

**k− DENTING POINTS AND k− SMOOTHNESS OF BANACH SPACES**


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Abstract. In this paper, the concepts of $k$−smoothness, $k$−very smoothness and $k$−strongly smoothness of Banach spaces are dealt with together briefly by introducing three types $k$−denting point regarding different topology of conjugate spaces of Banach spaces. In addition, the characterization of first type $w^*−k$ denting point is described by using the slice of closed unit ball of conjugate spaces.

1. Introduction

Throughout this paper, $(X, \| \cdot \|)$ will denote a real Banach space and $X^*$ will denote its conjugate space. Set

$U(X) = \{x : x \in X, \| x \| \leq 1\}$, $U(x_0, \delta) = \{x : x \in X, \| x - x_0 \| \leq \delta\}$,

$S(X) = \{x : x \in X, \| x \| = 1\}$, $S_x = \{f : f \in S(X^*), f(x) = 1 = \| x \|\}$.

For $f \in X^*$ and $\delta > 0$, set $F(f, \delta)$ will denote the slice $\{x \in U(X) : f(x) > 1 - \delta\}$. The symbol $x_n \overset{w^*}{\longrightarrow} x$ (resp. $x_n \overset{w}{\longrightarrow} x$, $x_n \rightarrow x$ ) will denote the sequence $\{x_n\}$ of $X$ which $w^*$ (resp. $w$, strong ) convergence to $x$ in $X$. $\sigma(X, w)$ will denote the weak topology of $X$ and the open (resp. compact, closed ) set regarding weak topology $\sigma(X, w)$ is said

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to be \( w \) open (resp. \( w \) compact, \( w \) closed) set. The symbol \( \sigma(X^*, w^*) \) will denote the weak* topology of \( X^* \) and the open (resp. compact, closed) set regarding weak* topology \( \sigma(X^*, w^*) \) is said to be \( w^* \) open (resp. \( w^* \) compact, \( w^* \) closed) set. The neighborhood regarding weak (weak*) topology is said to be \( w \) (\( w^* \)) neighborhood. The accumulation point regarding weak* topology is said to be \( w^* \) accumulation point. The symbol co\(M\) will denote the convex hull of set \( M \) and the symbol \( \overline{H}^w \) (resp. \( \overline{H}^{w^*} \)) will denote the \( w \) (resp. \( w^* \)) closure of set \( H \), where \( H \subset X^* \).

**Definition 1.1.** A point \( x^* \in S(X^*) \) is said to be first (resp. second) type weak* \( - k \) (in short \( w^* - k \)) denting point of \( U(X^*) \) if there is a \( x \in S(X) \) with \( x^*(x) = 1 \), \( \dim S_x \leq k \) such that for every norm (resp. \( w^* \)) open set \( V_{S_x} \), which includes set \( S_x \), we have \( S_x \cap \sigma^{w^*}(U(X^*) \setminus V_{S_x}) = \emptyset \).

**Definition 1.2.** A point \( x^* \in S(X^*) \) is said to be weak \(- k \) (in short \( w - k \)) denting point of \( U(X^*) \) if there is a \( x \in S(X) \) with \( x^*(x) = 1 \), \( \dim S_x \leq k \) such that for every \( w \) open set \( V_{S_x} \), which includes set \( S_x \), we have \( S_x \cap \sigma^w(U(X^*) \setminus V_{S_x}) = \emptyset \).

**Definition 1.3.** [4] Let \( X \) be a Banach space. A point \( x \in S(X) \) is said to be \( k \)-smooth point of \( X \) if the inequality \( \dim S_x \leq k \) holds for \( x \in S(X) \), where \( \dim S_x \) denote the linear dimension of \( S_x \). \( X \) is said to be \( k \)-smooth space if every point of \( S(X) \) is \( k \)-smooth point of \( X \).

**Definition 1.4.** [4, 9] Let \( X \) be a Banach space. \( X \) is said to be \( k \)-strongly (resp. \( k \)-very) smooth space if and only if \( X \) is \( k \)-smooth space and for any sequence \( \{f_n\} \subset S(X^*) \), \( x \in S(X) \) and \( f_n(x) \to 1 \) imply that \( \{f_n\} \) is relatively compact (resp. relatively \( w \) compact).

Let us recall the concepts of denting point and property \((G)\).

Let \( M \) be a subset of \( X \). A point \( x \in M \) is said to be denting point of \( M \) if \( x \not\in \sigma(M \setminus N(0, \epsilon)) \) holds for any \( \epsilon > 0 \). \( M \) is said to be dentable set if for any \( \epsilon > 0 \) there is a \( x_\epsilon \in M \) such that \( x_\epsilon \not\in \sigma(M \setminus N(x_\epsilon, \epsilon)) \), where \( N(x_\epsilon, \epsilon) = \{x \in X : \|x - x_\epsilon\| < \epsilon \} \). The concept of dentable set was first introduced by Rieffel in 1966 and the following important result has been given in [5]. That is, \( X \) has the Radon-Nikodym property whenever every bounded subset of \( X \) is dentable. This important result, later improved by Maynard [3] in 1973, is very simply. That is, \( X \) has the Radon-Nikodym property if and only if \( X \) is dentable.
The property (G) is given by Fan and Glicksberg [1] in 1955. Banach space $X$ has the property (G) if and only if for all $x \in S(X)$ and $\epsilon > 0$, we have $x \notin \overline{co}(H(x, \epsilon))$, where $H(x, \epsilon) = \{y : y \in X, \|y - x\| \geq \epsilon\}$. In 1993, the concept of strongly convex Banach spaces were introduced by Wu and Li, and the another important result connected to property (G) has been given in [7]. That is, $X$ is strongly convex space if and only if $X$ has the property (G), where $X$ is reflexive Banach space. Noticing that the connection with dentable set and property (G), the above important result can be motivated by the following restatement of property (G). That is, $X$ is strongly convex space if and only if every point of $S(X)$ is denting point of $U(X)$, where $X$ is reflexive Banach space. Up to now, this result is only a result has being known about describing the straight relations between dentability and convexity.

The concept of $w^*$ denting point of $U(X^*)$ was given in [1]. A point $x^* \in S(X^*)$ is said to be denting point of $U(X^*)$ if $x^* \notin \overline{co}(U(X^*) \setminus N(x^*, \epsilon))$ holds for each $\epsilon > 0$, where $N(x^*, \epsilon) = \{y^* : y^* \in X^*, \|y^* - x^*\| < \epsilon\}$. About the strongly smooth space which is the dual concept of strongly convex space, Shang, Cui and Fu [6] are greatly inspired to obtain the following important result: $X$ is strongly smooth spaces if and only if the point of $S(X^*)$ which attains its norm is the $w^*$ denting point of $U(X^*)$. Up to now, this important result is only a result has being known about describing the straight relations between dentability and smoothness also.

In this paper, the concepts of $k$--smoothness, $k$--very smoothness and $k$--strongly smoothness of Banach spaces are dealt with together by introducing three types $k$--denting point regarding different topology of conjugate spaces of Banach spaces. In fact, by using the skill of Banach spaces theory, we show that $X$ is $k$--smooth (resp. $k$--strongly smooth ) spaces if and only if each point of $S(X^*)$ which attains its norm is the second ( resp. first ) type $w^* - k$ denting point of $U(X^*)$; $X$ is $k$--very smooth spaces if and only if each point of $S(X^*)$ which attains its norm is the $w - k$ denting point of $U(X^*)$. Specially, as a simple consequence of these results, we obtain the main result of ref [6]. In fact, the first type weak$^* - 1$ denting point coincide with weak$^*$ denting point. Also, the characterization of first type $w^* - k$ denting point is described by using the slice of closed unit ball of conjugate spaces.
2. Main results

**Theorem 2.1.** $X$ is $k$–very smooth spaces if and only if each point of $S(X^*)$ which attains its norm is the $w – k$ denting point of $U(X^*)$.

**Proof.** Proof of necessity. Firstly, we will prove that if for all $x^* \in S(X^*)$, there exists $x \in S(X)$ such that $x^*(x) = 1$, $\dim S_x \leq k$, and $\{x^*_n\}_{n=1}^\infty \subset S(X^*)$ satisfying $x^*_n(x) \to 1(n \to \infty)$, then
\[
\{x^*_n\}_{n=1}^\infty \wedge S_x \neq \emptyset.
\]
In fact, by the $k$–very smoothness of $X$, we know that $\dim S_x \leq k$ and there exists a subsequence $\{x^*_n\}_{k=1}^\infty$ of $\{x^*_n\}_{n=1}^\infty$ such that $x^*_n \overset{w}{\to} y^*(k \to \infty)$. It follows that $x^*_n(x) \to y^*(x) = 1$, hence $\|y^*\| \geq 1$.

On the other hand, noticing that $U(X^*)$ is $w^*$ closed set, we know that $\|y^*\| \leq 1$. Moreover, we have $y^* \in S_x$. This shows that
\[
\{x^*_n\}_{n=1}^\infty \wedge S_x \neq \emptyset.
\]
Secondly, we will prove that for all $x^* \in S(X^*)$, there exists $x \in S(X)$ such that $x^*(x) = 1$, and for each $w$ open set $V_{S_x}$ which includes $S_x$ there exists a scalar $m > 0$ such that
\[
x^*(x) \geq z^*(x) + m, \text{ if } z^* \in U(X^*) \setminus V_{S_x}.
\]
If it is not true, then there exists $z^*_n \in U(X^*) \setminus V_{S_x}$ such that $z^*_n(x) \to x^*(x) = 1(n \to \infty)$, so we have
\[
\{z^*_n\}_{n=1}^\infty \wedge S_x \neq \emptyset, \{z^*_n\}_{n=1}^\infty \cap V_{S_x} = \emptyset,
\]
which is a contradiction.

Moreover, we have
\[
x^*(x) - m \geq \sup\{z^*(x) : z^* \in U(X^*) \setminus V_{S_x}\} = \sup\{z^*(x) : z^* \in \text{co}(U(X^*) \setminus V_{S_x})\} = \sup\{z^*(x) : z^* \in \overline{\text{co}}(U(X^*) \setminus V_{S_x})\}.
\]
This shows that $x^* \not\in \overline{\text{co}}(U(X^*) \setminus V_{S_x})$, hence $S_x \cap \overline{\text{co}}(U(X^*) \setminus V_{S_x}) = \emptyset$.

By Definition 2.1 we know that each point of $S(X^*)$ which attains its norm is the $w – k$ denting point of $U(X^*)$.

Proof of sufficiency.

Firstly, we will prove that $X$ is $k$–smooth spaces.

For all $x \in S(X)$, by Hahn-Banach theorem, there exists $x^* \in S(X^*)$ such that $x^*(x) = 1$, hence $x^*$ is a point of $S(X^*)$ which attains its norm. By the assumption of Theorem 2.1, we know that $x^*$ is $w – k$ denting point of $U(X^*)$. It follows that $\dim S_x \leq k$, this shows that $X$ is $k$–smooth spaces.
Secondly, we will prove that if
\[ x \in S(X), \{x_n^*\}_{n=1}^\infty \subset S(X^*), \quad x_n^*(x) \to 1(n \to \infty), \]
then \( \{x_n^*\}_{n=1}^\infty \) is relatively \( w \) compact set and there exist
\[ x^* \in S, \text{ net } \{x_\alpha^*\}_{\alpha \in \Delta} \subset \{x_n^*\}_{n=1}^\infty \]
such that \( x_n^* \overset{w^*}{\to} x^* \) (here, we may assume that \( x_n^* \neq x_m^* \) for all \( m \neq n \)).

Because \( U(X^*) \) is \( w^* \) compact set, so there exists \( x^* \in U(X^*) \) such that \( x^* \) become \( w^* \) accumulation point of \( \{x_n^*\}_{n=1}^\infty \).

Let \( \Delta = \{R_{x^*}: R_{x^*} \) is \( w^* \) neighborhood of point \( x^* \}\)
and define a order by inclusive relation, i.e., \( R_{x^*} \subset Q_{x^*} \) if and only if \( R_{x^*} \supset Q_{x^*} \). Then
\[ \{R_{x^*} \cap \{x_n^*\}_{n=1}^\infty : R_{x^*} \) is \( w^* \) neighborhood of point \( x^* \}\]
is a semi-ordered set. By Zermelo principle, there is a mapping \( f \) such that
\[ f(R_{x^*} \cap \{x_n^*\}_{n=1}^\infty) \in R_{x^*} \cap \{x_n^*\}_{n=1}^\infty. \]

Put \( x_\alpha^* = f(R_{x^*} \cap \{x_n^*\}_{n=1}^\infty) \), then \( \{x_\alpha^*\}_{\alpha \in \Delta} \subset \{x_n^*\}_{n=1}^\infty \) is a net. From \( x_n^*(x) \to 1(n \to \infty) \) and the structure of this net, we know that \( x_n^* \overset{w^*}{\to} x^* \) and \( x^* \in S. \)

It remains to prove that \( \{x_n^*\}_{n=1}^\infty \) is relatively \( w \) compact set.

Case 1°: If \( \{x_n^*\}_{n=1}^\infty \cap S = \emptyset \), then \( \{x_n^*\}_{n=1}^\infty \) must be a relatively \( w \) compact set. If it is not true, then any point of \( S \) is not \( w \) accumulation point of \( \{x_n^*\}_{n=1}^\infty \), i.e., for all \( x^* \in S \) there exists a \( w \) neighborhood \( V_{x^*} \)
of point 0 such that \( x^* + V_{x^*} \) does not contain any point of \( \{x_n^*\}_{n=1}^\infty \). We construct a \( w \) open set
\[ V_S = \cup_{x^* \in S} \{y^*: y^* \in x^* + V_{x^*}\}. \]
Obviously, \( V_S \) includes \( S \) and \( \{x_n^*\}_{n=1}^\infty \cap V_S = \emptyset \). Because \( U(X^*) \) is \( w^* \) compact set, so \( \overline{co}^{w^*}(U(X^*) \setminus V_S) \) is \( w^* \) compact set also. Noticing that \( S \) is \( w^* \) closed set, by separating theorem, we know that there exists \( y \in X \) such that
\[ y(S) > \sup y(\overline{co}^{w^*}(U(X^*) \setminus V_S)). \]
Moreover, we choose a scalar \( r > 0 \) such that
\[ y(S) - y(\overline{co}^{w^*}(U(X^*) \setminus V_S)) > r. \]
Obviously,
\[ \{x_n^*\}_{n=1}^\infty \subset \overline{co}^{w^*}(U(X^*) \setminus V_S). \]
On the other hand, by we have proved above, we know that there exists net \( \{x^*_\alpha\}_{\alpha \in \Delta} \subset \{x^*_n\}_{n=1}^\infty \), such that \( x^*_\alpha \overset{w^*}{\to} x^* \) and \( x^* \in S_x \). This contradicts that
\[
y(S_x) - y(\overline{co}^w(U(X^*) \setminus V_{S_x})) > r.
\]
Hence, we obtain the desired result that \( \{x^*_n\}_{n=1}^\infty \) is a relatively \( w \) compact set.

Case 2°: If \( \{x^*_n\}_{n=1}^\infty \cap S_x \neq \emptyset \), then by case 1° we know that \( \{x^*_n\}_{n=1}^\infty \cap S_x \) is a relatively \( w \) compact set. Because \( S_x \) is a bounded closed set of finite dimensional spaces, so \( \{x^*_n\}_{n=1}^\infty \cap S_x \) is a relatively \( w \) compact set. Noticing that
\[
\{x^*_n\}_{n=1}^\infty = (\{x^*_n\}_{n=1}^\infty \cap S_x) \cup (\{x^*_n\}_{n=1}^\infty \setminus S_x),
\]
we have
\[
\{x^*_n\}_{n=1}^\infty \overset{w^*}{\to} (\{x^*_n\}_{n=1}^\infty \cap S_x) \cup (\{x^*_n\}_{n=1}^\infty \setminus S_x).\]
Thus \( \{x^*_n\}_{n=1}^\infty \) is a relatively \( w \) compact set.

**Theorem 2.2.** \( X \) is \( k \)-strongly smooth spaces if and only if each point of \( S(X^*) \) which attains its norm is the first type \( w^* - k \) denting point of \( U(X^*) \).

**Proof.** Proof of necessity. Firstly, we will prove that if for all \( x^* \in S(X^*) \), there exists \( x \in S(X) \) such that \( x^*(x) = 1 \), \( \dim S_x \leq k \), and each norm open set \( V_{S_x} \) which includes \( S_x \) there exists a scalar \( r > 0 \) such that the inequality \( \text{dist}(z^*, S_x) \geq r \) holds for \( z^* \notin V_{S_x} \).

In fact, by the \( k \)-strongly smoothness of \( X \), we know that \( \dim S_x \leq k \). Because \( V_{S_x} \) is a norm open set which includes \( S_x \), so there exists \( \delta' > 0 \) such that \( U(x^*, \delta') \subset V_{S_x} \) holds for \( x^* \in S_x \) and such \( \delta' \) exists a minimum value \( \delta \). Obviously, \( \bigcup_{x^* \in S_x} U(x^*, \delta) \subset V_{S_x} \). Let \( r = \frac{\delta}{2} \), then we have \( \text{dist}(z^*, S_x) \geq r \). Otherwise, there exists \( x^* \in S_x \) such that \( \|z^* - x^*\| < r < \delta \), hence \( z^* \in \bigcup_{x^* \in S_x} U(x^*, \delta) \subset V_{S_x} \). This contradicts that \( z^* \notin V_{S_x} \).

Secondly, we will prove that for all \( x^* \in S(X^*) \), there exists \( x \in S(X) \) such that \( x^*(x) = 1 \), and for each norm open set \( V_{S_x} \) which includes \( S_x \) there exists a scalar \( m > 0 \) such that
\[
x^*(x) \geq z^*(x) + m, \text{ if } z^* \in U(X^*) \setminus V_{S_x}.
\]
If it is not true, then there exists \( z^*_n \in U(X^*) \setminus V_{S_x} \) such that \( z^*_n(x) \to x^*(x) = 1(n \to \infty) \). By the \( k \)-strongly smoothness of \( X \), we can deduce that \( \text{dist}(z^*_n, S_x) \to 0(n \to \infty) \). Otherwise, we may find a \( \epsilon_0 > 0 \) such that for every \( n_0 > 0 \), there exists \( n_k > n_0, k = 1, 2, \cdots \), satisfying
dist\((z^*_{n_k}, S_x) > \epsilon_0\). On the other hand, \(z^*_n(x) \to 1\) implies that \(z^*_{n_k}(x) \to 1\). Hence, by the \(k\)-strongly smoothness of \(X\) we know that \(\{z^*_{n_k}\}\) is a relatively compact set. It follows that there exists subsequence \(\{z^*_{n_{k_l}}\} \subset \{z^*_{n_k}\}\) such that \(z^*_{n_{k_l}} \to z^*_0\). Hence \(z^*_{n_{k_l}}(x) \to z^*_0(x) = 1\) and \(z^*_0 \in S_x\). Which leads to that \(dist(z^*_{n_{k_l}}, S_x) \to 0\). This contradicts that \(dist(z^*_{n_{k_l}}, S_x) > \epsilon_0\).

Moreover, we have
\[
x^*(x) - m \geq \sup\{z^*(x) : z^* \in U(X^*) \setminus V_{S_x}\}
\]
\[
= \sup\{z^*(x) : z^* \in co(U(X^*) \setminus V_{S_x})\}
\]
\[
= \sup\{z^*(x) : z^* \in co^w(U(X^*) \setminus V_{S_x})\}.
\]
This shows that \(x^* \notin co^w(U(X^*) \setminus V_{S_x})\), it follows that \(S_x \cap co^w(U(X^*) \setminus V_{S_x}) = \emptyset\). Hence, we obtain the desired result that each point of \(S(X^*)\) which attains its norm is the first type \(w^* - k\) denting point of \(U(X^*)\).

Proof of sufficiency. Suppose that \(x \in S(X)\), \(\{x^*_n\}_{n=1}^{\infty} \subset S(X^*)\), \(x^*_n(x) \to 1(n \to \infty)\). Greatly similarly to the proof of Theorem 2.1, by using the given conditions in Theorem 2.2, we can prove that there exists a net \(x^* \in S_x\{x^*_n\}_{n=1}^{\infty} \subset \{x^*_n\}_{n \in \Delta}\) such that \(x^*_n \xrightarrow{w^*} x^*\) and \(X\) is \(k\)-smooth spaces. Now we prove that \(\{x^*_n\}_{n=1}^{\infty}\) is a relatively compact set.

Case 1°: If \(\{x^*_n\}_{n=1}^{\infty} \cap S_x = \emptyset\), then \(\{x^*_n\}_{n=1}^{\infty}\) must be a relatively compact set. If it is not true, then any point of \(S_x\) is not accumulation point of \(\{x^*_n\}_{n=1}^{\infty}\). Hence, for all \(x^* \in S_x\) there is a \(\epsilon > 0\) such that the set \(\{y^* : \|y^* - x^*\| < \epsilon\}\) does not contain any point of \(\{x^*_n\}_{n=1}^{\infty}\). We construct a norm open set
\[
V_{S_x} = \cup_{x^* \in S_x} \{y^* : \|y^* - x^*\| < \epsilon\}.
\]
Obviously, \(V_{S_x}\) includes \(S_x\) and \(\cup_{x^* \in S_x} \{y^* : \|y^* - x^*\| < \epsilon\} \cap \{x^*_n\}_{n=1}^{\infty} = \emptyset\). Greatly similarly to the proof of Theorem 2.1, we can deduce that \(\{x^*_n\}_{n=1}^{\infty}\) is a relatively compact set.

Case 2°: If \(\{x^*_n\}_{n=1}^{\infty} \cap S_x \neq \emptyset\), then by case 1° we know that \(\{x^*_n\}_{n=1}^{\infty} \setminus S_x\) is a relatively compact set. Because \(S_x\) is a bounded closed set of finite dimensional spaces, so \(\{x^*_n\}_{n=1}^{\infty} \cap S_x\) is a relatively compact set. Noticing that
\[
\{x^*_n\}_{n=1}^{\infty} = (\{x^*_n\}_{n=1}^{\infty} \cap S_x) \cup (\{x^*_n\}_{n=1}^{\infty} \setminus S_x),
\]
we have
\[
\{x^*_n\}_{n=1}^{\infty} = (\{x^*_n\}_{n=1}^{\infty} \cap S_x) \cup (\{x^*_n\}_{n=1}^{\infty} \setminus S_x),
\]
Thus \(\{x^*_n\}_{n=1}^{\infty}\) is a relatively compact set.

When \(k = 1\), the first type \(w^* - 1\) denting point coincide with \(w^*\) denting point. It is well known that 1–strongly smooth space coincide
with usual strongly smooth spaces [8]. Hence we obtained the following corollary.

**Corollary 2.1.** [6] \( X \) is strongly smooth spaces if and only if each point of \( S(X^*) \) which attains its norm is the \( w^* \)-denting point of \( U(X^*) \).

In what follows, using the slice of closed unit ball of conjugate spaces \( X^* \), we will describe the characterization of first type \( w^* - k \) denting point.

**Theorem 2.3.** \( x^* \in S(X^*) \) is first \( w^* - k \) denting point of \( U(X^*) \) if and only if there exists \( x \in S(X) \) such that \( x^* \in S_x \), \( \dim S_x \leq k \) and for \( \forall \epsilon > 0 \), there exists slice

\[
F(x, \delta) = \{ z^* : z^* \in U(X^*), z^*(x) > 1 - \delta \}
\]

satisfying the inclusive relation

\[
F(x, \delta) \subset \{ y^* : y^* \in U(X^*), d(y^*, S_x) < \epsilon \}.
\]

**Proof.** Proof of necessity. Suppose that \( x^* \in S(X^*) \) is first \( w^* - k \) denting point of \( U(X^*) \), then there exists \( x \in S(X) \) such that \( x^* \in S_x \), \( \dim S_x \leq k \). Let

\[
H_{S_x} = \{ y^* : y^* \in U(X^*), d(y^*, S_x) < \epsilon \},
\]

then \( H_{S_x} \) is norm open set which includes \( S_x \), hence \( S_x \cap \text{co}w^*(U(X^*) \setminus H_{S_x}) = \emptyset \). Moreover, we can deduce that

\[
\sup x(\text{co}w^*(U(X^*) \setminus H_{S_x})) < 1.
\]

Otherwise, there exists sequence \( y^*_n \in \text{co}w^*(U(X^*) \setminus H_{S_x}) \) such that \( y^*_n(x) \to 1 \) \((n \to \infty)\). Let \( x^*_n = \frac{y^*_n}{\|y^*_n\|} \), then \( x^*_n(x) \to 1 \) \((n \to \infty)\). From the proof of Theorem 2.2, we know that \( x \) is \( k \)-smooth point of \( X \) and \( \{x^*_n\}_{n=1}^\infty \) is relatively compact set. Therefore, sequence \( \{x^*_n\}_{n=1}^\infty \) has the convergent subsequence, without loss of generality, let the convergent subsequence be \( \{x^*_n\}_{n=1}^\infty \) itself and suppose that \( x^*_n \to x^*_0 \) \((n \to \infty)\).

Clearly,

\[
x^*_n(x) \to 1 = x^*_0(x) \quad (n \to \infty), \quad x^*_0 \in S_x.
\]

On the other hand,

\[
\|y^*_n - x^*_0\| \leq \|\frac{y^*_n}{\|y^*_n\|} - y^*_n\| + \|\frac{y^*_n}{\|y^*_n\|} - x^*_0\| \to 0 \quad (n \to \infty),
\]

it follows that \( x^*_0 \) belong to the norm closure of set \( \text{co}w^*(U(X^*) \setminus H_{S_x}) \).

Noticing that this set is closed set regarding norm topology, we know that \( x^*_0 \in \text{co}w^*(U(X^*) \setminus H_{S_x}) \), hence \( x^*_0 \notin H_{S_x} \). It is impossible.

Let \( 1 - \delta = \sup x(\text{co}w^*(U(X^*) \setminus H_{S_x})) \). It is easy to see that if

\[
z^* \in F(x, \delta) = \{ z^* : z^* \in U(X^*), z^*(x) > 1 - \delta \},
\]

then \( z^* \notin \text{co}w^*(U(X^*) \setminus H_{S_x}) \). Hence \( z^* \in H_{S_x} \), this shows that \( F(x, \delta) \subset H_{S_x} \).
Proof of sufficiency. Suppose that there exists \( x \in S(X) \) such that \( x^* \in S_x \), \( \dim S_x \leq k \) and for \( \forall \epsilon > 0 \), there exists slice \( F(x, \delta) = \{ z^* : z^* \in U(X^*), z^*(x) > 1 - \delta \} \) satisfying the inclusive relation
\[
F(x, \delta) \subset \{ y^* : y^* \in U(X^*), d(y^*, S_x) < \epsilon \}.
\]
For the convenient, we denote \( \{ y^* : y^* \in U(X^*), d(y^*, S_x) < \epsilon \} \) by \( H_{S_x} \), then
\[
1 - \delta \geq \sup\{ z^*(x) : z^* \in \text{co}(U(X^*) \setminus H_{S_x}) \} = \sup\{ z^*(x) : z^* \in \overline{co}^w(U(X^*) \setminus H_{S_x}) \}.
\]
Moreover, we can deduce that \( S_x \cap \overline{co}^w(U(X^*) \setminus H_{S_x}) = \emptyset \) from the structure of \( S_x \). Hence \( x^* \in S(X^*) \) is first \( w^* - k \) denting point of \( U(X^*) \). \( \square \)

**Theorem 2.4.** \( X \) is \( k \)-smooth spaces if and only if each point of \( S(X^*) \) which attains its norm is the second type \( w^* - k \) denting point of \( U(X^*) \).

**Proof.** The sufficiency is immediate from the definition of \( k \)-smooth spaces. It remains to prove the necessity.

Firstly, we will prove that for all \( x^* \in S(X^*) \), there exists \( x \in S(X) \) such that \( x^*(x) = 1 \), and \( \{ x^*_n \}_{n=1}^\infty \subset S(X^*) \) satisfying \( x^*_n(x) \to 1(n \to \infty) \), then \( \{ x^*_n \}_{n=1}^\infty \cap S_x \neq \emptyset \).

If it is not true, then there exists \( w^* \) neighborhood \( V_{S_x} \) which includes \( S_x \) such that \( \{ x^*_n \}_{n=1}^\infty \cap S_x = \emptyset \). From the proof of sufficient of Theorem 2.2, we know that there exists net \( \{ x^*_\alpha \}_{\alpha \in \Delta} \subset \{ x^*_n \}_{n=1}^\infty \) satisfying \( x^*_\alpha \xrightarrow{w^*} x^* \), \( x^* \in S_x \). Hence \( \{ x^*_n \}_{n=1}^\infty \cap S_x \neq \emptyset \). This contradicts that \( \{ x^*_n \}_{n=1}^\infty \cap S_x = \emptyset \).

Secondly, we will prove that if for all \( x^* \in S(X^*) \), there exists \( x \in S(X) \) such that \( x^*(x) = 1 \), and each \( w^* \) open set \( V_{S_x} \) which includes \( S_x \) there exists a scalar \( m > 0 \) such that \( x^*(x) \geq z^*(x) + m \) holds for \( z^* \in U(X^*) \setminus V_{S_x} \).

If it is not true, then there exists \( z^*_n \in U(X^*) \setminus V_{S_x} \) such that \( z^*_n(x) \to x^*(x) = 1(n \to \infty) \). Hence we have \( \{ z^*_n \}_{n=1}^\infty \cap S_x \neq \emptyset \). On the other hand, for \( z^*_n \in U(X^*) \setminus V_{S_x} \), we have \( \{ z^*_n \}_{n=1}^\infty \cap V_{S_x} = \emptyset \). This contradicts that \( \{ z^*_n \}_{n=1}^\infty \cap S_x \neq \emptyset \).

Moreover, we have
\[
x^*(x) - m \geq \sup\{ z^*(x) : z^* \in U(X^*) \setminus V_{S_x} \} = \sup\{ z^*(x) : z^* \in \text{co}(U(X^*) \setminus V_{S_x}) \} = \sup\{ z^*(x) : z^* \in \overline{co}^w(U(X^*) \setminus V_{S_x}) \}.
\]
This shows that $x^* \not\in \overline{w^*}(U(X^*)\setminus V_{S_x})$, it follows that $S_x \cap \overline{w^*}(U(X^*)\setminus V_{S_x}) = \emptyset$. By the definition of $k-$ smooth spaces, we know that $\text{dim} S_x \leq k$. Hence, we obtain the desired result that each point of $S(X^*)$ which attains its norm is the second type $w^* - k$ denting point of $U(X^*)$. □

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