WHEN THE NAGATA RING $D(X)$ IS A SHARP DOMAIN

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Abstract. Let $D$ be an integral domain, $X$ be an indeterminate over $D$, $D[X]$ be the polynomial ring over $D$, and $D(X)$ be the Nagata ring of $D$. Let $[d]$ be the star operation on $D[X]$, which is an extension of the $d$-operation on $D$ as in [5, Theorem 2.3]. In this paper, we show that $D$ is a sharp domain if and only if $D[X]$ is a $[d]$-sharp domain, if and only if $D(X)$ is a sharp domain.

1. Introduction

Let $D$ be an integral domain and $*$ be a star operation on $D$. (The definitions related to star operations will be reviewed in Section 2.) As in [2], we say that $D$ is a *-sharp domain if whenever $I \supseteq AB$ with $I, A, B$ nonzero ideals of $D$, there exist nonzero ideals $H$ and $J$ of $D$ such that $I^* = (HJ)^*$, $H^* \supseteq A$, and $J^* \supseteq B$. Following [1], we say that a $d$-sharp domain is a sharp domain, i.e., $D$ is a sharp domain if whenever $I \supseteq AB$ with $I, A, B$ nonzero ideals of $D$, there exist ideals $A_0 \supseteq A$ and $B_0 \supseteq B$ of $D$ such that $I = A_0B_0$. Assume that $D$ is a *-sharp domain. It is known that $D$ is a $t$-sharp domain (and hence $D$ is a $PvMD$ whose prime $t$-ideals are maximal $t$-ideal); if $* = *_w$, then $D$ is a $P*MD$ whose maximal *-ideals have height-one; and $I_v$ is *-invertible for all nonzero fractional ideals $I$ of $D$ [2, Propositions 2.2, 3.1, 2.3 and
2.4]. Also, $D$ is a $v$-sharp domain if and only if $D$ is completely integrally closed [2, Corollary 2.6].

In [2, Theorem 3.7(a)], the authors showed that $D$ is a $t$-sharp domain if and only if $D[X]$, the polynomial ring over $D$, is a $t$-sharp domain. They then remarked that “we do not have a ‘$d$-analogue’ of [2, Theorem 3.7(a)] because a sharp domain has Krull dimension $\leq 1$”. In fact, if $D$ is a sharp domain, then $D$ is a Prüfer domain with dim($D$) $\leq 1$ [1, Theorem 11]. Hence, $D[X]$ is a sharp domain if and only if $D$ is a field. However, in this paper, we use the star operation $[d]$ on $D[X]$ (see Lemma 2) to prove the $d$-operation analogue of [2, Theorem 3.7(a)]. Precisely, we prove that if $*$ is a star operation on $D$ such that $*_w = *$, then $D$ is a $*$-sharp domain if and only if $D[X]$ is a $[*]$-sharp domain, if and only if $D[X]_{N_w}$ is a sharp domain, where $N_w = \{f \in D[X] \mid c(f)^* = D\}$. Let $D(X)$ be the Nagata ring of $D$, i.e., $D(X) = \{\frac{f}{g} \mid f, g \in D[X] \text{ and } c(g) = D\}$. As a corollary, we have that $D$ is a sharp domain if and only if $D[X]$ is a $[d]$-sharp domain, if and only if $D(X)$ is a sharp domain. Finally, we study when $D[X]$ is a $*$-sharp domain if $*$ is a star operation on $D[X]$ such that $*_w = *$.

2. Definitions related to star operations

Let $D$ be an integral domain with quotient field $K$, $F(D)$ be the set of nonzero fractional ideals of $D$, and $f(D)$ be the set of nonzero finitely generated fractional ideals of $D$; so $f(D) \subseteq F(D)$, and equality holds if and only if $D$ is Noetherian. We say that a mapping $* : F(D) \to F(D)$, $I \mapsto I^*$, is a star operation on $D$ if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in F(D)$: (i) $(aD)^* = aD$ and $(aI)^* = aI^*$, (ii) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$, and (iii) $(I^J)^* = I^*$. Given a star operation $*$ on $D$, two new star operations $*_f$ and $*_w$ on $D$ can be constructed as follows for all $I \in F(D)$: $I^*_f = \bigcup\{J^* \mid J \subseteq I \text{ and } J \in f(D)\}$ and $I^*_w = \{x \in K \mid xJ \subseteq I \text{ for some } J \in f(D)\}$ with $J^* = D$. Clearly, $(*_f)_{f} = *_{f}$ and $(*_w)_{w} = *_{w}$. An $I \in F(D)$ is called a $*$-ideal if $I^* = I$, and a $*$-ideal is a maximal $*$-ideal if it is maximal among proper integral $*$-ideals. Let $*_f\text{-Max}(D)$ be the set of maximal $*$-ideals of $D$. It is known that $*_f\text{-Max}(D) \neq \emptyset$ when $D$ is not a field; $*_f\text{-Max}(D) = *_w\text{-Max}(D)$ [3, Theorem 2.16]; and $I^*_w = \bigcap_{P \in \text{*f-Max}(D)} fDP$ for all $I \in F(D)$ [3, Corollary 2.10]. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$; then $I^{-1} \in F(D)$. We say
that \( I \in F(D) \) is \(*\)-invertible if \((II^{-1})^* = D \), and \( D \) is a Prüfer \(*\)-multiplication domain (P\(*\)MD) if every nonzero finitely generated ideal of \( D \) is \(*_f\)-invertible. Examples of star operations include the \( d\), \( v\)-, \( t\)-, and \( w\)-operations. The \( d\)-operation is the identity function of \( F(D) \), i.e., \( I^d = I \) for all \( I \in F(D) \), the \( v\)-operation is defined by \( I^v = (I^{-1})^{-1} \), the \( t\)-operation is defined by \( t = v_f \), and the \( w\)-operation is given by \( w = v_w \). For more on basic properties of star operations, see [8, Sections 32 and 34].

Let \( X \) be an indeterminate over \( D \), \( D[X] \) be the polynomial ring over \( D \), and \( c(f) \) be the ideal of \( D \) generated by the coefficients of \( f \in D[X] \). The next lemma is nice characterizations of P\(*\)MDs, which appear in [7, Theorem 3.1 and Proposition 3.15] in a more general setting of semistar operations.

**Lemma 1.** Let \(*\) be a star operation on \( D \) with \(*_f = *\). Then the following statements are equivalent.

1. \( D \) is a P\(*\)MD.
2. \( D \) is a P\(v\)MD and \(* = t\).
3. \( D \) is a P\(v\)MD and \(*_w = t\).
4. \( D[X]_{N_*} \) is a Prüfer domain, where \( N_* = \{ f \in D[X] \mid c(f)^* = D \} \).

In this case, \(*_f = *_w = t = w\).

Let \(*\) be a star operation on \( D \). Then there is a star operation \([*]\) on \( D[X] \), which is an extension of \(*_w\) to \( D[X] \) in the sense that \((I[X])[\cdot]\cap K = I^{*w}\) for each \( I \in F(D) \). We recall this result for easy reference of the reader.

**Lemma 2.** [5, Theorem 2.3] Let \( X, Y \) be two indeterminates over \( D \), \(*\) be a star operation on \( D \), and let

\[
\Delta = \{ Q \in \text{Spec}(D[X]) \mid Q \cap D = (0) \text{ with } htQ = 1 \text{ or } Q = (Q \cap D)[X] \text{ and } (Q \cap D)^{*f} \subsetneq D \}.
\]

Set \( S = D[X][Y] \setminus \left( \bigcup \{ Q[Y] \mid Q \in \Delta \} \right) \) and define

\[
A^{[\cdot]} = A[Y]_{S \cap K(X)} \text{ for all } A \in F(D[X]).
\]

1. The mapping \([*] : F(D[X]) \to F(D[X])\), given by \( A \mapsto A^{[\cdot]}\), is a star operation on \( D[X] \) such that \([*] = [\cdot]_f = [\cdot]_w\).
2. \([*] = [\cdot]_f = [\cdot]_w\).
3. \((ID[X])^{[\cdot]} \cap K = I^{*w}\) for all \( I \in F(D)\).
4. \((ID[X])^{[\cdot]} = I^{*w}D[X]\) for all \( I \in F(D)\).
5. $\text{Max}(D[X]) = \{Q \mid Q \in \text{Spec}(D[X]) \text{ such that } Q \cap D = (0),
\text{ht}(Q) = 1, \text{ and } \left(\sum_{g \in Q} c(g)\right)^* = D\}$ \cup \{P[X] \mid P \in \text{Max}(D)\}.

6. $[v]$ is the $w$-operation on $D[X]$.

Let $*$ be a star operation on $D$. It is known that $D$ is a P$*$MD if and only if $D[X]$ is a P$*$MD \cite{5, Corollary 2.5}; hence $D$ is a P$v$MD if and only if $D[X]$ is a P$v$MD. Also, since a P$d$MD is just the Pr"ufer domain, $D$ is a Pr"ufer domain if and only if $D[X]$ is a P$d$MD.

3. Main Results

Let $D$ be an integral domain with quotient field $K$, and we assume that $D \neq K$ in order to avoid the trivial case. Let $X$ be an indeterminate over $D$, $D[X]$ be the polynomial ring over $D$, and $N_v = \{f \in D[X] \mid c(f)^v = D\}$.

**Lemma 3.** Let $N_v = \{f \in D[X] \mid c(f)^v = D\}$, $I \in F(D)$, and $A \in F(D[X])$.

1. $ID[X]_{N_v} \cap K = I^w$, and hence $I^w D[X]_{N_v} = ID[X]_{N_v}$.
2. $A_{N_v} = (A^w)_{N_v}$.

**Proof.** (1) \cite[Lemma 2.1]{4}.

(2) It suffices to show that $A^w \subseteq A_{N_v}$. For this, let $0 \neq f \in A^w$. Then there is a nonzero finitely generated ideal $J$ of $D[X]$ such that $J^v = D[X]$ and $fJ \subseteq A$. Since $J^v = D[X]$, $J \nsubseteq P[X]$ for all $P \in t$-Max$(D)$, and hence $\left(\sum_{h \in J} c(h)\right)^t = D$. Hence, there is a $0 \neq g \in J$ with $c(g)^v = D$; so $g \in N_v$ and $fg \in A$. Thus, $f \in A_{N_v}$.

It is known that $D$ is a $t$-sharp domain if and only if $D[X]$ is a $t$-sharp domain, if and only if $D[X]_{N_v}$ is a sharp domain \cite[Theorem 3.7]{2}. We next generalize this result to an arbitrary star operation $*$ on $D$ with $*_w = *$.

**Theorem 4.** Let $*$ be a star operation on $D$ such that $*_w = *$. Then the following statements are equivalent.

1. $D$ is a $*$-sharp domain.
2. $D[X]$ is a $[\ast]$-sharp domain.
3. $D[X]_{N_*}$ is a sharp domain, where $N_* = \{f \in D[X] \mid c(f)^* = D\}$.

In this case, $* = t = w$ on $D$ and $[\ast] = t = w$ on $D[X]$. 
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**Proof.** (1) $\Rightarrow$ (2) If $D$ is a $*$-sharp domain, then $D$ is a $P$*$MD$ [2, Proposition 2.3], and hence $D$ is a $PvMD$ with $*=t=w$ on $D$ by Lemma 1. Hence, $D$ is a $t$-sharp domain, and thus $D[X]$ is a $t$-sharp domain. However, note that $D[X]$ is a $PvMD$ and $[t]=w=t$ on $D[X]$ by Lemmas 1 and 2; so $[*]=t$. Thus, $D[X]$ is a $[*]$-sharp domain.

(2) $\Rightarrow$ (3) Assume that $D[X]$ is a $[*]$-sharp domain. Then $D[X]$ is a $PvMD$ and $[*]=t=w$ on $D[X]$; hence $I^*=(ID[X])[*]\cap K=(ID[X])^t\cap K=I^tD[X]\cap K=I^t$ for all $I\in F(D)$ by Lemma 2 and [9, Proposition 4.3]. Thus, $D$ is a $PvMD$ and $*=t=w$ on $D$.

Let $I_{N*}=(A_{N*})(B_{N*})$ with $I,A,B$ nonzero ideals of $D[X]$, and let $C=ID[X]_{N*}\cap D[X]$. Then $C\supseteq AB$, and hence by assumption, there exist nonzero ideals $A_0$ and $B_0$ of $D[X]$ such that $C^w=(A_0B_0)^w$, $(A_0)^w\supseteq A$, and $(B_0)^w\supseteq B$. Thus, by Lemma 3, $I_{N*}=C_{N*}=(C^w)_{N*}=(A_0B_0)^w_{N*}=(A_0B_0)_{N*}=(A_0)_{N*}((B_0)_{N*})$, $(A_0)_{N*}=(A_0)^w_{N*}\supseteq A_{N*}$, and $(B_0)_{N*}=(B_0)^w_{N*}\supseteq B_{N*}$. Thus, $D[X]_{N*}$ is a sharp domain.

(3) $\Rightarrow$ (1) Suppose that $D[X]_{N*}$ is a sharp domain. Then $D[X]_{N*}$ is a Prüfer domain [2, Theorem 11], and hence $D$ is a $P$*$MD$ by Lemma 1 and every ideal of $D[X]_{N*}$ is extended from $D$ [10, Theorem 3.1]. Let $I\supseteq AB$ with $I,A,B$ nonzero ideals of $D$. Then $ID[X]_{N*}\supseteq AD[X]_{N*}BD[X]_{N*}$, and hence by assumption, there exist nonzero ideals $H$ and $J$ of $D$ such that

$$ID[X]_{N*}=(HD[X]_{N*})(JD[X]_{N*})=(HJ)D[X]_{N*},$$

$HD[X]_{N*}\supseteq AD[X]_{N*}$, and $JD[X]_{N*}\supseteq BD[X]_{N*}$. Thus, by Lemma 3, $I^*=ID[X]_{N*}\cap K=(HJ)D[X]_{N*}\cap K=(HJ)^*$, $H^*=HD[X]_{N*}\cap K\supseteq AD[X]_{N*}\cap K\supseteq A$, and $J^*=JD[X]_{N*}\cap K\supseteq BD[X]_{N*}\cap K\supseteq B$. Thus, $D$ is a $*$-sharp domain.

The next result is the $d$-operation analogue of [2, Theorem 3.7(a)] that $D$ is a $t$-sharp domain if and only if $D[X]$ is a $t$-sharp domain.

**Corollary 5.** The following statements are equivalent for an integral domain $D$.

1. $D$ is a sharp domain.
2. $D[X]$ is a $[d]$-sharp domain.
3. $D(X)$ is a sharp domain.

**Proof.** This follows directly from Theorem 4 because $D(X)=D[X]_{N*}$.

$\square$
Let \(*\) be a star operation on \(D[X]\), and let \(I^*= (ID[X])^* \cap K\) for all \(I \in F(D)\). Then it is easy to see that \(*\) is a star operation on \(D\) (cf. [6, Lemma 5]). A nonzero prime ideal \(Q\) of \(D[X]\) is said to be an upper to zero in \(D[X]\) if \(Q \cap D = (0)\); so each upper to zero in \(D[X]\) has height-one. Let \(\ast = \ast_f\), and note that if every upper to zero in \(D[X]\) is a maximal \(*\)-ideal, then \(\ast\text{-Max}(D[X]) = t\text{-Max}(D[X])\) [11, Theorem 2.9], and hence \(\ast_w = w\) on \(D[X]\).

**Corollary 6.** Let \(\ast\) be a star operation on \(D[X]\) with \(\ast_w = \ast\), and let \(\ast\) be the star operation on \(D\) defined by \(I^*= (ID[X])^* \cap K\) for all \(I \in F(D)\). Suppose that every upper to zero in \(D[X]\) is a maximal \(*\)-ideal. Then \(D[X]\) is a \(\ast\)-sharp domain if and only if \(D\) is a \(\ast\)-sharp domain.

**Proof.** \((\Rightarrow)\) Assume that \(D[X]\) is a \(\ast\)-sharp domain. Then \(D[X]\) is a \(PvMD\) and \(\ast = t = w\) on \(D[X]\) as in the proof of Theorem 4, and hence \(D\) is a \(PvMD\) and \(\ast = t = w\) on \(D\) [6, Lemma 5]. Note that \([t] = w\) on \(D[X]\); so \([\ast] = \ast\). Thus, \(D\) is a \(\ast\)-sharp domain by Theorem 4.

\((\Leftarrow)\) Suppose that \(D\) is a \(\ast\)-sharp domain, and note that \(\ast_w = \ast\) [6, Lemma 5] because \(\ast_w = \ast\). Hence, \(D[X]\) is a \([\ast]\)-sharp domain and \([\ast] = t = w\) on \(D[X]\) by Theorem 4. Since every upper to zero in \(D[X]\) is a maximal \(*\)-ideal, \(\ast = w = \([\ast]\) on \(D[X]\). Thus, \(D[X]\) is a \(\ast\)-sharp domain. 

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**References**


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