LIPSCHITZ CONTINUOUS AND COMPACT COMPOSITION OPERATOR ACTING BETWEEN SOME WEIGHTED GENERAL HYPERBOLIC-TYPE CLASSES


Abstract. In this paper, we study Lipschitz continuous, the boundedness and compactness of the composition operator $C_{\phi}$ acting between the general hyperbolic Bloch type-classes $B_{p,\log,\alpha}^{*}$ and general hyperbolic Besov-type classes $F_{p,\log}^{*}(p,q,s)$. Moreover, these classes are shown to be complete metric spaces with respect to the corresponding metrics.

1. Introduction

Let $\phi$ be an analytic self-map of the open unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the classes of analytic functions in the unit disc $\mathbb{D}$. Let $B(\mathbb{D})$ be a subset of $H(\mathbb{D})$ denote the classes of all the hyperbolic function classes in $\mathbb{D}$, such that $|f(z)| < 1$. A function $f \in B(\mathbb{D})$ belongs to $\alpha$-Bloch space $B_{\alpha}^{*}$, $0 < \alpha < \infty$ if

$$\| f \|_{B_{\alpha}^{*}} = \sup_{z \in \mathbb{D}} (1 - |z|)^{\alpha} |f'(z)| < \infty.$$ 

The little $\alpha$-Bloch space $B_{\alpha,0}^{*}$ consisting of all $f \in B_{\alpha}$ such that

$$\lim_{|z| \to 1-} (1 - |z|^{2}) |f'(z)| = 0.$$ 

Received March 14, 2016. Revised December 2, 2016. Accepted December 7, 2016.

2010 Mathematics Subject Classification: 47B38, 46E15.

Key words and phrases: metric space, the general hyperbolic Bloch-type classes $B_{p,\log,\alpha}^{*}$, the general hyperbolic Besov-type classes $F_{p,\log}^{*}(p,q,s)$.


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If \((X, d)\) is a metric space, we denote the open and closed balls with center \(x\) and radius \(r > 0\) by
\[
B(x, r) := \{y \in X : d(y, x) < r\} \quad \text{and} \quad \overline{B}(x, r) := \{y \in X : d(x, y) \leq r\},
\]
respectively.

Hyperbolic function classes are usually defined by using either the hyperbolic derivative \(f^* (z) = \frac{|f'(z)|}{1-|f(z)|^2}\) of \(f \in B(\mathbb{D})\), or the hyperbolic distance \(\rho(f(z), 0) := \frac{1}{2} \log \left( \frac{1+|f(z)|}{1-|f(z)|} \right)\) between \(f(z)\) and zero.

2. Preliminaries and basic concepts

The hyperbolic \(B^*_\alpha\) (see [3]) is defined as the set of \(f \in B(\mathbb{D})\) for which
\[
B^*_\alpha = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha f^*(z) < \infty \}.
\]

The little hyperbolic Bloch space \(B^*_{\alpha,0}\) is a subspace of \(B^*_\alpha\) consisting of all \(f \in B^*_\alpha\) such that
\[
\lim_{|z| \to 1} (1-|z|^2)^\alpha f^*(z) = 0.
\]

Quite recently, the author in [3] gave the following definitions for \((p, \alpha)\)-Bloch spaces \(B^*_{p,\alpha}\) and \(B^*_{p,\alpha,0}\) for \(f \in H(\mathbb{D})\)
\[
\|f\|_{B^*_{p,\alpha}} = \frac{p}{2} \sup_{z \in \mathbb{D}} |f(z)|^{\frac{p}{2}-1} |f'(z)| (1-|z|^2)^\alpha < \infty,
\]
and
\[
\lim_{|z| \to 1} |f(z)|^{\frac{p}{2}-1} |f'(z)| (1-|z|^2)^\alpha = 0,
\]
where \(2 \leq p < \infty\) and \(0 < \alpha < 1\).

Also in [3], the first author introduced the following generalized hyperbolic derivative:
\[
f^*_p (z) = \frac{p}{2} \frac{|f(z)|^{\frac{p}{2}-1} |f'(z)|}{1-|f(z)|^p}, \quad f(z) \in H(\mathbb{D}),
\]
when \(p = 2\) we obtain the usual hyperbolic derivative as defined above.

A function \(f \in B(\mathbb{D})\) is said to belong to the generalized \((p, \alpha)\) hyperbolic Bloch-type class \(B^*_{p,\alpha}\) if
\[
\|f\|_{B^*_{p,\alpha}} = \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha f^*_p (z) < \infty,
\]
the little generalized \((p, \alpha)\) hyperbolic Bloch-type class \(B^*_{p,\alpha,0}\) consists of all \(f \in B^*_{p,\alpha}\) such that
\[
\lim_{|z| \to 1} (1 - |z|^2)^{\alpha} f_p^*(z) = 0.
\]

**Remark 2.1.** It should be remarked that, the Schwarz-Pick lemma implies \(B^*_{p,\alpha} \equiv B(D)\) for all \(1 \leq \alpha < \infty\) with \(\|f\|_{B^*_{p,\alpha}} \leq 1\), hence the class \(B^*_{p,\alpha}\) is of interest only when \(0 < \alpha < 1\).

Denote by
\[
g(z, a) = \log \left| \frac{1 - a\overline{z}}{z - a} \right| = \log \left| \varphi_a(z) \right|
\]
the Green’s function of \(D\) with logarithmic singularity at \(a \in D\).

Now, we give the following definitions of the generalized hyperbolic Bloch-type classes \(B^*_{p,\log,\alpha}\) and the generalized hyperbolic Besov-type classes \(F^*_{p,\log, (p,q,s)}\):

**Definition 2.1.** Let \(2 \leq p, \alpha < \infty\), the generalized hyperbolic Bloch-type classes \(B^*_{p,\log,\alpha}\) consisting of all \(f \in B(D)\) such that
\[
\|f\|_{B^*_{p,\log,\alpha}} = \sup_{z \in D} f_p^*(z)(1 - |z|^2)^{\alpha} \left( \log \frac{2}{1 - |z|^2} \right) < \infty,
\]
the little generalized \((p, \log, \alpha)\) hyperbolic Bloch-type classes \(B^*_{p,\log,\alpha,0}\) consists of all \(f \in B^*_{p,\log,\alpha}\) such that
\[
\lim_{|z| \to 1} f_p^*(z)(1 - |z|^2)^{\alpha} \left( \log \frac{2}{1 - |z|^2} \right) = 0.
\]

**Definition 2.2.** Let \(2 \leq p < \infty, 0 < s < \infty\) and \(-2 < q < \infty\), the hyperbolic class \(F^*_{p,\log, (p,q,s)}\) consists of all functions \(f \in B(D)\) for which
\[
\|f\|_{F^*_{p,\log, (p,q,s)}} = \sup_{\alpha \in D} \int_{D} (f_p^*(z))^{p}(1 - |z|^2)^{q} g^s(z, a) \left( \log \frac{2}{1 - |z|^2} \right)^{p} dA(z) < \infty.
\]
Moreover, we say that \(f \in F^*_{p,\log, (p,q,s)}\) belongs to the class \(F^*_{p,\log,0} (p,q,s)\) if
\[
\lim_{|a| \to 1} \int_{D} (f_p^*(z))^{p}(1 - |z|^2)^{q} g^s(z, a) \left( \log \frac{2}{1 - |z|^2} \right)^{p} dA(z) = 0.
\]
Note that the hyperbolic classes are not linear spaces, since they consist of functions that are self-maps of $\mathbb{D}$. Thus, the result in this paper is a generalization of the recent results of Pérez-González, Rättyä and Taskinen [9]. The study of composition operator $C_\phi$ acting on spaces of analytic functions has engaged many analysts for many years (see e.g. [1,2,4–8,12]).

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \leq C\|f\|_X$ for all maps $f \in X$. By elementary functional analysis, it is well-known that a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in $X$ to sets in $Y$ which have compact closure. For Banach spaces $X$ and $Y$ contained in $B(\mathbb{D})$ or $H(\mathbb{D})$, $T : X \to Y$ is compact if and only if for each bounded sequence $(x_n) \in X$, the sequence $(Tx_n) \in Y$ contains a subsequence converging to a function $f \in Y$.

Two quantities $A$ and $B$ are said to be equivalent if there exist two finite positive constants $C_1$ and $C_2$ such that $C_1B \leq A \leq C_2B$, written as $A \approx B$. Throughout this paper, the letter $C$ denotes different positive constants which are not necessarily the same from line to line.

Now, we introduce the following definitions:

**Definition 2.3.** A composition operator $C_\phi : \mathcal{B}_{p,\log,\alpha}^* \to \mathcal{F}_{p,\log}(p, q, s)$ is said to be bounded, if there is a positive constant $C$ such that $\|C_\phi f\|_{\mathcal{F}_{p,\log}^*(p, q, s)} \leq C\|f\|_{\mathcal{B}_{p,\log,\alpha}^*}$ for all $f \in \mathcal{B}_{p,\log,\alpha}^*$.

**Definition 2.4.** A composition operator $C_\phi : \mathcal{B}_{p,\log,\alpha}^* \to \mathcal{F}_{p,\log}^*(p, q, s)$ is said to be compact, if it maps any ball in $\mathcal{B}_{p,\log,\alpha}^*$ onto a pre-compact set in $\mathcal{F}_{p,\log}^*(p, q, s)$.

The following lemma follows by standard arguments similar to the result in (see [11]). Hence we omit the proof.

**Lemma 2.1.** Assume $\phi$ is a holomorphic mapping from $\mathbb{D}$ into itself and let $2 \leq p < \infty$, $0 < \alpha < 1$, $0 < s < \infty$, and $-2 < q < \infty$. Then the composition operator $C_\phi : \mathcal{B}_{p,\log,\alpha}^* \to \mathcal{F}_{p,\log}^*(p, q, s)$ is compact if and only if for any bounded sequence $(f_n)_{n \in \mathbb{N}} \in \mathcal{B}_{p,\log,\alpha}^*$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ we have

$$\lim_{n \to \infty} \|C_\phi f_n\|_{\mathcal{F}_{p,\log}^*(p, q, s)} = 0.$$
Theorem 2.1. Let $0 < p, s < \infty$, $-2 < q < \infty$, $0 < r < 1$, $\alpha = \frac{q+2}{p}$ and $q + s > -1$. If
\[
(f^*(a))^p \leq \frac{1}{\pi r^2} \int_{D(0, r)} \left( \frac{|f'(\varphi_a(w))|}{1 - |f(\varphi_a(w))|^2} \right)^p dA(w).
\]
Then the following are equivalent:

(A) $f \in B^*_p, log, \alpha,$

(B) $f \in F^*_p, log(p, q, s),$

(C) $\sup_{a \in D} \left( \log \frac{2}{1 - |a|^2} \right)^p \int_D (f^*_p(z))^p(1 - |z|^2)^{ap-2}(1 - |\varphi_a(z)|^2)^s dA(z) < \infty,$

(D) $\sup_{a \in D} \left( \log \frac{2}{1 - |a|^2} \right)^p \int_D (f^*_p(z))^p(1 - |z|^2)^{ap-2} g^s(z, a) dA(z) < \infty.$

Proof. The proof is similar to the main results in [10].

Now, we can find a natural metric on the generalized hyperbolic $(p, log, \alpha)$-Bloch class $B^*_p, log, \alpha$ and the class $F^*_p, log(p, q, s)$.

Let $2 \leq p < \infty, 0 < s < \infty, -2 < q < \infty$, and $0 < \alpha < 1$.
First, we can find a natural metric in $B^*_p, log, \alpha$ by defining
\[
d(f, g; B^*_p, log, \alpha) := d_{B^*_p, log, \alpha}(f, g) + \|f - g\|_{B^*_p, log, \alpha} + |f(0) - g(0)|^\frac{2}{p},
\]

\[
d_{B^*_p, log, \alpha}(f, g) := \sup_{a \in D} \left| \frac{f'(z)|f(z)|^{\frac{\alpha}{2} - 1}}{1 - |f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{\alpha}{2} - 1}}{1 - |g(z)|^p} \right| (1 - |z|^2)^\alpha \left( \log \frac{2}{1 - |z|^2} \right).
\]

For $f, g \in F^*_p, log(p, q, s)$, define their distance by
\[
d(f, g; F^*_p, log(p, q, s)) := d_{F^*_p, log(p, q, s)}(f, g) + \|f - g\|_{F^*_p, log(p, q, s)} + |f(0) - g(0)|,
\]

where
\[
d_{F^*_p, log(p, q, s)}(f, g) := \left( \sup_{z \in D} \left( \log \frac{2}{1 - |a|^2} \right)^p \int_D \left| f^*_p(z) - g^*_p(z) \right|^p(1 - |z|^2)^q g^s(z, a) dA(z) \right)^{\frac{1}{p}}.
\]

Now we prove the following results.
PROPOSITION 2.1. The class $\mathcal{B}^*_{p,\log,\alpha}$ equipped with the metric $d(\cdot, \cdot; \mathcal{B}^*_{p,\log,\alpha})$ is a complete metric space. Moreover, $\mathcal{B}^*_{p,\log,\alpha,0}$ is a closed (and therefore complete) subspace of $\mathcal{B}^*_{p,\log,\alpha}$.

Proof. For $f, g, h \in \mathcal{B}^*_{p,\log,\alpha}$. Then
\begin{itemize}
  \item $d(f, g; \mathcal{B}^*_{p,\log,\alpha}) \geq 0$,
  \item $d(f, f; \mathcal{B}^*_{p,\log,\alpha}) = 0$,
  \item $d(f, g; \mathcal{B}^*_{p,\log,\alpha}) = 0$ implies $f = g$.
\end{itemize}

Hence, $d$ is a metric on $\mathcal{B}^*_{p,\log,\alpha}$.

For the completeness proof, let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in the metric space $(\mathcal{B}^*_{p,\log,\alpha}, d)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m) < \varepsilon$, for all $n, m > N$. Since $f_n \in B(D)$ such that $f_n$ converges to $f$ uniformly on compact subsets of $D$. Let $m > N$ and
\[
f^*_{n,p}(z) = \frac{p}{2} \frac{|f^*_m(z)|^{p-1} |f^*_m(z)|}{1 - |f^*_m(z)|^p}.
\]

Then the uniform convergence yields
\[
|f^*_p(z) - f^*_{n,p}(z)| (1 - |z|^2)^\alpha \left( \log \frac{2}{1 - |z|^2} \right) = \lim_{n \to \infty} |f^*_p(z) - f^*_{n,p}(z)| (1 - |z|^2)^\alpha \left( \log \frac{2}{1 - |z|^2} \right) \leq \lim_{n \to \infty} d(f_n, f_m; \mathcal{B}^*_{p,\log,\alpha}) \leq \varepsilon.
\]

This yields
\[
\|f^*\|_{\mathcal{B}^*_{p,\log,\alpha}} \leq \varepsilon + \|f_m\|_{\mathcal{B}^*_{p,\log,\alpha}}.
\]

Thus $f \in \mathcal{B}^*_{p,\log,\alpha}$ as desired. Moreover, (1) and the completeness of the $(p, \log, \alpha)$-Bloch-space imply that $(f_n)_{n=1}^\infty$ converges to $f$ with respect to the metric $d$. The second part of the assertion follows by (1).

PROPOSITION 2.2. The class $F^*_{p,\log}(p, q, s)$ equipped with the metric $d(\cdot, \cdot; F^*_{p,\log}(p, q, s))$ is a complete metric space. Moreover, $F^*_{p,\log,0}(p, q, s)$ is a closed (and therefore complete) subspace of $F^*_{p,\log}(p, q, s)$.

Proof. For $f, g, h \in F^*_{p,\log}(p, q, s)$. Then
\begin{itemize}
  \item $d(f, g; F^*_{p,\log}(p, q, s)) \geq 0$,
  \item $d(f, f; F^*_{p,\log}(p, q, s)) = 0$,
\end{itemize}
Lipschitz continuous and compact composition operator

- $d(f, g; F_{p, \log}^*(p, q, s)) = 0$ implies $f = g$.
- $d(f, g; F_{p, \log}^*(p, q, s)) = d(g, f; F_{p, \log}^*(p, q, s))$.
- $d(f, h; F_{p, \log}^*(p, q, s)) \leq d(f, g; F_{p, \log}^*(p, q, s)) + d(g, h; F_{p, \log}^*(p, q, s))$.

Hence, $d$ is metric on $F_{p, \log}^*(p, q, s)$.

For the completeness proof, let $(f_n)_{n=0}^\infty$ be a Cauchy sequence in the metric space $F_{p, \log}^*(p, q, s)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m) < \varepsilon$, for all $n, m > N$. Since $f_n \in B(\mathbb{D})$ such that $f_n$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$. Let $m > N$ and $0 < r < 1$. Let

$$f_{m,p}^*(z) = \frac{p}{2} \frac{|f_m(z)|^{p-1} |f_m(z)|}{1 - |f_m(z)|^p}.$$

Then Fatou’s lemma yields

$$\int_{D(0,r)} (f_{p}^*(z) - f_{m,p}^*(z))(1 - |z|^2)^q g^*(z, a)dA(z)$$

$$= \int_{D(0,r)} \lim_{n \to \infty} |f_{n,p}^*(z) - f_{m,p}^*(z)|^p (1 - |z|^2)^q g^*(z, a)dA(z)$$

$$\leq \lim_{n \to \infty} \int_{\mathbb{D}} |f_{n,p}^*(z) - f_{m,p}^*(z)|^p (1 - |z|^2)^q g^*(z, a)dA(z) \leq \varepsilon^p.$$

By letting $r \to 1^-$, it follows from the above inequality and $(a + b)^p \leq 2^p(a^p + b^p)$ that

$$\int_{\mathbb{D}} (f_{p}^*(z))^p (1 - |z|^2)^q g^*(z, a)dA(z)$$

$$\leq 2^p \varepsilon^p + 2^p \int_{\mathbb{D}} (f_{m,p}^*(z))^p (1 - |z|^2)^q g^*(z, a)dA(z).$$

This yields

$$\|f\|_{F_{p, \log}^*(p, q, s)}^p \leq 2^p \varepsilon^p + 2^p \|f_m\|_{F_{p, \log}^*(p, q, s)}^p,$$

and thus $f \in F_{p, \log}^*(p, q, s)$. We also find that $f_n \to f$ with respect to the metric of $F_{p, \log}^*(p, q, s)$. The second part of the assertion follows by (2).
3. Lipschitz continuous and boundedness of $C_\phi$

For $0 < \alpha < 1$, $2 \leq p < \infty$. Let $f, g \in B_{p,\log,\alpha}^*$. Then, we will suppose that

$$
(f_\alpha^*(z) + g_\alpha^*(z)) \geq \frac{C}{(1 - |z|^2)\alpha(\log \frac{2}{1-|z|^2})} > 0,
$$

for some constant $C$ and for each $z \in \mathbb{D}$.

Let $0 < \alpha < 1$, $0 < s < \infty$, and $-2 < q < \infty$. We define the following notation:

$$\psi_\phi(\alpha, p, q, s; a) = \ell^p(a) \int_{\mathbb{D}} \frac{|\phi'(z)|p(1 - |z|^2)^q}{(1 - |\phi(z)|^p)^\alpha(\log \frac{2}{1-|\phi(z)|^p})^q} g^s(z, a) dA(z),$$

where $\ell^p(a) = (\log \frac{2}{1-|a|^2})^p$.

Now, we give the following result.

**Theorem 3.1.** Assume $\phi$ is a holomorphic mapping from $\mathbb{D}$ into itself and let $0 < \alpha < 1$, $2 \leq p < \infty$, $0 < s < \infty$, $-2 < q < \infty$. Suppose that (3) is satisfied. Then the following statements are equivalent:

(i) $C_\phi : B_{p,\log,\alpha}^* \to F_{p,\log}^*(p, q, s)$ is bounded;
(ii) $C_\phi : B_{p,\log,\alpha}^* \to F_{p,\log}^*(p, q, s)$ is Lipschitz continuous;
(iii) $\sup_{a \in \mathbb{D}} \psi_\phi(\alpha, p, q, s; a) < \infty$.

**Proof.** To prove (i) $\Leftrightarrow$ (iii), first assume that (iii) holds and that $f \in B_{p,\log,\alpha}^*$, then, we obtain

$$
\sup_{a \in \mathbb{D}} (\log \frac{2}{1-|a|^2})^p \int_{\mathbb{D}} ((f_\alpha^* \circ \phi)^*(z))^p(1 - |z|^2)^q g^s(z, a) dA(z)
$$

$$
= \sup_{a \in \mathbb{D}} (\log \frac{2}{1-|a|^2})^p \int_{\mathbb{D}} (f_\alpha^*(\phi(z)))^p|\phi'(z)|^p(1 - |z|^2)^q g^s(z, a) dA(z)
$$

$$
\leq \|f\|_{B_{p,\log,\alpha}^*}^p \sup_{a \in \mathbb{D}} \psi_\phi(\alpha, p, q, s; a) < \infty.
$$

Hence, it follows that (i) holds.

Conversely, assuming that (i) holds, then there exists a constant $C$ such that

$$
\|C_\phi f\|_{F_{p,\log}^*(p, q, s)} \leq C \|f\|_{B_{p,\log,\alpha}^*}.
$$
For giving $f \in \mathcal{B}^*_{p,\log,\alpha}$, the function $f_t(z) = f(tz)$, where $0 < t < 1$, belongs to $\mathcal{B}^*_{p,\log,\alpha}$ with the property $\|f_t\|_{\mathcal{B}^*_{p,\log,\alpha}} \leq \|f\|_{\mathcal{B}^*_{p,\log,\alpha}}$. Let $f$, $g$ be the functions from (3), we have

$$f^*_p(z) + g^*_p(z) \geq \frac{C}{(1 - |z|^2)\alpha(\log \frac{2}{1 - |z|^2})} > 0$$

for all $z \in \mathbb{D}$, then

$$\frac{|\phi'(z)|}{(1 - |\phi(z)|^2)^{\alpha}(\log \frac{2}{1 - |z|^2})} \leq (f \circ \phi)^*(z) + (g \circ \phi)^*(z),$$

thus,

$$\ell^p(a) \int_{\mathbb{D}} \frac{|t\phi'(z)|^p}{(1 - |t\phi(z)|^2)^{\alpha}(\log \frac{2}{1 - |z|^2})^p} (1 - |z|^2)^q g^*(z, a) dA(z)$$

$$\leq \ell^p(a) \int_{\mathbb{D}} \left( ((f \circ \phi)^*(z))^p + ((g \circ \phi)^*(z))^p \right) (1 - |z|^2)^q g^*(z, a) dA(z)$$

$$\leq C(\|C_f\|_{F^*_{p,\log}(p,q,s)} + \|C_g\|_{F^*_{p,\log}(p,q,s)})$$

$$\leq C \|C_f\|_p^p(\|f\|_{\mathcal{B}^*_{p,\log,\alpha}}^p + \|g\|_{\mathcal{B}^*_{p,\log,\alpha}}^p),$$

so (iii) is satisfied.

To prove (ii) $\iff$ (iii), assume first that $C : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p,q,s)$ is Lipschitz continuous, that is, there exists a positive constant $C$ such that

$$d(f \circ \phi, g \circ \phi; F^*_{p,\log}(p,q,s)) \leq C \|f - g\|_{\mathcal{B}^*_{p,\log,\alpha}}, \quad \text{for all } f, g \in \mathcal{B}^*_{p,\log,\alpha}.$$

Taking $g = 0$, we get

$$\|f \circ \phi\|_{F^*_{p,\log}(p,q,s)} \leq C(\|f\|_{\mathcal{B}^*_{p,\log,\alpha}} + \|f\|_{\mathcal{B}^*_{p,\log,\alpha}} + |f(0)|^\frac{p}{q}), \quad \text{for all } f \in \mathcal{B}^*_{p,\log,\alpha}.$$

The assertion (iii) for $\alpha = 1$, follows by choosing $f(z) = z$ in (4).

If $0 < \alpha < 1$ and $(\log \frac{2}{1 - |z|^2}) \approx (\log \frac{2}{1 - |z|^2})$ then

$$|f(z)|^\frac{p}{q} \leq \frac{2}{p} \int_0^{|z|} |f(s)|^\frac{p}{q} f'(s) ds + |f(0)|^\frac{p}{q}$$

$$\leq \frac{2}{p} \left[ \|f\|_{\mathcal{B}^*_{p,\log,\alpha}} \left( \log \frac{2}{1 - |z|^2} \right) \int_0^{|z|} ds \right] \frac{|s|}{(1 - s^2)^\alpha} + |f(0)|^\frac{p}{q}$$

$$\leq C \|f\|_{\mathcal{B}^*_{p,\log,\alpha}} \frac{1}{1 - \alpha} + \frac{2}{p} |f(0)|^\frac{p}{q}$$
Moreover, from (3), for \( f, g \in B^*_{p,\log,\alpha} \), we deduce that
\[
(\|f^\prime(z)\| + |g^\prime(z)|)(1 - |z|^2)^\alpha (\log \frac{2}{1 - |z|^2}) \geq C > 0, \quad \text{for all } z \in \mathbb{D}.
\]
Therefore,
\[
\|f\|_{B^*_{p,\log,\alpha}} + \|g\|_{B^*_{p,\log,\alpha}} + \|f\|_{B_{p,\log,\alpha}} + \|g\|_{B_{p,\log,\alpha}} + \|f(0)\|^\frac{q}{p} + |g(0)|^\frac{q}{p}
\geq C \int \mathbb{D} \frac{|\phi^\prime(z)|^p(1 - |z|^2)^q}{(1 - (\phi(z))^p)^\alpha (\log \frac{2}{1 - |z|^2})^p} g^s(z, a)dA(z),
\]
for which the assertion (iii) follows.
Assume now that (iii) is satisfied, we have
\[
d(f \circ \phi, g \circ \phi; F^*_{p,\log}(p, q, s)) = d_{F^*_{p,\log}(p, q, s)}(f \circ \phi, g \circ \phi)
+ \|f \circ \phi - g \circ \phi\|_{F_{p,\log}(p, q, s)} + \|f(0) - g(0)\|^\frac{q}{p}.
\]
\[
\leq d_{B^*_{p,\log,\alpha}}(f, g) \left( \sup_{a \in \mathbb{D}} \int \mathbb{D} \frac{|\phi^\prime(z)|^p(1 - |z|^2)^q}{(1 - (\phi(z))^p)^\alpha (\log \frac{2}{1 - |z|^2})^p} g^s(z, a)dA(z) \right)^\frac{1}{p}
+ \|f - g\|_{B^*_{p,\log,\alpha}} \left( \sup_{a \in \mathbb{D}} \int \mathbb{D} \frac{|\phi^\prime(z)|^p(1 - |z|^2)^q}{(1 - (\phi(z))^p)^\alpha (\log \frac{2}{1 - |z|^2})^p} g^s(z, a)dA(z) \right)^\frac{1}{p}
+ \|f - g\|_{B_{p,\log,\alpha}} \left( \sup_{a \in \mathbb{D}} \int \mathbb{D} \frac{|\phi^\prime(z)|^p(1 - |z|^2)^q}{(1 - (\phi(z))^p)^\alpha (\log \frac{2}{1 - |z|^2})^p} g^s(z, a)dA(z) \right)^\frac{1}{p}
\leq C d(f, g; B^*_{p,\log,\alpha}).
\]
Thus \( C_\phi : B^*_{p,\log,\alpha} \rightarrow F_{p,\log}(p, q, s) \) is Lipschitz continuous and the proof is established.

\[
\text{Remark 3.1. We know that a composition operator } C_\phi : B^*_{p,\log,\alpha} \rightarrow F^*_{p,\log}(p, q, s) \text{ is said to be bounded if there is a positive constant } C \text{ such that } \|C_\phi f\|_{F^*_{p,\log}(p, q, s)} \leq C \|f\|_{B^*_{p,\log,\alpha}}, \text{ for all } f \in B^*_{p,\log,\alpha}. \text{ Theorem 3.1 shows that } C_\phi : B^*_{p,\log,\alpha} \rightarrow F^*_{p,\log}(p, q, s) \text{ is bounded if and only if it is Lipschitz continuous, that is, if there exists a positive constant } C \text{ such that } d(f \circ \phi, g \circ \phi; F^*_{p,\log}(p, q, s)) \leq C d(f, g; B^*_{p,\log,\alpha}) \text{, for all } f, g \in B^*_{p,\log,\alpha}.
\]
By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, since the boundedness is trivially also equivalent to the Lipschitz-continuity. Our result for
composition operator in hyperbolic spaces is the correct and natural generalization of the linear operator theory.

4. Compactness of $C_{\phi} : \mathcal{B}_{p,\log,\alpha}^* \to F_{p,\log}^*(p, q, s)$

Recall that a composition operator $C_{\phi} : \mathcal{B}_{p,\log,\alpha}^* \to F_{p,\log}^*(p, q, s)$ is said to be compact, if it maps any ball in $\mathcal{B}_{p,\log,\alpha}^*$ onto a pre-compact set in $F_{p,\log}^*(p, q, s)$.

Now, we give the following important results.

**Proposition 4.1.** Assume $\phi$ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $2 \leq p < \infty$, $-2 < q < \infty$, $0 < \alpha < 1$ and $0 \leq s < \infty$. If $C_{\phi} : \mathcal{B}_{p,\log,\alpha}^* \to F_{p,\log}^*(p, q, s)$ is compact, it maps closed balls onto compact sets.

**Proof.** If $B \subset \mathcal{B}_{p,\log,\alpha}^*$ is a closed ball and $g \in F_{p,\log}^*(p, q, s)$ belongs to the closure of $C_{\phi}(B)$, we can find a sequence $(f_n)_{n=1}^{\infty} \subset B$ such that $f_n \circ \phi$ converges to $g \in F_{p,\log}^*(p, q, s)$ as $n \to \infty$. But $(f_n)_{n=1}^{\infty}$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^{\infty}$ converging uniformly on the compact subsets of $\mathbb{D}$ to an analytic function $f$. As in earlier arguments of Proposition 2.1, we get a positive estimate which shows that $f$ must belong to the closed ball $B$. On the other hand, also the sequence $(f_{n_j} \circ \phi)_{j=1}^{\infty}$ converges uniformly on compact subsets to an analytic function, which is $g \in F_{p,\log}^*(p, q, s)$. We get $g = f \circ \phi$, i.e. $g$ belongs to $C_{\phi}(B)$. Thus, this set is closed and also compact.

Compactness of composition operator acting between $\mathcal{B}_{p,\log,\alpha}^*$ and $F_{p,\log}^*(p, q, s)$ classes can be characterized in the following result.

**Theorem 4.1.** Assume $\phi$ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $2 \leq p < \infty$, $-2 < q < \infty$, $0 < \alpha < 1$ and $0 \leq s < \infty$. Then the following statements are equivalent:

(i) $C_{\phi} : \mathcal{B}_{p,\log,\alpha}^* \to F_{p,\log}^*(p, q, s)$ is compact.

(ii) $\lim_{r \to 1^-} \sup_{a \in \mathbb{D}} \psi_{\phi}(\alpha, p, q, s; a) = 0$.

**Proof.** We first assume that (ii) holds. Let $B := \bar{B}(g, \delta) \subset \mathcal{B}_{p,\log,\alpha}^*$, $g \in \mathcal{B}_{p,\log,\alpha}^*$ and $\delta > 0$, be a closed ball, and let $(f_n)_{n=1}^{\infty} \subset B$ be any sequence. We show that its image has a convergent subsequence in $F_{p,\log}^*(p, q, s)$, which proves the compactness of $C_{\phi}$ by definition.
Again, \((f_n)_n \in B(\mathbb{D})\) is normal, hence, there is a subsequence \((f_{n_j})_j\) which converges uniformly on the compact subsets of \(\mathbb{D}\) to an analytic function \(f\). By Cauchy formula for the derivative of an analytic function, also the sequence \((f'_{n_j})_j\) converges uniformly on the compact subsets of \(\mathbb{D}\) to \(f'\). It follows that also the sequences \((f_{n_j} \circ \phi)_j\) and \((f'_{n_j} \circ \phi)_j\) converge uniformly on the compact subsets of \(\mathbb{D}\) to \(f \circ \phi\) and \(f' \circ \phi\), respectively. Moreover, \(f \in B \subset B_{p,\log,\alpha}^*\) since for any fixed \(R, 0 < R < 1\), the uniform convergence yield

\[
\begin{align*}
\sup_{|z| \leq R} \left| \frac{f'(z)}{1 - |f(z)|^p} - \frac{g'(z)}{1 - |g(z)|^p} \right| (1 - |z|^2)^\alpha \left( \log \frac{2}{1 - |z|^2} \right) \\
+ \sup_{|z| \leq R} \left| f'(z) - g'(z) \right| (1 - |z|^2)^\alpha \left( \log \frac{2}{1 - |z|^2} \right) \\
+ |f(0) - g(0)|^2 < \delta.
\end{align*}
\]

Hence, \(d(f, g; B^*_{p,\log,\alpha}) \leq \delta\).

Let \(\varepsilon > 0\). Since (ii) is satisfied, we may fix \(r, 0 < r < 1\), such that

\[
\sup_{a \in D} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^p)\rho^\alpha \left( \log \frac{2}{1 - |\phi(z)|^p} \right)^\alpha \left( 1 - |\phi(z)|^2 \right)^\eta} g\phi(z, a) dA(z) \leq \varepsilon.
\]

By the uniform convergence, we may fix \(N_1 \in \mathbb{N}\) such that

\[
|f_{n_j} \circ \phi(0) - f \circ \phi(0)| \leq \varepsilon, \quad \text{for all } j \geq N_1.
\]  

The condition (ii) is known to imply the compactness of \(C_\phi : B_{p,\log,\alpha} \to F_{p,\log,\alpha}^\alpha\), hence possibly to passing once more to a subsequence and adjusting the notations, we may assume that

\[
\|f_{n_j} \circ \phi - f \circ \phi\|_{F_{p,\log,\alpha}^\alpha} \leq \varepsilon, \quad \text{for all } j \geq N_2; \quad N_2 \in \mathbb{N}.
\]
Since \((f_{n_j})_{j=1}^\infty \subset B\) and \(f \in B\), it follows that
\[
\sup_{a \in D} \ell^p(a) \int_{|\phi(z)| > r} \left[ (f_{p,n_j} \circ \phi)^* (z) - (g_p \circ \phi)^* (z) \right]^p (1 - |z|^2)^q g^*(z, a) \, dA(z)
\leq \frac{p}{2} \sup_{a \in D} \ell^p(a) \int_{|\phi(z)| > r} L(f_{n_j}, g, \phi)(1 - |z|^2)^q g^*(z, a) \, dA(z)
\leq d_{\ell_p, \log, a} (f_{n_j}, g) \sup_{a \in D} \ell^p(a) \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p (1 - |z|^2)^q}{(1 - |\phi(z)|^p)^a p \log \left( \frac{2}{1 - |z|^2} \right)} g^*(z, a) \, dA(z),
\]
where
\[
L(f_{n_j}, g, \phi) = \left| \frac{(f_{n_j} \circ \phi)'(z) ((f_{n_j} \circ \phi)(z))^q}{1 - |(f_{n_j} \circ \phi)(z)|^p} - \frac{(g \circ \phi)'(z) ((g_n \circ \phi)(z))^q}{1 - |(g \circ \phi)(z)|^p} \right|^p.
\]

hence,
\[
\sup_{a \in D} \ell^p(a) \int_{|\phi(z)| > r} \left[ (f_{p,n_j} \circ \phi)^* (z) - (g_p \circ \phi)^* (z) \right]^p (1 - |z|^2)^q g^*(z, a) \, dA(z) \leq C \varepsilon.
\] (7)

On the other hand, by the uniform convergence on the compact disc \(D\), we can find an \(N_3 \in \mathbb{N}\) such that for all \(j \geq N_3\),
\[
L_1(f_{n_j}, g, \phi) = \left| \frac{(f_{n_j}'(\phi(z))) ((f_{n_j} \circ \phi)(z))^q}{1 - |(f_{n_j} \circ \phi)(z)|^p} - \frac{(g_n'(\phi(z))) ((g_n \circ \phi)(z))^q}{1 - |(g \circ \phi)(z)|^p} \right| \leq \varepsilon.
\]

For all \(z\) with \(|\phi(z)| \leq r\). Hence, for such \(j\),
\[
\sup_{a \in D} \ell^p(a) \int_{|\phi(z)| \leq r} \left[ (f_{p,n_j} \circ \phi)^* (z) - (g_p \circ \phi)^* (z) \right]^p (1 - |z|^2)^q g^*(z, a) \, dA(z)
\leq \sup_{a \in D} \ell^p(a) \int_{|\phi(z)| \leq r} L_1(f_{n_j}, g, \phi)|\phi'(z)|^p (1 - |z|^2)^q g^*(z, a) \, dA(z)
\leq \varepsilon \left( \sup_{a \in D} \ell^p(a) \int_{|\phi(z)| \leq r} \frac{|\phi'(z)|^p (1 - |\phi(z)|^p)^a p \log \left( \frac{2}{1 - |z|^2} \right)}{1 - |\phi(z)|^p} g^*(z, a) dA(z) \right)^\frac{1}{p} \leq C \varepsilon,
\] (8)

where \(C\) is bounded which is obtained from (iii) of Theorem 3.1. Combining (5), (6), (7) and (8) we deduce that \(f_{n_j} \to f\) in \(F_{p, \log}^*(p, q, s)\).
For the converse direction, let $f_n(z) := \frac{1}{2} n^{\alpha - 1} z^n$ for all $n \in \mathbb{N}$, $n \geq 2$.

$$\|f\|_{B^*_{p, \log, \alpha}} = \frac{p}{2} \sup_{a \in \mathbb{D}} \frac{n^{\alpha p} |z|^{\frac{np}{2} - 1} (1 - |z|^2)^\alpha}{1 - 2^{-p} (p^\alpha - 1) |z|^{np}} \leq (2^{p-1} + 1) \sup_{a \in \mathbb{D}} n^{\alpha p} |z|^{\frac{np}{2} - 1} (1 - |z|^2)^\alpha$$

Then the sequence $(f_n)_{n=1}^\infty$ belongs to the ball $B(0; (2^{p-1} + 1)) \subset B^*_{p, \log, \alpha}$ (see [3]). We are assuming that $C_\phi$ maps the closed ball $B(0; (2^{p-1} + 1)) \subset B^*_{p, \log, \alpha}$ into a compact subset of $F^*_{p, \log, (p, q, s)}$, hence, there exists an unbounded increasing subsequence $(n_j)_{j=1}^\infty$ such that the image subsequence $(C_\phi f_{n_j})_{j=1}^\infty$ converges with respect to the norm. Since, both $(f_n)_{n=1}^\infty$ and $(C_\phi f_{n_j})_{j=1}^\infty$ converge to the zero function uniformly on compact subsets of $\mathbb{D}$, the limit of the latter sequence must be 0. Hence,

$$\lim_{j \to \infty} \|n_j^{\alpha - 1} \phi^{n_j}\|_{F^*_{p, \log, (p, q, s)}} = 0. \quad (9)$$

Now let $r_j = 1 - \frac{1}{n_j}$. For all numbers $a$, $r_j \leq a < 1$, we have the following estimate

$$\frac{n_j^\alpha a^{n_j - 1}}{1 - a^{n_j}} \geq \frac{1}{e(1 - a)^\alpha}. \quad (see \ [3, 9]) \quad (10)$$

Using (10) we deduce

$$\|n_j^{\alpha - 1} \phi^{n_j}\|_{F^*_{p, \log, (p, q, s)}} \geq \frac{p}{2} \sup_{a \in \mathbb{D}} \left( \int_{|\phi(z)| \geq r_j} \frac{n_j^\alpha (\phi(z))^{n_j - 1} |\phi^{n_j}(z)|^{\frac{np}{2} - 1} |\phi'(z)|^p}{1 - |\phi^{n_j}(z)|^p} \right)^p \times (1 - |z|^2)^q g^s(z, a) \, dA(z) \geq \frac{Cp}{2(2e)^p} \sup_{a \in \mathbb{D}} \left( \int_{|\phi(z)| \geq r_j} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^p)^{pa}} (1 - |z|^2)^q g^s(z, a) \, dA(z) \right). \quad (11)$$

From (9) and (11), the condition (ii) follows. The proof is therefore completed.

5. Acknowledgements

The authors would like to thank the referees for their useful comments, which improved the original manuscript.
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