STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR A MODIFIED SYSTEM OF VARIATIONAL INEQUALITIES AND A FINITE FAMILY OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, a new iterative scheme based on the extra-gradient-like method for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of solutions of modified variational inequalities in Banach spaces. A strong convergence theorem for this iterative scheme in Banach spaces is established. Our results extend recent results announced by many others.

1. Introduction

Let \((E, \| \cdot \|)\) be a Banach space and \(C\) be a nonempty closed convex subset of \(E\). Recall that a mapping \(T : C \to C\) is said to be nonexpansive if

\[\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.\]

We denote by \(F(T)\) the set of fixed points of \(T\).

Received June 29, 2015. Revised September 5, 2015. Accepted September 7, 2015.
2010 Mathematics Subject Classification: 49J30, 49J40, 47J25, 49H09.
Key words and phrases: Fixed point; inverse strongly accretive mapping; variational inequality; nonexpansive mapping.
This work was supported by Dong-eui University Grant (2015AA049).
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Let $A, B : C \to E$ be two nonlinear mappings, $I$ be the identity mapping. We consider the modified system of nonlinear variational inequalities for finding $(x^*, y^*) \in C \times C$ such that

$$(1.1) \begin{cases} 
  \langle x^* - (I - \lambda_1 A)(ax^* + (1 - a)y^*), j(x - x^*) \rangle \geq 0, & \forall x \in C, \\
  \langle y^* - (I - \lambda_2 B)x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C,
\end{cases}$$

where $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$, $J$ is the normalized duality mapping, $j \in J$.

In the case $a = 0$, problem (1.1) reduces to the following general system of nonlinear variational inequalities for finding $(x^*, y^*) \in C \times C$ such that

$$$(1.2) \begin{cases} 
  \langle \lambda_1 Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall x \in C, \\
  \langle \lambda_2 Bx^* + y^* - x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C,
\end{cases}$$

which was considered by Wang and Yang [12], Yao et al. [13].

In particular, if $A = B$, then problem (1.2) reduces to the following system of variational inequalities for finding $(x^*, y^*) \in C \times C$ such that

$$(1.3) \begin{cases} 
  \langle \lambda_1 Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall x \in C, \\
  \langle \lambda_2 Ax^* + y^* - x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C,
\end{cases}$$

which was studied by Qin et al. [6].

If $x^* = y^*$ in (1.3), then (1.3) reduces to

$$(1.4) \quad \langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C,$$

which was considered by Aoyama et al. [1].

If $E = H$ is a real Hilbert space and $A, B : C \to H$ are nonlinear mappings, then (1.1) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$(1.5) \begin{cases} 
  \langle x^* - (I - \lambda_1 A)(ax^* + (1 - a)y^*), x - x^* \rangle \geq 0, & \forall x \in C, \\
  \langle y^* - (I - \lambda_2 B)x^*, x - x^* \rangle \geq 0, & \forall x \in C.
\end{cases}$$

Aoyama et al. [1] proved that an element $x^* \in C$ is a solution of the variational inequality (1.4) if and only if $x^* \in C$ is a fixed point of the mapping $Q_C(I - \lambda A)$, where $\lambda > 0$ is a constant and $Q_C$ is a sunny nonexpansive retraction from $E$ onto $C$. 
Recently, Qin et al. [6] studied the problem of finding a common element in fixed point set of a nonexpansive mapping and solution set of a variational inequality for an inverse strongly accretive mapping. More precisely, they proved the following theorem.

**Theorem 1.1.** Let $E$ be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant $K$, $C$ be a nonempty closed convex subset of $E$ and $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$. Let $A : C \to E$ be an $\alpha$-inverse strongly accretive mapping and $S : C \to C$ be a nonexpansive mapping with a fixed point. Assume that $\mathcal{F} = F(S) \cap F(D) \neq \phi$, where $Dx = Q_C[Q_C(x - \mu Ax) - \lambda AQ_C(x - \mu Ax)]$ for all $x \in C$. Let $\{x_n\}$ be a sequence generated in the following manner:

\begin{equation}
\begin{aligned}
x_1 &= u \in C, \\
y_n &= Q_C(x_n - \mu Ax_n), \\
x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n [\delta Sx_n + (1 - \delta)Q_C(y_n - \lambda Ay_n)], \quad n \geq 1.
\end{aligned}
\end{equation}

where $\delta \in (0, 1)$, $\lambda, \mu \in (0, \frac{\alpha}{K^2})$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that

(a) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1$;
(b) $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = Q_{\mathcal{F}u}$ and $(\bar{x}, \bar{y})$, where $\bar{y} = Q_C(\bar{x} - \mu A\bar{x})$, is a solution of the problem (1.3).

Motivated and inspired by the research work going on this field, in this paper, we consider the problem of convergence of an iterative algorithm for a modified system of nonlinear variational inequalities and a finite family of nonexpansive mappings. We prove the strong convergence of the purposed iterative scheme in uniformly convex and 2-uniformly smooth Banach spaces.

2. Preliminaries

Let $C$ be a nonempty closed convex subset of a Banach space $E$ with its dual space $E^*$. Let $\langle \cdot, \cdot \rangle$ denote the dual pair between $E$ and $E^*$. Let
$2^E$ denote the family of all the nonempty subsets of $E$. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \}, \quad \forall x \in E.$$ 

In particular, $J = J_2$ is the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$ and $J_q$ is single-valued if $E^*$ is strictly convex or $E$ is uniformly smooth. If $E = H$ is a Hilbert space, $J = I$, the identity mapping.

Let $B = \{ x \in E : \|x\| = 1 \}$. A Banach space $E$ is said to be uniformly convex if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in B$,

$$\|x - y\| \geq \varepsilon \quad \text{implies} \quad \|\frac{x+y}{2}\| \leq 1 - \delta.$$ 

It is known that a uniformly convex Banach space is reflexive and strictly convex. $E$ is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in B$. The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$ 

A Banach space $E$ is called uniformly smooth if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$. $E$ is called $q$-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_E(t) \leq c t^q, \quad q > 1.$$ 

If $E$ is $q$-uniformly smooth, then $q \leq 2$ and $E$ is uniformly smooth.

**Definition 2.1.** Let $A : C \to E$ be a mapping. $A$ is said to be

(i) accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0$$

for all $x, y \in C$.

(ii) $\zeta$-inverse strongly accretive if there exist $j(x - y) \in J(x - y)$ and a constant $\zeta > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \zeta \|Ax - Ay\|^2$$

for all $x, y \in C$. 
**Definition 2.2.** Let $C$ be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself and let $\eta_1, \cdots, \eta_N$ be real numbers such that $0 \leq \eta_i \leq 1$ for every $i = 1, \cdots, N$. Define a mapping $S : C \to C$ as follows:

$$U_1 = \eta_1 T_1 + (1 - \eta_1) I,$$

$$U_2 = \eta_2 T_2 U_1 + (1 - \eta_2) U_1,$$

$$U_3 = \eta_3 T_3 U_2 + (1 - \eta_3) U_2,$$

$$\vdots$$

$$U_{N-1} = \eta_{N-1} T_{N-1} U_{N-2} + (1 - \eta_{N-1}) U_{N-2},$$

$$S = U_N = \eta_N T_N U_{N-1} + (1 - \eta_N) U_{N-1}.$$  

Such a mapping $S$ is called the $K$-mapping generated by $T_1, \cdots, T_N$ and $\eta_1, \cdots, \eta_N$.

Let $D$ be a subset of $C$ and $Q$ be a mapping of $C$ into $D$. Then $Q$ is said to be sunny if

$$Q[Q(x) + t(x - Q(x))] = Q(x),$$

whenever $Q(x) + t(x - Q(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q$ of $C$ into itself is called a retraction if $Q^2 = Q$. If a mapping $Q$ of $C$ into itself is a retraction, then $Q(z) = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of $Q$. A subset $D$ of $C$ is called a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$.

In order to prove our main results in the next section, we also need the following lemmas.

**Lemma 2.1.** ([10]) Let $E$ be a real 2-uniformly smooth Banach space. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + 2\|Ky\|^2, \quad \forall x, y \in E,$$

where $K$ is the 2-uniformly smooth constant of $E$.

**Lemma 2.2.** ([5]) Let $C$ be a closed convex subset of a strictly convex Banach space $E$. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings of $C$ into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\eta_1, \cdots, \eta_N$ be real numbers such that $0 < \eta_i < 1$ for every $i = 1, \cdots, N-1$ and $0 < \eta_N \leq 1$. 

Let $S$ be the $K$-mapping generated by $T_1, \cdots, T_N$ and $\eta_1, \cdots, \eta_N$. Then $F(S) = \cap_{i=1}^{N} F(T_i)$.

**Remark 2.1.** It is easy to see that the $K$-mapping is a nonexpansive mapping.

**Lemma 2.3.** ([9]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ for all integer $n \geq 0$ and

$$\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$ Then $\lim_{n \to \infty} \|x_n - z_n\| = 0$.

**Lemma 2.4.** ([8]) Let $E$ be a uniformly smooth Banach space, $C$ be a closed convex subset of $E$ and $D : C \to C$ be a nonexpansive mapping with $F(D) \neq \phi$. For each fixed point $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $x \mapsto tu + (1 - t)Dx$ converges strongly as $t \to 0$ to a point of $F(D)$. Define $Q : C \to F(D)$ by $Q(u) = \lim_{t \to 0} x_t$. Then $Q$ is the unique sunny nonexpansive retraction from $C$ onto $F(D)$, that is, $Q$ satisfy the property:

$$\langle u - Q(u), j(y - Q(u)) \rangle \leq 0, \quad \forall u \in C, y \in F(D).$$

**Lemma 2.5.** ([2]) Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $\{S_k\}$ be a sequence of nonexpansive mappings of $C$ into $E$ and $\{\beta_k\}$ be a sequence of positive real numbers such that $\sum_{k=1}^{\infty} \beta_k = 1$. If $\bigcap_{k=1}^{\infty} F(S_k) \neq \phi$, then the mapping $S = \sum_{k=1}^{\infty} \beta_k S_k$ is nonexpansive and $F(S) = \bigcap_{k=1}^{\infty} F(S_k)$.

**Lemma 2.6.** ([11]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \alpha_n) a_n + \beta_n,$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

(a) $\{\alpha_n\} \subset [0, 1], \sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\limsup_{n \to \infty} \frac{\beta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

**Lemma 2.7.** ([7]) Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$ and let $Q_C$ be a retraction from $E$ onto $C$. Then the following are equivalent:
(i) $Q_C$ is both sunny and nonexpansive;
(ii) $\langle x - Q_C(x), j(y - Q_C(x)) \rangle \leq 0$ for all $x \in E$ and $y \in C$.

**Lemma 2.8.** ([3]) In a Banach space $E$, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle,$$

where $j(x + y) \in J(x + y)$.

**Lemma 2.9.** ([3]) Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$. Let $Q_C : E \to C$ be a sunny nonexpansive retraction, $A, B : C \to E$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$, the following statements are equivalent:

(a) $(x^*, y^*) \in C \times C$ is a solution of problem (1.1).
(b) $x^*$ is a fixed point of the mapping $G : C \to C$ defined by

$$G(x) = Q_C(I - \lambda_1 A)(ax + (1 - a)Q_C(I - \lambda_2 B)x),$$

where $y^* = Q_C(I - \lambda_2 B)x^*$.

**Proof.** (a)$\Rightarrow$(b). Let $(x^*, y^*) \in C \times C$ be a solution of problem (1.1). For every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$, we have

$$\begin{cases} 
  \langle x^* - (I - \lambda_1 A)(ax^* + (1 - a)y^*), j(x - x^*) \rangle \geq 0, & \forall x \in C, \\
  \langle y^* - (I - \lambda_2 B)x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C.
\end{cases}$$

From Lemma 2.7, we have

$$\begin{cases} 
  x^* = Q_C(I - \lambda_1 A)(ax^* + (1 - a)y^*), \\
  y^* = Q_C(I - \lambda_2 B)x^*.
\end{cases}$$

It implies that

$$x^* = Q_C(I - \lambda_1 A)(ax^* + (1 - a)Q_C(I - \lambda_2 B)x^*) = G(x^*).$$

Hence, we have $x^* \in F(G)$, where $y^* = Q_C(I - \lambda_2 B)x^*$.

(b)$\Rightarrow$(a). Let $x^* \in F(G)$ and $y^* = Q_C(I - \lambda_2 B)x^*$. Then, we have

$$x^* = G(x^*) = Q_C(I - \lambda_1 A)(ax^* + (1 - a)Q_C(I - \lambda_2 B)x^*)$$

$$= Q_C(I - \lambda_1 A)(ax^* + (1 - a)y^*).$$
From Lemma 2.7, we have
\[
\begin{aligned}
&\begin{cases}
(x^* - (I - \lambda_1 A)(ax^* + (1 - a)y^*), j(x - x^*)) \geq 0, & \forall x \in C, \\
(y^* - (I - \lambda_2 B)x^*, j(x - y^*)) \geq 0, & \forall x \in C.
\end{cases}
\end{aligned}
\]

Hence, we have \((x^*, y^*) \in C \times C\) is a solution of (1.1). \(\square\)

3. Main results

Now we state and prove our main results.

**Theorem 3.1.** Let \(E\) be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniformly smooth constant \(K\), \(H\) be a nonempty closed convex subset of \(E\) and \(Q_{C_H}\) be a sunny nonexpansive retraction from \(E\) onto \(H\). Let \(A, B : H \to E\) be \(\zeta_1, \zeta_2\)-inverse strongly accretive mappings, respectively. Define the mapping \(G : H \to H\) by \(G(x) = Q_{C_H}(I - \lambda_1 A)(ax + (1 - a)Q_{C_H}(I - \lambda_2 B)x)\) for all \(x \in H\), \(\lambda_1, \lambda_2 > 0\) and \(a \in [0, 1]\). Let \(S : H \to H\) be the \(K\)-mapping generated by \(T_1, T_2, \cdots, T_N\) and \(\eta_1, \eta_2, \cdots, \eta_N\), where \(\eta_i \in (0, 1)\), for \(i = 1, 2, \cdots, N - 1\), and \(\eta_N \in (0, 1]\) with \(F = \cap_{i=1}^N F(T_i) \cap F(G) \neq \emptyset\). Suppose that \(\{x_n\}\) is the sequence generated by
\[
\begin{aligned}
x_n + 1 = \begin{cases}
x_1, u \in C, \\
y_n = Q_{C_H}(I - \lambda_2 B)x_n, \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n [\delta Sx_n + (1 - \delta)Q_{C_H}(ax_n + (1 - a)y_n) - \lambda_1 A(ax_n + (1 - a)y_n)], & \forall n \geq 1,
\end{cases}
\end{aligned}
\]
where \(\lambda_1 \in (0, \frac{\alpha_1}{K})\), \(\lambda_2 \in (0, \frac{\alpha_2}{K})\) and \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) are sequences in \([0, 1]\). Assume that the following conditions hold:
\begin{enumerate}
\item \((i)\) \(\alpha_n + \beta_n + \gamma_n = 1\),
\item \((ii)\) \(\lim_{n \to \infty} \alpha_n = 0\) and \(\sum_{n=1}^{\infty} \alpha_n = \infty\),
\item \((iii)\) \(0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1\).
\end{enumerate}
Then \(\{x_n\}\) converges strongly to \(x_0 = Q_F u\) and \((x_n, y_n)\) is a solution of (1.1), where \(y_0 = Q_{C_H}(I - \lambda_2 B)x_0\).

**Proof.** First, we show that \(Q_{C_H}(I - \lambda_1 A)\) and \(Q_{C_H}(I - \lambda_2 B)\) are nonexpansive mappings for \(\lambda_1 \in (0, \frac{\alpha_1}{K})\), \(\lambda_2 \in (0, \frac{\alpha_2}{K})\). Let \(x, y \in H\). Since \(A\) is an \(\zeta_1\)-inverse strongly accretive mapping and \(\lambda_1 \leq \frac{\zeta_1}{K^2}\), we have from
Lemma 2.1 that

\[ \| (I - \lambda_1 A)x - (I - \lambda_2 A)y \|^2 \]
\[ \leq \| x - y \|^2 - 2\lambda_1 \langle Ax - Ay, j(x - y) \rangle + 2K^2 \lambda_1^2 \| Ax - Ay \|^2 \]
\[ \leq \| x - y \|^2 - 2\lambda_1 \zeta_1 \| Ax - Ay \|^2 + 2K^2 \lambda_1^2 \| Ax - Ay \|^2 \]
\[ = \| x - y \|^2 + 2\lambda_1 (\lambda_1 K^2 - \zeta_1) \| Ax - Ay \|^2 \]
(3.2)
\[ \leq \| x - y \|^2 . \]

Thus \( (I - \lambda_1 A) \) is a nonexpansive mapping. So is \( (I - \lambda_2 B) \). Hence \( Q_C(I - \lambda_1 A), Q_C(I - \lambda_2 B) \) are nonexpansive mappings. It is easy to see that the mapping \( G \) is a nonexpansive mapping. This show from Remark 2.1 that \( \mathcal{F} = F(S) \cap F(G) \) is closed and convex. Let \( x^* \in \mathcal{F} \). Then we have \( x^* = Sx^* \) and

\[ x^* = Gx^* \]
\[ = Q_C(I - \lambda_1 A)(ax^* + (1 - a)Q_C(I - \lambda_2 B)x^*) \]}

Putting \( w_n = Q_C(I - \lambda_1 A)(ax_n + (1 - a)y_n) \) and \( y^* = Q_C(I - \lambda_2 B)x^* \), we can rewrite (3.1) by

\[ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n (\delta S x_n + (1 - \delta) w_n) \]

and \( x^* = Q_C(I - \lambda_1 A)(ax^* + (1 - a)y^*) \). Since \( Q_C(I - \lambda_1 A) \) and \( Q_C(I - \lambda_2 B) \) are nonexpansive, we have

(3.3)
\[ \| w_n - x^* \| \]
\[ = \| Q_C(I - \lambda_1 A)(ax_n + (1 - a)y_n) - Q_C(I - \lambda_1 A)(ax^* + (1 - a)y^*) \| \]
\[ \leq \| ax_n + (1 - a)y_n - (ax^* + (1 - a)y^*) \| \]
\[ \leq a \| x_n - x^* \| + (1 - a) \| y_n - y^* \| \]
\[ \leq a \| x_n - x^* \| + (1 - a) \| x_n - x^* \| \]
\[ = \| x_n - x^* \| . \]
It follows from the definition of $x_n$ and (3.3) that

\[
\|x_{n+1} - x^*\| \\
= \|\alpha_n u + \beta_n x_n + \gamma_n (\delta Sx_n + (1 - \delta)w_n) - x^*\| \\
\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n[\delta\|Sx_n - x^*\| + (1 - \delta)\|w_n - x^*\|] \\
\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n[\delta\|x_n - x^*\| + (1 - \delta)\|x_n - x^*\|] \\
= \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}.
\]

So, \(\{x_n\}\) is bounded. Hence \(\{y_n\}, \{w_n\}\) and \(\{Sx_n\}\) are also bounded. And we have

(3.4)  
\[
\|w_{n+1} - w_n\| \\
= \|Q_C(I - \lambda_1 A)(ax_{n+1} + (1 - a)y_{n+1}) - Q_C(I - \lambda_1 A)(ax_n + (1 - a)y_n)\| \\
\leq a\|x_{n+1} - x_n\| + (1 - a)\|y_{n+1} - y_n\| \\
\leq a\|x_{n+1} - x_n\| + (1 - a)\|x_{n+1} - x_n\| \\
= \|x_{n+1} - x_n\|.
\]

Next, we will show that

(3.5)  
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

Let

(3.6)  
\[
x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n, \quad \forall n \geq 1,
\]
where \( z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \) for each \( n \geq 1 \). Since \( x_{n+1} - \beta_n x_n = \alpha_n u + \gamma_n [\delta S x_n + (1 - \delta) w_n] \) and (3.6), we have

\[
\begin{align*}
& z_{n+1} - z_n \\
& = \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
& = \frac{\alpha_{n+1} u + \gamma_{n+1} [\delta S x_{n+1} + (1 - \delta) w_{n+1}]}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n [\delta S x_n + (1 - \delta) w_n]}{1 - \beta_n} \\
& \quad - \frac{\gamma_{n+1} [\delta S x_{n+1} + (1 - \delta) w_{n+1}]}{1 - \beta_{n+1}} + \frac{\gamma_n [\delta S x_n + (1 - \delta) w_n]}{1 - \beta_n} \\
& = \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) u \\
& \quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} [\delta (S x_{n+1} - S x_n) + (1 - \delta)(w_{n+1} - w_n)] \\
& \quad + \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) [\delta S x_n + (1 - \delta) w_n] \\
& = \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) u \\
& \quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} [\delta (S x_{n+1} - S x_n) + (1 - \delta)(w_{n+1} - w_n)] \\
& \quad + \left( \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right) [\delta S x_n + (1 - \delta) w_n].
\end{align*}
\]

It follows from (3.4) that

\[
\| z_{n+1} - z_n \| \\
\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \| u \| \\
+ \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \| \delta (S x_{n+1} - S x_n) + (1 - \delta)(w_{n+1} - w_n) \| \\
+ \left| \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right| \| \delta S x_n + (1 - \delta) w_n \|
\]
\[
\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \left[ \|u\| + \|Sx_n\| + \|w_n\| \right] + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left[ \|Sx_{n+1} - Sx_n\| + (1 - \delta) \|w_{n+1} - w_n\| \right]
\]
\[
\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \left[ \|u\| + \|Sx_n\| + \|w_n\| \right] + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \left[ \|x_{n+1} - x_n\| + (1 - \delta) \|x_{n+1} - x_n\| \right]
\]
\[
= \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \left[ \|u\| + \|Sx_n\| + \|w_n\| \right] + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|x_{n+1} - x_n\|
\]
\[
\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \left[ \|u\| + \|Sx_n\| + \|w_n\| \right] + \|x_{n+1} - x_n\|.
\]

From the conditions (ii) and (iii), we have
\[
\limsup_{n \to \infty} \left( \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) \leq 0.
\]

From Lemma 2.3 and (3.6), we have
\[
\lim_{n \to \infty} \|z_n - x_n\| = 0.
\]

Since \(x_{n+1} - x_n = (1 - \beta_n)(z_n - x_n)\), we obtain
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}
\]

Next, we will show that
\[
\limsup_{n \to \infty} \langle u - x_0, j(x_n - x_0) \rangle \leq 0,
\]
where \(x_0 = Q_{\mathcal{F}} u\). To show this inequality, define a mapping \(D : C \to C\) by
\[
Dx = \delta Sx + (1 - \delta)Q_C (I - \lambda_1 A)(ax + (1 - a)Q_C (I - \lambda_2 B)x)
\]
\[
= \delta Sx + (1 - \delta)Gx, \quad \forall x \in C
\]

From Lemma 2.2 and 2.5, we have \(D\) is a nonexpansive mapping with
\[
F(D) = F(S) \cap F(G) = \cap_{i=1}^N F(T_i) \cap F(G) = \mathcal{F}. \tag{3.8}
\]
From the nonexpansiveness of the mapping $D$ and the definition of $x_n$, we have
\[
\|x_n - Dx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Dx_n\| \\
\leq \|x_n - x_{n+1}\| + \alpha_n\|u - Dx_n\| + \beta_n\|x_n - Dx_n\|.
\]
This implies that
\[
(1 - \beta_n)\|x_n - Dx_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|u - Dx_n\|.
\]
From the conditions (ii), (iii) and (3.7), we have
\[
\lim_{n \to \infty} \|x_n - Dx_n\| = 0.
\]
Let $x_t$ be the fixed point of the contraction $x \mapsto tu + (1 - t)Dx$, where $t \in (0, 1)$. That is,
\[
x_t = tu + (1 - t)Dx_t.
\]
From the definition of $x_t$, we have
\[
\|x_t - x_n\|^2 = \|t(u - x_n) + (1 - t)(Dx_t - x_n)\|^2 \\
= (1 - t)\|\langle Dx_t - Dx_n, j(x_t - x_n) \rangle + \langle Dx_n - x_n, j(x_t - x_n) \rangle \| \\
+ t\langle u - x_t, j(x_t - x_n) \rangle + \langle x_t - x_n, j(x_t - x_n) \rangle \\
\leq (1 - t)(\|x_t - x_n\|^2 + \|Dx_n - x_n\|\|x_t - x_n\|) \\
+ t\langle u - x_t, j(x_t - x_n) \rangle + t\|x_t - x_n\|^2 \\
= \|x_t - x_n\|^2 + (1 - t)\|Dx_n - x_n\|\|x_t - x_n\| \\
+ t\langle u - x_t, j(x_t - x_n) \rangle.
\]
(3.10) implies that
\[
\langle u - x_t, j(x_n - x_t) \rangle \leq \frac{1 - t}{t} \|Dx_n - x_n\|\|x_t - x_n\|.
\]
From (3.9) and (3.11), we have
\[
\limsup_{n \to \infty} \langle u - x_t, j(x_n - x_t) \rangle \leq 0.
\]
From Lemma 2.4 and (3.8), we see that \( Q_{F(D)}u = \lim_{t \to 0} x_t \) and \( F(D) = F \). It follows that \( \lim_{t \to 0} x_t = x_0 = Q_F(u) \). Since

\[
\langle u - x_0, j(x_n - x_0) \rangle \\
= \langle u - x_0, j(x_n - x_0) \rangle - \langle u - x_0, j(x_n - x_t) \rangle \\
+ \langle u - x_0, j(x_n - x_t) \rangle - \langle u - x_t, j(x_n - x_t) \rangle \\
+ \langle u - x_t, j(x_n - x_t) \rangle \\
= \langle u - x_0, j(x_n - x_0) - j(x_n - x_t) \rangle + \langle x_t - x_0, j(x_n - x_t) \rangle \\
+ \langle u - x_t, j(x_n - x_t) \rangle \\
= \|u - x_0\|\|j(x_n - x_0) - j(x_n - x_t)\| + \|x_t - x_0\|\|x_n - x_t\| \\
+ \langle u - x_t, j(x_n - x_t) \rangle,
\]

it follows that

\[
\limsup_{n \to \infty} \langle u - x_0, j(x_n - x_0) \rangle \leq \limsup_{n \to \infty} \|u - x_0\|\|j(x_n - x_0) - j(x_n - x_t)\| \\
+ \|x_t - x_0\| \limsup_{n \to \infty} \|x_n - x_t\| \\
+ \limsup_{n \to \infty} \langle u - x_t, j(x_n - x_t) \rangle.
\]

(3.13)

Since \( j \) is norm-to-norm uniformly continuous on a bounded subset of \( E \), (3.12) and (3.13), we have

\[
\limsup_{n \to \infty} \langle u - x_0, j(x_n - x_0) \rangle = \limsup_{t \to 0} \limsup_{n \to \infty} \langle u - x_0, j(x_n - x_0) \rangle \\
\leq 0.
\]

(3.14)

Finally, we will show that the sequence \( \{x_n\} \) converges strongly to \( x_0 \in F \). From the definition of \( x_n \) and Lemma 2.8, we have

\[
\|x_{n+1} - x_0\|^2 \\
= \|\alpha_n(u - x_0) + \beta_n(x_n - x_0) + \gamma_n(Dx_n - x_0)\|^2 \\
\leq \|\beta_n(x_n - x_0) + \gamma_n(Dx_n - x_0)\|^2 + 2\alpha_n\langle u - x_0, j(x_{n+1} - x_0) \rangle \\
\leq (\beta_n\|x_n - x_0\| + \gamma_n\|x_n - x_0\|)^2 + 2\alpha_n\langle u - x_0, j(x_{n+1} - x_0) \rangle \\
(3.15) \\
\leq (1 - \alpha_n)\|x_n - x_0\|^2 + 2\alpha_n\langle u - x_0, j(x_{n+1} - x_0) \rangle.
\]

From the condition (ii), (3.14) and Lemma 2.6 to (3.15), we obtain that

\[
\lim_{n \to \infty} \|x_n - x_0\| = 0.
\]

This completes the proof. \( \square \)
Remark 3.1. (1) If we take $a = 0$, then the iterative scheme (3.1) reduces to the following scheme:

\begin{equation}
\begin{cases}
x_1, u \in C, \\
y_n = Q_C(I - \lambda_2 B)x_n, \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta Sx_n + (1 - \delta)Q_C(y_n - \lambda_1 Ay_n)], \quad \forall n \geq 1,
\end{cases}
\end{equation}

(3.16)

From Theorem 3.1, we obtain that the sequence $\{x_n\}$ generated by (3.16) converges strongly to $x_0 = Q_{\cap_{i=1}^N F(T_i) \cap F(G)}u$, where the mapping $G : C \to C$ defined by $G(x) = Q_C(I - \lambda_1 A)Q_C(I - \lambda_2 B)x$ for all $x \in C$ and $(x_0, y_0)$ is a solution of (1.2), where $y_0 = Q_C(I - \lambda_2 B)x_0$.

(2) If we take $x_1 = u$, $A = B$, $N = 1$, $\eta_1 = 1$ and $T_1 = S : C \to C$ is a nonexpansive mapping, then the iterative scheme (3.16) reduces to the following scheme:

\begin{equation}
\begin{cases}
x_1 = u \\
y_n = Q_C(I - \lambda_2 A)x_n, \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta Sx_n + (1 - \delta)Q_C(y_n - \lambda_1 Ay_n)], \quad \forall n \geq 1,
\end{cases}
\end{equation}

(3.17)

which is (1.6). From Theorem 3.1, we obtain that the sequence $\{x_n\}$ generated by (3.17) converges strongly to $x_0 = Q_{F(S) \cap F(G)}u$, where the mapping $G : C \to C$ defined by $G(x) = Q_C(I - \lambda_1 A)Q_C(I - \lambda_2 A)x$ for all $x \in C$ and $(x_0, y_0)$ is a solution of (1.3), where $y_0 = Q_C(I - \lambda_2 A)x_0$.

Remark 3.2. (i) We note that all Hilbert spaces and $L^p(p \geq 2)$ spaces are 2-uniformly smooth.

(ii) If $E = H$ is a Hilbert space, then a sunny nonexpansive retraction $Q_C$ is coincident with the metric projection $P_C$ from $H$ onto $C$.

(iii) It is well known that the 2-uniformly smooth constant $K = \frac{\sqrt{2}}{2}$ in Hilbert spaces.

From Theorem 3.1 and Remark 3.3, we can obtain the following result immediately.

Corollary 3.1. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $P_C$ be the metric projection from $H$ onto $C$. Let $A, B : C \to H$ be $\zeta_1, \zeta_2$-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by

\[ G(x) = P_C(I - \lambda_1 A)(ax + (1 - a)P_C(I - \lambda_2 B)x) \]
for all $x \in C$, $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1)$. Let $S : C \to C$ be the $K$-mapping generated by $T_1, T_2, \cdots , T_N$ and $\eta_1, \eta_2, \cdots , \eta_N$, where $\eta_i \in (0, 1)$ for $i = 1, 2, \cdots , N - 1$ and $\eta_N \in (0, 1]$ with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(G) \neq \phi$.

Suppose that $\{x_n\} \text{is the sequence generated by }$

$$
\begin{cases}
x_1, u \in C, \\
y_n = P_C(I - \lambda_2 B)x_n, \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[Sx_n] \\
\quad + (1 - \delta) P_C(ax_n + (1 - a)y_n - \lambda_1 A(ax_n + (1 - a)y_n))], \quad \forall n \geq 1,
\end{cases}
$$

where $\lambda_1 \in (0, 2\zeta_1)$, $\lambda_2 \in (0, 2\zeta_2)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $x_0 = P_F u$ and $(x_0, y_0)$ is a solution of (1.5), where $y_0 = P_C(I - \lambda_2 B)x_0$.

**Remark 3.3.** We can see easily that Aoyama et al. [1], Iiduka and Takahashi [4], Yao and Yao [14], Qin et al. [6], Wang and Yang [12]’s results are special cases of Theorem 3.1.

**Completing interests**

The author declares that he has no competing interests.

**References**


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