^a_\infty$ of $k$.

The layers $k^c_n$ of the cyclotomic $\mathbb{Z}_p$-extension are well understood. Since the Hilbert class field of $k$ is an anti-cyclotomic extension of $k$, determination of the first layer of the anti-cyclotomic $\mathbb{Z}_p$-extension becomes complicated as the $p$-rank of the $p$-Hilbert class field of $k$ becomes larger. In the papers [3,5,6], using Kummer theory and class field theory, we constructed the first layer $k^a_1$ of the anti-cyclotomic $\mathbb{Z}_p$-extension of $k$ under the assumption that the 3-part of Hilbert class field $H_k$ of $k$ is 3-elementary. A criterion on linearly disjointness of $k^a_1$ and $H_k$ over $k$ is

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proved in [4] under the assumption. In this paper, we prove the criterion without the assumption. See Corollary 1 of this paper.

Contrary to the case of the cyclotomic $\mathbb{Z}_p$-extension, the lambda invariant $\lambda_p(k^\infty_n)$ of the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field is not well known. Few examples of computation of $\lambda_p(k^\infty_n)$ are given. Following the idea of Fujii [1], we give an example of $k$ with $\lambda_3(k^\infty_n) \geq 8$.

2. Proof of Theorems

Let $p$ be an odd prime number. Throughout this section, we denote by $H_k$, $h_k$, $A_k$, and $M_k$ the $p$-part of Hilbert class field, the $p$-class number, $p$-part of ideal class group, and the maximal abelian $p$-extension of an imaginary quadratic field $k$ unramified outside above $p$, respectively. The first layer of the anti-cyclotomic $\mathbb{Z}_p$-extension of $k$ may be or may not be contained in the $p$-Hilbert class field of $k$. The following theorem and the criterion in [4] gives an answer for this question. We define $\text{rank}_{\mathbb{Z}/p\mathbb{Z}} A$ to be the dimension of $A/A^p$ over $\mathbb{Z}/p\mathbb{Z}$ for any abelian group $A$. Note that $K \cap H_k = k^\infty_n \cap H_k$.

**Theorem 1.** Let $d \not\equiv 3 \mod 9$ be a square free positive integer, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field. Let $L$ be a subfield of $H_k$ which satisfies the following properties:

$$H_k \cap k^\infty_n = k^\infty_n \leq L(n \geq 1), \quad \text{Gal}(L/k) \text{ is cyclic}.$$ 

Then

$$L = k^\infty_n.$$ 

**Proof.** Assume that $k^\infty_n \neq L$. Then there exists a ramified extension of $k$ of degree $p$ which becomes unramified over $k^\infty_n$. By class field theory, we see that

$$\text{Gal}(M_k/H_k) \simeq \left( \prod_{p|p} U_{1,p} \right),$$

where $U_{1,p}$ is the local units of $k$ which is congruent to 1 mod $p$. However, by the condition of Theorem 1, there is no $p$-torsion point in $\prod_{p|p} U_{1,p}$, which contradicts to the fact that the ramified extension of $k$ of degree $p$ exists. This completes the proof. \qed
By Theorem 1 one can easily prove the following corollary, which was proved in [4] with the assumption that $A_{\mathbb{Q}(\sqrt{-d})}$ is 3-elementary, without the assumption. In fact, the following equivalence

$$H_k \cap k_\infty^a = k \iff \text{rank}_{\mathbb{Z}/3} X_{k,\chi} = 1 + \text{rank}_{\mathbb{Z}/3} A_k$$

in [4] holds without the assumption by Theorem 1. Here

$$X_k := \text{Gal}(M_k/k)/p\text{Gal}(M_k/k)$$

and $X_{k,\chi}$ be the $\chi$-component of $X_k$ for the nontrivial character $\chi$ of $\text{Gal}(k/\mathbb{Q})$.

**COROLLARY 1.** Let $d \neq 3 \pmod{9}$ be a square free positive integer, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field and $k_\infty^a$ the anti-cyclotomic $\mathbb{Z}_3$-extension over $k$. Then

$$H_k \cap k_\infty^a = k \iff \text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-3d})} = \text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})}.$$ 

By following the idea of Fujii [1], we give an example of an imaginary quadratic field with large invariant $\lambda_3(k_\infty^a)$.

**THEOREM 2.**

$$\lambda_3(k_\infty^a) \geq 8,$$

where $k = \mathbb{Q}(\sqrt{-1423})$.

**Proof.** Denote by $K_3^a$ the compositum of all $\mathbb{Z}_3$-extensions of $k_3^a$. First note that the class number of $\mathbb{Q}(\sqrt{3*1423})$ is one. Hence, by Theorem 3 below, $H_k \subset k_\infty^a$. Since the class number of $k$ is 9, $H_k = k_3^a$. By simple computation, we see that 3 stays prime in $k$. The definition of anticyclotomic extension and class field theory shows that $\mathfrak{p}_3$, the prime of $k$ above 3, splits completely in $k_3^a$. Note that the $\mathbb{Z}_3$-rank of $\text{Gal}(K_3^a/k_3^a)$ is 10. Since the inertia group of primes of $k_3^a$ above 3 is isomorphic to $\mathbb{Z}_3^2$ and $K/k$ is abelian, the extension $K_3^a/K$ is unramified everywhere. Hence the maximal abelian 3-extension of $k_3^a$ contains $K_3^a$, and the galois group of $K_3^a$ over $K$ is isomorphic to $\mathbb{Z}_3^8$. This completes the proof. \(\square\)

The following theorem is given in [2].

**THEOREM 3.** If $p = 3$ and $d \neq 3 \pmod{9}$, then $H_k \subset k_\infty^a$ if and only if the class number of $\mathbb{Q}(\sqrt{3d})$ is not divisible by 3.
References


Jangheon Oh
Faculty of Mathematics and Statistics
Sejong University
Seoul 143-747, Korea
E-mail: oh@sejong.ac.kr