WEAKLY SUBNORMAL WEIGHTED SHIFTS NEED NOT BE 2-HYPONORMAL

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Abstract. In this paper we give an example which is a weakly subnormal weighted shift but not 2-hyponormal. Also, we show that every partially normal extension of an isometry \( T \) needs not be 2-hyponormal even though p.n.e.\((T)\) is weakly subnormal.

Let \( \mathcal{H} \) and \( \mathcal{K} \) be complex Hilbert spaces, let \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) be the set of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K} \) and write \( \mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H}) \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be normal if \( T^*T = TT^* \), hyponormal if \( T^*T \geq TT^* \), and subnormal if \( T = N|_\mathcal{H} \), where \( N \) is normal on some Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \). If \( T \) is subnormal then \( T \) is also hyponormal.

The Bram-Halmos criterion for subnormality states that an operator \( T \) is subnormal if and only if

\[
\sum_{i,j} (T^i x_j, T^j x_i) \geq 0
\]

for all finite collections \( x_0, x_1, \ldots, x_k \in \mathcal{H} \) ([1], [5, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

\[
\begin{pmatrix}
I & T^* & \cdots & T^*^k \\
T & T^*T & \cdots & T^*^kT \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^*T^k & \cdots & T^*^kT^k
\end{pmatrix} \geq 0 \quad \text{(all } k \geq 1).\]

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Condition (1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1) for $k = 1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (1) for all $k$. Let $[A, B] := AB - BA$ denote the commutator of two operators $A$ and $B$, and define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (2) is equivalent to the positivity of the $(k + 1) \times (k + 1)$ operator matrix in (1); the Bram-Halmos criterion can be then rephrased as saying that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([7], [6]).

On the other hand, note that an operator $T$ is subnormal if and only if there exist operators $A$ and $B$ such that $\hat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$\begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases}$$

The operator $\hat{T}$ is called a normal extension of $T$. We also say that $\hat{T}$ in $L(K)$ is a minimal normal extension (briefly, m.n.e.) of $T$ if $K$ has no proper subspace containing $H$ to which the restriction of $\hat{T}$ is also a normal extension of $T$. It is known that

$$\hat{T} = \text{m.n.e.}(T) \iff K = \sqrt{\{ \hat{T}^*h : h \in H, \ n \geq 0 \}},$$

and the m.n.e.(T) is unique.

An operator $T \in L(H)$ is said to be weakly subnormal if there exist operators $A \in L(H', H)$ and $B \in L(H')$ such that the first two conditions in (3) hold:

$$[T^*, T] = AA^* \quad \text{and} \quad A^*T = BA^*,$$

or equivalently, there is an extension $\hat{T}$ of $T$ such that

$$\hat{T}^*\hat{T}f = \hat{T}\hat{T}^*f \quad \text{for all } f \in H.$$

The operator $\hat{T}$ is called a partially normal extension (briefly, p.n.e.) of $T$. We also say that $\hat{T}$ in $L(K)$ is a minimal partially normal extension (briefly, m.p.n.e.) of $T$ if $K$ has no proper subspace containing $H$ to
which the restriction of \( \hat{T} \) is also a partially normal extension of \( T \). It is known ([4, Lemma 2.5 and Corollary 2.7]) that
\[
\hat{T} = \text{m.p.n.e.}(T) \iff \mathcal{K} = \sqrt{\{ \hat{T}^* h : h \in \mathcal{H}, n = 0, 1 \}},
\]
and the m.p.n.e.\( (T) \) is unique. For convenience, if \( \hat{T} = \text{m.p.n.e.}(T) \) is also weakly subnormal then we write \( \hat{T}^{(2)} := \hat{T} \) and more generally,
\[
\hat{T}^{(n)} := \hat{T}^{(n-1)},
\]
which will be called the \( n \)-th minimal partially normal extension of \( T \).

It was ([4], [3]) shown that
\[
(5) \quad 2\text{-hyponormal} \implies \text{weakly subnormal} \implies \text{hyponormal}
\]
and the converses of both implications in 5 are not true in general. It was ([4]) known that
\[
(6) \quad T \text{ is weakly subnormal} \implies T(\ker [T^*, T]) \subseteq \ker [T^*, T]
\]
and it was ([3]) known that if \( \hat{T} := \text{m.p.n.e.}(T) \) then for any \( k \geq 1 \),
\[
T \text{ is } (k + 1)\text{-hyponormal} \iff T \text{ is weakly subnormal and } \hat{T} \text{ is } k\text{-hyponormal}.
\]
So, in particular, one can see that
\[
(7) \quad \text{if } T \text{ is subnormal then } \hat{T} \text{ is subnormal.}
\]

Recall that given a bounded sequence of positive numbers \( \alpha : \alpha_0, \alpha_1, \ldots \) (called weights), the (unilateral) weighted shift \( W_\alpha \) associated with \( \alpha \) is the operator on \( \ell^2(\mathbb{Z}_+) \) defined by \( W_\alpha e_n := \alpha_n e_{n+1} \) for all \( n \geq 0 \), where \( \{e_n\}_{n=0}^\infty \) is the canonical orthonormal basis for \( \ell^2 \). It is straightforward to check that \( W_\alpha \) can never be normal, and that \( W_\alpha \) is hyponormal if and only if \( \alpha_n \leq \alpha_{n+1} \) for all \( n \geq 0 \). The moments of \( \alpha \) are given as
\[
\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0. \end{cases}
\]

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [5, III.8.16]): \( W_\alpha \) is subnormal if and only if there exists a probability measure \( \xi \) supported in \([0, \|W_\alpha\|^2]\) such that \( \gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int t^k \, d\xi(t) \) \( (k \geq 1) \).

In a talk at Kyoto University entitled ‘On 2-hyponormal operators’, W.Y. Lee posed the following question.
QUESTION 1. Is every weakly subnormal weighted shift 2-hyponormal?

In this paper we negatively answer to the Question 1. To do so, we need the next Lemma.

**Lemma 2.** ([2, Corollary 6]) Let $W_\alpha$ be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then $\alpha$ is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.

**Example 3.** If $W_\alpha$ is the weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$, where

$$\alpha_0 = a, \quad \alpha_1 = b, \quad \alpha_n = 1 \quad (n \geq 2, \quad a < b < 1)$$

then $W_\alpha$ is weakly subnormal, but $W_\alpha$ is not 2-hyponormal.

**Proof.** For the weak subnormality, let

$$A := \begin{pmatrix} a & 0 & 0 \\ 0 & \sqrt{b^2 - a^2} & 0 \\ 0 & 0 & \sqrt{1 - b^2} \end{pmatrix} \oplus 0$$

and

$$B := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{b^2 - a^2} & 0 \\ 0 & 0 & b\sqrt{\frac{1-b^2}{b^2-a^2}} \end{pmatrix} \oplus 0.$$ 

Observe that $[W_\alpha^*, W_\alpha] = A^2 = AA^*$ and $A^*W_\alpha = BA^*$. Thus, $\hat{W}_\alpha := \begin{pmatrix} W_\alpha & A \\ 0 & B \end{pmatrix}$ is a partially normal extension of $W_\alpha$ (cf. [4, Theorem 5.4]).

Since $\alpha$ has two equal weights, by Lemma 2 $W_\alpha$ cannot be 2-hyponormal without being flat. Thus, $W_\alpha$ is not 2-hyponormal.

**Remark 4.** In particular, the weighted shift $W_\alpha$ in Example 3 is a partially normal extension of the unilateral shift $U$: indeed, observe that

$$W_\alpha \cong \begin{pmatrix} U & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ & \vdots & \vdots & \vdots \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{p.n.e.}(U).$$
So, we need not expect that every partially normal extension of an isometry $T$ is 2-hyponormal even though p.n.e.($T$) is weakly subnormal.

References


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