COMPLEX FACTORIZATIONS OF THE GENERALIZED FIBONACCI SEQUENCES \{q_n\}

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Abstract. In this note, we consider a generalized Fibonacci sequence \{q_n\}. Then give a connection between the sequence \{q_n\} and the Chebyshev polynomials of the second kind \(U_n(x)\). With the aid of factorization of Chebyshev polynomials of the second kind \(U_n(x)\), we derive the complex factorizations of the sequence \{q_n\}.

1. Introduction

For any integer \(n \geq 0\), the well-known Fibonacci sequence \(\{F_n\}\) is defined by the second order linear recurrence relation \(F_{n+2} = F_{n+1} + F_n\), where \(F_0 = 0\) and \(F_1 = 1\). The Fibonacci sequence has been generalized in many ways, for example, by changing the recurrence relation (see [8]), by changing the initial values (see [4, 5]), by combining these two techniques (see [3]), and so on.

In [2], Edson and Yayenie defined a further generalized Fibonacci sequence \(\{q_n\}\) depending on two real parameters used in a non-linear (piecewise linear) recurrence relation, namely,

\[
q_n = a^{1-\xi(n)}b^{\xi(n)}q_{n-1} + q_{n-2} \quad (n \geq 2)
\]
with initial values $q_0 = 0$ and $q_1 = 1$, where $a$ and $b$ are positive real numbers and

$$
\xi(n) = \begin{cases} 
0 & \text{if } n \text{ is even} \\
1 & \text{if } n \text{ is odd}
\end{cases}
$$

is the parity function. Also, the authors showed that the terms of the sequence $\{q_n\}$ are given by the extended Binet’s formula

$$
q_n = \left( \frac{a^{1-\xi(n)} - \beta^n}{(ab)^{n-\xi(n)/2}} \right) \alpha^n - \beta^n,
$$

where $\alpha$ and $\beta$ are roots of the quadratic equation $x^2 - abx - ab = 0$ and $\alpha > \beta$.

These sequences arise in a natural way in the study of continued fractions of quadratic irrationals and combinatorics on words or dynamical system theory. Some well-known sequences are special cases of this generalization. The Fibonacci sequence is a special case of $\{q_n\}$ with $a = b = 1$. When $a = b = 2$, we obtain the Pell’s sequence $\{P_n\}$. Even further, if we set $a = b = k$ for some positive integer $k$, we obtain the $k$-Fibonacci sequence $\{F_{k,n}\}$.


In [9], the authors obtained complex factorization formulas for the Fibonacci, Pell and $k$-Fibonacci numbers by using the determinants of sequences of tridiagonal matrices. They used the $n \times n$ tridiagonal matrices

$$
\begin{pmatrix}
1 & 2i \\
-i & 1 & i \\
& -i & 1 & \ddots & \ddots \\
& & \ddots & \ddots & i \\
& & & -2i & 1
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 2i \\
-i & 2 & i \\
& -i & 2 & \ddots & \ddots \\
& & \ddots & \ddots & i \\
& & & -2i & 2
\end{pmatrix}, \quad
\begin{pmatrix}
k & i \\
& i & k & i \\
& & i & k & \ddots & \ddots \\
& & & \ddots & \ddots & i \\
& & & & & i & k
\end{pmatrix},
$$
respectively, to prove that
\[ F_n = \prod_{k=1}^{n-1} \left( 1 - 2i \cos \frac{\pi k}{n} \right), \quad P_n = \prod_{k=1}^{n-1} \left( 2 - 2i \cos \frac{\pi k}{n} \right), \]
\[ F_{k,n} = \prod_{j=1}^{n-1} \left( k - 2i \cos \frac{\pi j}{n} \right) \]
for any integer \( n \geq 2 \), where \( i = \sqrt{-1} \).

In this paper, we give a connection between the sequence \( \{q_n\} \) and the Chebyshev polynomials of the second kind. With the aid of factorization of Chebyshev polynomials of the second kind, we derive the complex factorizations of the sequence \( \{q_n\} \).

2. Chebyshev polynomials of the second kind

Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. Chebyshev polynomials of the second kind \( U_n(x) \) defined by setting \( U_0(x) = 1, U_1(x) = 2x \) and the recurrence relation
\[ U_n(x) = 2x U_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \ldots . \]  

Hsiao [6] gave a complete factorization of Chebyshev polynomials of the first kind. Rivlin [10] adapts Hsiao’s proof for the Chebyshev polynomials of the second kind \( U_n(x) \) as follows
\[ U_n(x) = \frac{\sin((n + 1) \cos^{-1} x)}{\sin(\cos^{-1} x)}, \]
or
\[ U_n(x) = 2^n \prod_{k=1}^{n} \left( x - \cos \left( \frac{k\pi}{n+1} \right) \right). \]
Now, the first few numbers $q_n$ and Chebyshev polynomials of the second kind $U_n(x)$ are

\[
\begin{align*}
q_0 &= 0 : U_0(x) = 1 \\
q_1 &= 1 : U_1(x) = 2x \\
q_2 &= a : U_2(x) = 4x^2 - 1 \\
q_3 &= ab + 1 : U_3(x) = 8x^3 - 4x \\
q_4 &= a^2b + 2a : U_4(x) = 16x^4 - 12x^2 + 1 \\
q_5 &= a^2b^2 + 3ab + 1 : U_5(x) = 32x^5 - 32x^3 + 6x \\
q_6 &= a^3b^2 + 4a^2b + 3a : U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1.
\end{align*}
\]

3. Complex factorizations of the sequence $\{q_n\}$

In this section, we give a connection between the sequence $\{q_n\}$ and the Chebyshev polynomials of the second kind $U_n(x)$. With the aid of factorization (5) and (6) of Chebyshev polynomials of the second kind, we derive the complex factorizations of the sequence $\{q_n\}$.

Lemma 3.1. The sequence $\{q_n\}$ satisfies

\[
q_{n+1} = a \frac{\xi(m)}{x} b \frac{\xi(n)}{x} i^n U_n \left( -\frac{\sqrt{ab}}{2} i \right), \quad n \geq 1,
\]

where $i = \sqrt{-1}$ and $a, b$ are positive real numbers.

Proof. First, note that

\[
\begin{align*}
\xi(m + n) &= \xi(m) + \xi(n) - 2\xi(m)\xi(n), \\
\xi(n + 1) &= \xi(n - 1).
\end{align*}
\]

We prove the identity (7) by induction on $n$. When $n = 1$, we have

\[
a \frac{\xi(1)}{x} b \frac{\xi(1)}{x} i U_1 \left( -\frac{\sqrt{ab}}{2} i \right) = a \frac{1}{2} b \frac{1}{2} i 2 \left( -\frac{\sqrt{ab}}{2} i \right) = a = q_2.
\]

Next we assume the identity (7) holds for all positive integers less than or equal to $n$, that is,

\[
q_k = a \frac{\xi(k-1)}{x} b \frac{\xi(k-1)}{x} i^{k-1} U_{k-1} \left( -\frac{\sqrt{ab}}{2} i \right) \quad (1 \leq k \leq n).
\]
Complex factorizations of the generalized Fibonacci sequences \{q_n\}

Then we have

\[ a^{\frac{\xi(n)}{2}} b^{-\frac{\xi(n)}{2}} i^n U_n \left( -\frac{\sqrt{ab}}{2} i \right) \]

\[ = a^{\frac{\xi(n)}{2}} b^{-\frac{\xi(n)}{2}} i^n \left\{ 2 \left( -\frac{\sqrt{ab}}{2} i \right) U_{n-1} \left( -\frac{\sqrt{ab}}{2} i \right) - U_{n-2} \left( -\frac{\sqrt{ab}}{2} i \right) \right\} \]

\[ = a^{1-\xi(n-1)} b^{\xi(n-1)} \left( a^{\frac{\xi(n-1)}{2}} b^{-\frac{\xi(n-1)}{2}} i^{n-1} U_{n-1} \left( -\frac{\sqrt{ab}}{2} i \right) \right) \]

\[ + a^{\frac{\xi(n-2)}{2}} b^{-\frac{\xi(n-2)}{2}} i^{n-2} U_{n-2} \left( -\frac{\sqrt{ab}}{2} i \right) \]

\[ = a^{1-\xi(n-1)} b^{\xi(n-1)} q_n + q_{n-1} \quad (\because (4)) \]

\[ = a^{1-\xi(n-1)} b^{\xi(n-1)} q_n + q_{n-1} \quad (\because (2), (8)) \]

\[ = a^{1-\xi(n-1)} b^{\xi(n-1)} q_n + q_{n-1} \quad (\because (9)) \]

\[ = q_{n+1} \quad (\because (1)). \]

Therefore the identity (7) holds for all integers \( n \geq 1 \).

**Theorem 3.2.** The sequence \{q_n\} satisfies

\[ q_{n+1} = a^{\frac{\xi(n)}{2}} b^{-\frac{\xi(n)}{2}} i^n \frac{\sin \left( (n + 1) \cos^{-1} \left( -\frac{\sqrt{ab}}{2} i \right) \right)}{\sin \left( \cos^{-1} \left( -\frac{\sqrt{ab}}{2} i \right) \right)} \]

or

\[ q_{n+1} = a^{\frac{\xi(n)}{2}} b^{-\frac{\xi(n)}{2}} \prod_{k=1}^{n} \left( \sqrt{ab} - 2i \cos \left( \frac{k\pi}{n+1} \right) \right) \]

where \( i = \sqrt{-1} \) and \( a, b \) are positive real numbers.

**Proof.** Using (7) in Lemma 3.1, (5) and (6), we obtain (11) and (12). \( \square \)
Acknowledgments. At first, the author obtained Theorem 3.2 similar to [9] using the determinants of sequences of tridiagonal matrices

\[ M_n(a, b) = \begin{pmatrix}
ab & bi \\
ai & ab & bi \\
ai & ab & i & \ddots & bi \\
\ddots & \ddots & bi \\
ai & ab & i & \ddots & bi
\end{pmatrix}.
\]

Then, the referee suggested to simplify the proof by using the connection between the sequence \( \{q_n\} \) and the Chebyshev polynomials of the second kind \( U_n(x) \). His advice gave a nice perspective. The author is very grateful to the referee.

References

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