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COMPLEX FACTORIZATIONS OF THE GENERALIZED FIBONACCI SEQUENCES $\{q_n\}$

Sang Pyo Jun

Abstract. In this note, we consider a generalized Fibonacci sequence $\{q_n\}$. Then give a connection between the sequence $\{q_n\}$ and the Chebyshev polynomials of the second kind $U_n(x)$. With the aid of factorization of Chebyshev polynomials of the second kind $U_n(x)$, we derive the complex factorizations of the sequence $\{q_n\}$.

1. Introduction

For any integer $n \geq 0$, the well-known Fibonacci sequence $\{F_n\}$ is defined by the second order linear recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has been generalized in many ways, for example, by changing the recurrence relation (see [8]), by changing the initial values (see [4, 5]), by combining of these two techniques (see [3]), and so on.

In [2], Edson and Yayenie defined a further generalized Fibonacci sequence $\{q_n\}$ depending on two real parameters used in a non-linear (piecewise linear) recurrence relation, namely,

$$q_n = a^{1-\xi(n)} b^{\xi(n)} q_{n-1} + q_{n-2} \ (n \geq 2) \tag{1}$$

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with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are positive real numbers and

$$\xi(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \tag{2}$$

is the parity function. Also, the authors showed that the terms of the sequence $\{q_n\}$ are given by the extended Binet's formula

$$q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\frac{n-\xi(n)}{2}}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{3}$$

where α and β are roots of the quadratic equation $x^2 - abx - ab = 0$ and $\alpha > \beta$.

These sequences arise in a natural way in the study of continued fractions of quadratic irrationals and combinatorics on words or dynamical system theory. Some well-known sequences are special cases of this generalization. The Fibonacci sequence is a special case of $\{q_n\}$ with $a = b = 1$. When $a = b = 2$, we obtain the Pell's sequence $\{P_n\}$. Even further, if we set $a = b = k$ for some positive integer k, we obtain the k-Fibonacci sequence $\{F_{k,n}\}$.

Using the extended Binet's formula (3), Edson and Yayenie [2] derived a number of mathematical properties including generalizations of Cassini's, Catalan's and d'Ocagne's identities for the Fibonacci sequence, Yayenie [11] obtained numerous new identities of $\{q_n\}$, and Zhang and Wu [12] studied the partial infinite sums of reciprocal of $\{q_n\}$. Jang and Jun [7] give linearlization of the sequence $\{q_n\}$.

In [9], the authors obtained complex factorization formulas for the Fibonacci, Pell and k-Fibonacci numbers by using the determinants of sequences of tridiagonal matrices. They used the $n \times n$ tridiagonal matrices

$$\begin{pmatrix} 1 & 2i & & & \\ -i & 1 & i & & \\ & -i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & -2i & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2i & & & \\ -i & 2 & i & & \\ & -i & 2 & \ddots & \\ & & \ddots & \ddots & i \\ & & & -2i & 2 \end{pmatrix}, \quad \begin{pmatrix} k & i & & & \\ i & k & i & & \\ & i & k & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & k \end{pmatrix},$$

respectively, to prove that

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2i \cos \frac{\pi k}{n}\right), \; P_n = \prod_{k=1}^{n-1} \left(2 - 2i \cos \frac{\pi k}{n}\right),$$

$$F_{k,n} = \prod_{j=1}^{n-1} \left(k - 2i \cos \frac{\pi j}{n}\right)$$

for any integer $n \geq 2$, where $i = \sqrt{-1}$.

In this paper, we give a connection between the sequence $\{q_n\}$ and the Chebyshev polynomials of the second kind. With the aid of factorization of Chebyshev polynomials of the second kind, we derive the complex factorizations of the sequence $\{q_n\}$.

2. Chebyshev polynomials of the second kind

Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. Chebyshev polynomials of the second kind $U_n(x)$ defined by setting $U_0(x) = 1$, $U_1(x) = 2x$ and the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \; n = 2, 3, \cdots . \tag{4}$$

Hsiao [6] gave a complete factorization of Chebyshev polynomials of the first kind. Rivlin [10] adapts Hsiao's proof for the Chebyshev polynomials of the second kind $U_n(x)$ as follows

$$U_n(x) = \frac{\sin((n+1)\cos^{-1} x)}{\sin(\cos^{-1} x)}, \tag{5}$$

or

$$U_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos\left(\frac{k\pi}{n+1}\right)\right). \tag{6}$$

Now, the first few numbers q_n and Chebyshev polynomials of the second kind $U_n(x)$ are

$$\begin{aligned}
q_0 = 0 \quad &: \quad U_0(x) = 1 \\
q_1 = 1 \quad &: \quad U_1(x) = 2x \\
q_2 = a \quad &: \quad U_2(x) = 4x^2 - 1 \\
q_3 = ab + 1 \quad &: \quad U_3(x) = 8x^3 - 4x \\
q_4 = a^2b + 2a \quad &: \quad U_4(x) = 16x^4 - 12x^2 + 1 \\
q_5 = a^2b^2 + 3ab + 1 \quad &: \quad U_5(x) = 32x^5 - 32x^3 + 6x \\
q_6 = a^3b^2 + 4a^2b + 3a \quad &: \quad U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1.
\end{aligned}$$

3. Complex factorizations of the sequence $\{q_n\}$

In this section, we give a connection between the sequence $\{q_n\}$ and the Chebyshev polynomials of the second kind $U_n(x)$. With the aid of factorization (5) and (6) of Chebyshev polynomials of the second kind, we derive the complex factorizations of the sequence $\{q_n\}$.

LEMMA 3.1. *The sequence* $\{q_n\}$ *satisfies*

$$q_{n+1} = a^{\frac{\xi(n)}{2}} b^{-\frac{\xi(n)}{2}} i^n U_n\left(-\frac{\sqrt{ab}}{2}i\right), \ n \geq 1, \tag{7}$$

where $i = \sqrt{-1}$ *and* a, b *are positive real numbers.*

Proof. First, note that

$$\xi(m+n) = \xi(m) + \xi(n) - 2\xi(m)\xi(n), \tag{8}$$

$$\xi(n+1) = \xi(n-1). \tag{9}$$

We prove the identity (7) by induction on n. When $n = 1$, we have

$$a^{\frac{\xi(1)}{2}} b^{-\frac{\xi(1)}{2}} iU_1\left(-\frac{\sqrt{ab}}{2}i\right) = a^{\frac{1}{2}} b^{-\frac{1}{2}} i2\left(-\frac{\sqrt{ab}}{2}i\right) = a = q_2.$$

Next we assume the identity (7) holds for all positive integers less than or equal to n, that is,

$$q_k = a^{\frac{\xi(k-1)}{2}} b^{-\frac{\xi(k-1)}{2}} i^{k-1} U_{k-1}\left(-\frac{\sqrt{ab}}{2}i\right) \ (1 \leq k \leq n). \tag{10}$$

Then we have

$$
\begin{aligned}
& a^{\frac{\xi(n)}{2}} b^{-\frac{\xi(n)}{2}} i^n U_n\left(-\frac{\sqrt{ab}}{2} i\right) \\
= \; & a^{\frac{\xi(n)}{2}} b^{-\frac{\xi(n)}{2}} i^n \left\{2\left(-\frac{\sqrt{ab}}{2} i\right) U_{n-1}\left(-\frac{\sqrt{ab}}{2} i\right)-U_{n-2}\left(-\frac{\sqrt{ab}}{2} i\right)\right\} \\
& \qquad (\because (4)) \\
= \; & a^{1-\xi(n-1)} b^{\xi(n-1)}\left(a^{\frac{\xi(n-1)}{2}} b^{-\frac{\xi(n-1)}{2}} i^{n-1} U_{n-1}\left(-\frac{\sqrt{ab}}{2} i\right)\right) \\
& +a^{\frac{\xi(n-2))}{2}} b^{-\frac{\xi(n-2))}{2}} i^{n-2} U_{n-2}\left(-\frac{\sqrt{ab}}{2} i\right) \; (\because (2),(8)) \\
= \; & a^{1-\xi(n-1)} b^{\xi(n-1)} q_n+q_{n-1} \; (\because (10)) \\
= \; & a^{1-\xi(n+1)} b^{\xi(n+1)} q_n+q_{n-1} \; (\because (9)) \\
= \; & q_{n+1} \; (\because (1)).
\end{aligned}
$$

Therefore the identity (7) holds for all integers $n \geq 1$. □

THEOREM 3.2. *The sequence $\{q_n\}$ satisfies*

$$
q_{n+1}=a^{\frac{\xi(n)}{2}} b^{\frac{-\xi(n)}{2}} i^n \frac{\sin \left((n+1) \cos^{-1}\left(-\frac{\sqrt{ab}}{2} i\right)\right)}{\sin \left(\cos^{-1}\left(-\frac{\sqrt{ab}}{2} i\right)\right)}, \; n \geq 0, \tag{11}
$$

or

$$
q_{n+1}=a^{\frac{\xi(n)}{2}} b^{-\frac{\xi(n)}{2}} \prod_{k=1}^n\left(\sqrt{ab}-2 i \cos \left(\frac{k \pi}{n+1}\right)\right), \; n \geq 1, \tag{12}
$$

where $i=\sqrt{-1}$ and a, b are positive real numbers.

Proof. Using (7) in Lemma 3.1, (5) and (6), we obtain (11) and (12). □

Acknowledgments. At first, the author obtained Theorem 3.2 similar to [9] using the determinants of sequences of tridiagonal matrices

$$M_n(a,b) = \begin{pmatrix} ab & bi & & & \\ ai & ab & bi & & \\ & ai & ab & \ddots & \\ & & \ddots & \ddots & bi \\ & & & ai & ab \end{pmatrix}. \tag{13}$$

Then, the referee suggested to simplify the proof by using the connection between the sequence $\{q_n\}$ and the Chebyshev polynomials of the second kind $U_n(x)$. His advice gave a nice perspective. The author is very grateful to the referee.

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Sang Pyo Jun
Information Communication
Namseoul University
Cheonan 331-707, Korea
E-mail: spjun7129@naver.com