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SPECTRAL THEOREMS ASSOCIATED TO THE DUNKL OPERATORS

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Abstract. In this paper, we characterize the support for the Dunkl transform on the generalized Lebesgue spaces via the Dunkl resolvent function. The behavior of the sequence of L^p_k-norms of iterated Dunkl potentials is studied depending on the support of their Dunkl transform. We systematically develop real Paley-Wiener theory for the Dunkl transform on $\mathbb{R}^d$ for distributions, in an elementary treatment based on the inversion theorem. Next, we improve the Roe's theorem associated to the Dunkl operators.

1. Introduction

We consider the differential-difference operators $T_j,\ j = 1, 2, ..., d$, attached to a root system $\mathcal{R}$ and a multiplicity function k, introduced by Dunkl in [7], and called the Dunkl operators in the literature.

The Dunkl theory is based on the Dunkl kernel $K(i\lambda, .)$, $\lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$T_j u(x) = i\lambda_j u(x), \quad j = 1, 2, ..., d,$$

satisfying the normalizing condition $u(0) = 1$.

With the Dunkl kernel $K(i\lambda, .)$, Dunkl defined in [9] the Dunkl transform $\mathcal{F}_D$ and established some of its properties (see also [11]).

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The spectral theorems associated with the partial derivatives operators are of the most useful subjects in harmonic analysis. In this paper we present only three subjects.

The first subject of these theorems is one of the fundamental questions in Fourier analysis and abstract harmonic analysis: the real Paley-Wiener theorem. The fundamental theorem given by Bang (cf. [4]) can be stated as follows. Let f be a C^∞-function on $\mathbb{R}$ such that for all $n \in \mathbb{N}$, the function $\frac{d^n}{dx^n} f$ belongs to the Lebesgue space $L^p(\mathbb{R})$, then the limit $R_f = \lim_{n\to\infty} \|\frac{d^n}{dx^n} f\|_p^{1/n}$ exists and we have

$$R_f := \sup\Big\{|\lambda| : \ \lambda \in supp\,\mathcal{F}(f)\Big\},$$

where $\mathcal{F}(f)$ is the classical Fourier transform of f. Next the analogue of this theorem was established and improved for many other integral transforms, for examples (cf. [1, 6, 14, 16–18, 23]).

The second subject concerning spectral theorems is the study of tempered distributions with spectral gaps. More precisely a tempered distributions on $\mathbb{R}$ whose Fourier transform is supported in an interval $[-M, M]$, where $M > 0$, can be characterized by the behaviour of its successive derivatives. On the other hand, a tempered distribution on $\mathbb{R}$ whose Fourier transform vanishes in an interval $(-M, M)$, where $M > 0$, can be characterized by the behaviour of a particular sequence of successive antiderivatives. This subject was studied for many other integral transforms, for examples (cf. [2, 3, 17, 18]).

Third subject of spectral theorems, is Roe's theorem. The fundamental theorem given by Roe (cf. [19]) can be stated as follows. If a function and all its derivatives and integrals are absolutely uniformly bounded, then the function is a sine function with period 2π. This result has been studied and generalized, see [2, 13, 15–18, 21] including generalizations to differential and differential-difference operators with constant coefficients in higher dimensions.

Motivated by the treatment in the Euclidean setting, we will derive in this paper new real Paley-Wiener theorems for the Dunkl transform, on the generalized Lebesgue spaces and on the tempered distribution space $\mathcal{S}'(\mathbb{R}^d)$, and we improve the Roe's theorem in the context of the Dunkl operators.

The outline of this paper is as follow: In §2 we recall the main results about the harmonic analysis associated with the Dunkl operators. In §3 we introduce the Dunkl resolvent function as a solution of the generalized Poisson's equation associated to the Dunkl-Laplace operator on the generalized Wiener space. Next, we extend the definition of the Dunkl resolvent function on some subspace of the generalized Lebesgue space $L^2_k(\mathbb{R}^d)$. The §4 is devoted to characterize the support for the Dunkl transform on the generalized Lebesgue space $L^2_k(\mathbb{R}^d)$ via the Dunkl resolvent function. In §5 we solve the previous problem on the generalized Lebesgue space $L^p_k(\mathbb{R})$. In §6 we study the generalized tempered distributions with spectral gaps. Finally, the purpose of the last section is to improve and generalize a version of Roe's theorem for Dunkl operators from [15].

2. Preliminaries

This section gives an introduction to the theory of Dunkl operators, Dunkl kernel and Dunkl transform. Main references are [7–9, 11].

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle , \rangle$ and $||x|| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha. \tag{2.1}$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R \cap \mathbb{R}.\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R. We fix a positive root system $R_+ = \{\alpha \in R : \langle \alpha, \beta\rangle > 0\}$ for some $\beta \in \mathbb{R}^d\backslash \cup_{\alpha\in R} H_\alpha$. We will assume that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R_+$. A function $k : R \longrightarrow \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha). \tag{2.2}$$

Moreover, let ω_k denotes the weight function

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}, \tag{2.3}$$

which is invariant and homogeneous of degree 2γ. We introduce the Mehta-type constant

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{||x||^2}{2}} \omega_k(x)\, dx. \tag{2.4}$$

In the following we denote by

$C(\mathbb{R}^d)$	the space of continuous functions on $\mathbb{R}^d$.
$C^p(\mathbb{R}^d)$	the space of functions of class C^p on $\mathbb{R}^d$.
$C^p_b(\mathbb{R}^d)$	the space of bounded functions of class C^p.
$\mathcal{E}(\mathbb{R}^d)$	the space of C^∞-functions on $\mathbb{R}^d$.
$\mathcal{S}(\mathbb{R}^d)$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.
$D(\mathbb{R}^d)$	the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.
$\mathcal{S}'(\mathbb{R}^d)$	the space of temperate distributions on $\mathbb{R}^d$.

The Dunkl operators $T_j,\ j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d). \tag{2.5}$$

Some properties of the T_j, $j = 1, ..., d$, are given in the following :

For all f and g in $C^1(\mathbb{R}^d)$ with at least one of them is W-invariant, we have

$$T_j(fg) = (T_j f)g + f(T_j g), \quad j = 1, ..., d. \tag{2.6}$$

For f in $C^1_b(\mathbb{R}^d)$ and g in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x)\omega_k(x)\, dx = -\int_{\mathbb{R}^d} f(x) T_j g(x)\omega_k(x)\, dx,\ j = 1, ..., d. \tag{2.7}$$

We define the Dunkl-Laplace operator $\triangle_k$ on $\mathbb{R}^d$ by

$$\triangle_k f(x) := \sum_{j=1}^d T_j^2 f(x) = \triangle f(x) + 2\sum_{\alpha\in R^+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\Big),$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and nabla operators on $\mathbb{R}^d$, respectively.

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x, y) = y_j u(x, y), & j = 1, ..., d, \\ u(0, y) = 1, \end{cases} \tag{2.8}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.
The Dunkl kernel possesses the following properties:

i) For $z,t \in \mathbb{C}^d$, we have $K(z,t) = K(t,z)$; $K(z,0) = 1$ and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}$.

ii) For all $\nu \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ we have

$$|D_z^\nu K(x,z)| \leq ||x||^{|\nu|} \exp(||x|| \, ||\mathrm{Re} z||), \tag{2.9}$$

with

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d}} \quad \text{and} \quad |\nu| = \nu_1 + \cdots + \nu_d.$$

In particular for all $x,y \in \mathbb{R}^d$:

$$|K(-ix,y)| \leq 1.$$

Notation. We denote by $L_k^p(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\begin{aligned} ||f||_{L_k^p(\mathbb{R}^d)} &:= \Big(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x)\, dx \Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leq p < \infty, \\ ||f||_{L_k^\infty(\mathbb{R}^d)} &:= \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < \infty. \end{aligned}$$

The Dunkl transform of a function f in $L_k^1(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \frac{1}{c_k} \int_{\mathbb{R}^d} f(x) K(-iy,x) \omega_k(x) dx, \quad \text{for all } y \in \mathbb{R}^d. \tag{2.10}$$

In the following we give some properties of this transform (cf. [9,11]).

i) For f in $L_k^1(\mathbb{R}^d)$ we have

$$||\mathcal{F}_D(f)||_{L_k^\infty(\mathbb{R}^d)} \leq \frac{1}{c_k} ||f||_{L_k^1(\mathbb{R}^d)}. \tag{2.11}$$

ii) Inversion formula: Let f be a function in $L_k^1(\mathbb{R}^d)$, such that $\mathcal{F}_D(f) \in L_k^1(\mathbb{R}^d)$. Then

$$\mathcal{F}_D^{-1}(f)(x) = \mathcal{F}_D(f)(-x), \quad a.e.\ x \in \mathbb{R}^d. \tag{2.12}$$

iii) For f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\mathcal{F}_D(T_j f)(y) = i y_j \mathcal{F}_D(f)(y), \quad \text{for all } j = 1, ..., d \text{ and } y \in \mathbb{R}^d. \tag{2.13}$$

Proposition 2.1. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself. If we put for f in $\mathcal{S}(\mathbb{R}^d)$*

$$\overline{\mathcal{F}_D}(f)(y) = \mathcal{F}_D(f)(-y), \quad y \in \mathbb{R}^d, \tag{2.14}$$

we have

$$\mathcal{F}_D\overline{\mathcal{F}_D} = \overline{\mathcal{F}_D}\mathcal{F}_D = Id.$$

Proposition 2.2. i) **Plancherel formula for $\mathcal{F}_D$.**
For all f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2\omega_k(x)\, dx = \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2\omega_k(\xi)\, d\xi. \tag{2.15}$$

ii) **Plancherel theorem for $\mathcal{F}_D$.**
The Dunkl transform $f \to \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

Definition 2.1. Let y be in $\mathbb{R}^d$. The Dunkl translation operator $f \mapsto \tau_y f$ is defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}_D(\tau_y f)(x) = K(ix, y)\mathcal{F}_D(f)(x), \quad \text{for all } x \in \mathbb{R}^d. \tag{2.16}$$

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [22, 24]).

Definition 2.2. The Dunkl convolution product of f and g in $\mathcal{S}(\mathbb{R}^d)$ is the function $f *_D g$ defined by

$$f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y)g(y)\omega_k(y)dy, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.17}$$

This convolution is commutative and associative and satisfies the following properties (see [22]).

Proposition 2.3. i) *For f and g in $D(\mathbb{R}^d)$(resp. $\mathcal{S}(\mathbb{R}^d)$) the function $f *_D g$ belongs to $D(\mathbb{R}^d)$(resp. $\mathcal{S}(\mathbb{R}^d)$) and we have*

$$\mathcal{F}_D(f *_D g)(y) = \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y), \quad \text{for all } y \in \mathbb{R}^d. \tag{2.18}$$

ii) *Let $1 \leq p, q, r \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If f is in $L^p_k(\mathbb{R}^d)$ and g is a radial element of $L^q_k(\mathbb{R}^d)$, then $f *_D g \in L^r_k(\mathbb{R}^d)$ and we have*

$$\|f *_D g\|_{L^r_k(\mathbb{R}^d)} \leq C \|f\|_{L^p_k(\mathbb{R}^d)} \|g\|_{L^q_k(\mathbb{R}^d)}. \tag{2.19}$$

iii) *Let $W = \mathbb{Z}^d_2$. We have the same result for all $f \in L^p_k(\mathbb{R}^d)$ and $g \in L^q_k(\mathbb{R}^d)$.*

DEFINITION 2.3. i) The Dunkl transform of a distribution τ in $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{F}_D(\tau), \phi \rangle = \langle \tau, \mathcal{F}_D(\phi) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d). \tag{2.20}$$

ii) The Dunkl transform of f in $L^p_k(\mathbb{R}^d)$ denoted also by $\mathcal{F}_D(f)$, is defined by

$$\langle \mathcal{F}_D(f), \phi \rangle = \langle \mathcal{F}_D(\mathcal{T}_f), \phi \rangle = \langle \mathcal{T}_f, \mathcal{F}_D(\phi) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

Thus from (2.20) we have

$$\langle \mathcal{F}_D(f), \phi \rangle = \int_{\mathbb{R}^d} f(x) \mathcal{F}_D(\phi)(x) \omega_k(x) dx.$$

PROPOSITION 2.4. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}'(\mathbb{R}^d)$ onto itself.*

Let τ be in $\mathcal{S}'(\mathbb{R}^d)$. We define the distribution $T_j\tau$, $j = 1, ..., d$, by

$$\langle T_j\tau, \psi \rangle \quad = \quad -\langle \tau, T_j\psi \rangle, \ \text{ for all } \psi \in \mathcal{S}(\mathbb{R}^d).$$

Thus we deduce

$$\langle \triangle_k \tau, \psi \rangle = \langle \tau, \triangle_k \psi \rangle, \ \text{ for all } \psi \in \mathcal{S}(\mathbb{R}^d). \tag{2.21}$$

These distributions satisfy the following properties

$$\mathcal{F}_D(T_j\tau) \quad = \quad iy_j \mathcal{F}_D(\tau), \quad j = 1, ..., d. \tag{2.22}$$

$$\mathcal{F}_D(\triangle_k \tau) \quad = \quad -||y||^2 \mathcal{F}_D(\tau). \tag{2.23}$$

In the following $\mathcal{T}_f$ will be denoted by f.

3. Generalized Poisson's equation for the Dunkl-Laplace operator

For $x, y \in \mathbb{R}^d$ and $t > 0$, we put

$$p_t(x, y) = \frac{1}{(2t)^{\gamma + \frac{d}{2}} c_k} e^{-\frac{||x||^2 + ||y||^2}{4t}} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right). \tag{3.1}$$

For fixed $y \in \mathbb{R}^d$, the function $u(x, t) = p_t(x, y)$ is solution of the heat equation:

$$\frac{\partial u}{\partial t}(x, t) - \triangle_k u(x, t) = 0, \ \text{ on } \mathbb{R}^d \times (0, \infty).$$

The function p_t has the following properties

$$\forall\, t>0, \quad \int_{\mathbb{R}^d} p_t(x,y)\omega_k(y)dy = 1. \tag{3.2}$$

$$\forall\, t>0, \quad p_t(x,y) \leq \frac{1}{(2t)^{\gamma+\frac{d}{2}}c_k} e^{-\frac{(\|x\|-\|y\|)^2}{4t}}. \tag{3.3}$$

$$\forall\, t>0, \quad p_t(x,y) = \frac{1}{c_k^2}\int_{\mathbb{R}^d} e^{-t\|\xi\|^2} K(ix,\xi)K(-iy,\xi)\omega_k(\xi)d\xi. \tag{3.4}$$

Proposition 3.1. *Let $\mu \in \mathbb{C}$ with $Re\mu \geq 0$. The integral $G_\mu(x,y) = \int_0^\infty e^{-t\mu} p_t(x,y)dt$ is finite for $x \neq y$ in $\mathbb{R}^d$ and $Re\mu > 0$. If $Re\mu = 0$, the function G_μ is finite for $x \neq y$ in $\mathbb{R}^d$ if and only if $2\gamma + d > 2$.*

Proof. The proof follows immediately from the relation (3.3). □

Notation. We define the generalized Wiener space $W_k(\mathbb{R}^d)$ as follows:

$$W_k(\mathbb{R}^d) := \Big\{ f \in L_k^1(\mathbb{R}^d) \ : \ \mathcal{F}_D(f) \in L_k^1(\mathbb{R}^d) \Big\}.$$

Let $\mu \in \mathbb{C}$, we say that μ satisfies the hypothesis (H) if

$$(H) \begin{cases} \quad Re\mu > 0, \\ or \\ \quad Re\mu = 0 \ and \quad 2\gamma + d > 2. \end{cases}$$

Proposition 3.2. *Let $\mu \in \mathbb{C}$, we assume that μ satisfies the hypothesis (H). Let f be in $W_k(\mathbb{R}^d)$. The function*

$$R_\mu f(x) = \int_{\mathbb{R}^d} G_\mu(x,y) f(y)\omega_k(y)dy,$$

called the Dunkl resolvent function of f, is bounded, of class C^2 and satisfies the generalized Poisson's equation

$$(-\triangle_k + \mu)u = f.$$

Proof. Let us first prove that R_μ is well defined and bounded. We can assume $f \geq 0$ and $\mu \geq 0$. Fubini's theorem for positive functions gives that

$$R_\mu f(x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-t\mu} p_t(x,y) f(y)\omega_k(y)dydt.$$

From (3.4), we can write

$$R_\mu f(x) = \frac{1}{c_k^2}\int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-t(\mu+||\xi||^2)}K(ix,\xi)K(-iy,\xi)f(y)\omega_k(\xi)\omega_k(y)d\xi dy dt.$$

Now using relation (2.9) and the hypothesis on f, we first obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-t(\mu+||\xi||^2)}|K(ix,\xi)K(-iy,\xi)f(y)|\omega_k(\xi)\omega_k(y)d\xi dy \leq$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-t(\mu+||\xi||^2)}f(y)\omega_k(\xi)\omega_k(y)d\xi dy < \infty.$$

From Fubini's theorem and relation (2.10), we get

$$R_\mu f(x) = \frac{1}{c_k}\int_0^\infty \int_{\mathbb{R}^d} e^{-t(\mu+||\xi||^2)}K(ix,\xi)\mathcal{F}_D(f)(\xi)\omega_k(\xi)d\xi dt. \tag{3.5}$$

So

$$\begin{aligned}|R_\mu f(x)| &\leq \frac{1}{c_k}\int_{\mathbb{R}^d}\int_0^\infty e^{-t(\mu+||\xi||^2)}|\mathcal{F}_D(f)(\xi)|\omega_k(\xi)d\xi dt\\ &\leq \frac{1}{c_k}\int_{\mathbb{R}^d}\frac{|\mathcal{F}_D(f)(\xi)|}{\mu+\|\xi\|^2}\,\omega_k(\xi)d\xi < \infty.\end{aligned}$$

Thus the function $R_\mu f$ is well defined and bounded on $\mathbb{R}^d$. Now, if we apply Fubini's theorem to the equality (3.5), we obtain

$$R_\mu f(x) = \frac{1}{c_k}\int_{\mathbb{R}^d} K(ix,\xi)\frac{\mathcal{F}_D(f)(\xi)}{\mu+\|\xi\|^2}\,\omega_k(\xi)d\xi. \tag{3.6}$$

Moreover it is easy to see that the preceding equality is true for $\mu \in \mathbb{C}$ such that μ satisfies the hypothesis (H). Using relation (2.8), the fact that $\triangle_{k,x}K(ix,\xi) = -||\xi||^2K(ix,\xi)$ and the hypothesis on f, the theorem of derivation under the integral sign gives that

$$(-\triangle_{k,x}+\mu)R_\mu f(x) = \frac{1}{c_k}\int_{\mathbb{R}^d} K(ix,\xi)\mathcal{F}_D(f)(\xi)\,\omega_k(\xi)d\xi.$$

Thus we obtain the result from relation (2.12). $\square$

Remark 3.1. If $\mu = 0$ the function $R_0 f$ is called the Dunkl potential of f. (Cf. [10]).

Definition 3.1. Let $\mu \in \mathbb{C}$. We denote by

$$\mathcal{B}_{k,\mu}(\mathbb{R}^d) = \left\{ f \in L_k^2(\mathbb{R}^d) :\ \frac{\mathcal{F}_D(f)(\xi)}{\mu+\|\xi\|^2} \in L_k^2(\mathbb{R}^d)\right\}.$$

REMARK 3.2. i) From (3.6) we see that for $\mu \in \mathbb{C}$ such that μ satisfies the hypothesis (H)

$$R_\mu f = \mathcal{F}_D^{-1}\Big(\frac{\mathcal{F}_D(f)(\xi)}{\mu + \|\xi\|^2}\Big), \ f \in W_k(\mathbb{R}^d). \tag{3.7}$$

ii) It is easy to see that $W_k(\mathbb{R}^d) \subset L_k^2(\mathbb{R}^d)$.

iii) As

$$\Big|\Big|\frac{\mathcal{F}_D(f)}{\mu + \|\xi\|^2}\Big|\Big|^2_{L^2_k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \frac{|\mathcal{F}_D(f)(\xi)|^2}{(Re\mu + \|\xi\|^2)^2 + (Im\mu)^2}\omega_k(\xi)d\xi.$$

Then we deduce that:
- If $Im\mu \neq 0$, we have $\mathcal{B}_{k,\mu}(\mathbb{R}^d) = L_k^2(\mathbb{R}^d)$.
- If $Im\mu = 0$ and $Re\mu > 0$, we have $\mathcal{B}_{k,\mu}(\mathbb{R}^d) = L_k^2(\mathbb{R}^d)$.
- If $Im\mu = 0$ and $Re\mu \leq 0$, we have $\mathcal{B}_{k,\mu}(\mathbb{R}^d) \neq \emptyset$. Indeed, let φ be in $L_k^2(\mathbb{R}^d)$, we put $f = \mathcal{F}_D(1_{B^c(0,\sqrt{-Re\mu}+1)}\varphi)$. Thus it is clear that f belongs to $\mathcal{B}_{k,\mu}(\mathbb{R}^d)$.

DEFINITION 3.2. Let $\mu \in \mathbb{C}$. Let f be in $\mathcal{B}_{k,\mu}(\mathbb{R}^d)$, we extend the definition of the generalized resolvent function $R_\mu f$ as the inverse Dunkl-Plancherel transform of $\frac{\mathcal{F}_D(f)}{\mu+\|\xi\|^2}$.

PROPOSITION 3.3. *Let $\mu \in \mathbb{C}$. If f is in $\mathcal{B}_{k,\mu}(\mathbb{R}^d)$, then we have*

$$(-\triangle_k + \mu)R_\mu f = f. \tag{3.8}$$

Proof. Let f be in $\mathcal{B}_{k,\mu}(\mathbb{R}^d)$. For all φ in $\mathcal{S}(\mathbb{R}^d)$ we have from the relations (2.20), (2.21)

$$\begin{aligned}\langle \mathcal{F}_D((-\triangle_k + \mu)R_\mu f), \varphi\rangle &= \langle (-\triangle_k + \mu)R_\mu f, \mathcal{F}_D(\varphi)\rangle \\ &= \langle R_\mu f, (-\triangle_k + \mu)\mathcal{F}_D(\varphi)\rangle.\end{aligned}$$

Now using relation (2.23) we obtain

$$\langle \mathcal{F}_D((-\triangle_k + \mu)R_\mu f), \varphi\rangle = \int_{\mathbb{R}^d} R_\mu f(y)\mathcal{F}_D\Big((||x||^2 + \mu)\varphi\Big)(y)\omega_k(y)dy.$$

Finally the Dunkl-Plancherel formula gives that

$$\begin{aligned}\langle \mathcal{F}_D((-\triangle_k + \mu)R_\mu f), \varphi\rangle &= \int_{\mathbb{R}^d} \frac{\mathcal{F}_D(f)(x)}{||x||^2 + \mu}(||x||^2 + \mu)\varphi(x)\omega_k(x)dx \\ &= \langle \mathcal{F}_D(f), \varphi\rangle.\end{aligned}$$

Thus

$$\mathcal{F}_D((-\triangle_k + \mu)R_\mu f) = \mathcal{F}_D(f).$$

The result is proved. □

Corollary 3.4. *Let f be in $L^2_k(\mathbb{R}^d)$ such that* $\dfrac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n}$ *belongs to $L^2_k(\mathbb{R}^d)$ for some positive integer n, then $(-\triangle_k+\mu)^n(R^n_\mu f)=f$, where $R^n_\mu=R_\mu\circ\ldots\circ R_\mu$.*

Proof. From Proposition 3.3 it follows that if f is in $\mathcal{B}_{k,\mu}(\mathbb{R}^d)$, then

$$(-\triangle_k+\mu)(R_\mu f) \;=\; f.$$

Hence by induction, one can show that if $\dfrac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n}$ is in $L^2_k(\mathbb{R}^d)$ for some positive integer n, then

$$(-\triangle_k+\mu)^n R^n_\mu f=f.$$

□

4. Characterization for the support of the Dunkl transform on $L^2_k(\mathbb{R}^d)$ via the Dunkl resolvent function

Let $\mu\in\mathbb{C}$. We begin this section by the following definition and remark.

$$\sigma_\mu=\inf\Big\{|\mu+\|\xi\|^2| \,:\, \xi\in supp\mathcal{F}_D(f)\Big\},\ \tilde{\sigma}_\mu=\inf\Big\{|\mu+\|\xi\|^2| \,:\, \xi\in\mathbb{R}^d\Big\}.$$

Remark 4.1. It is easy to see that

i) $\sigma_\mu\ge\tilde{\sigma}_\mu$ and $\tilde{\sigma}_\mu=\begin{cases}|Im\mu| & \text{if } \ Re\mu\le 0\\ |\mu| & \text{if } \ Re\mu>0\end{cases}$.

ii) When Re$\mu\le 0$ the condition $\sigma_\mu>\tilde{\sigma}_\mu$ implies that $\mathcal{F}_D(f)$ vanishes on some neighborhood of ξ_0 with $||\xi_0||^2=$ -Reμ.

iii) When Re$\mu>0$, the condition $\sigma_\mu>\tilde{\sigma}_\mu$ implies that $\mathcal{F}_D(f)$ vanishes on some neighborhood of 0.

Lemma 4.1. *Let f be in $L^2_k(\mathbb{R}^d)$ such that* $\dfrac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n}$ *belongs to $L^2_k(\mathbb{R}^d)$ for any $n\in\mathbb{N}$. Then*

$$\lim_{n\to\infty}\left\|\frac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n}\right\|^{1/n}_{L^2_k(\mathbb{R}^d)}=\frac{1}{\sigma_\mu} \tag{4.1}$$

where we set $\dfrac{1}{0}=\infty$, for the sake of convention.

Proof. We divide the proof into two cases.
First case : $\sigma_\mu = 0$. Then for any $\varepsilon > 0$,

$$\int_{|\,\|\xi\|^2+\mu|<\varepsilon} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi > 0.$$

Therefore,

$$\begin{aligned}
\Big\| \frac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n} \Big\|^2_{L^2_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \frac{|\mathcal{F}_D(f)(\xi)|^2}{|\mu+\|\xi\|^2|^{2n}} \omega_k(\xi) d\xi \\
&\geq \int_{|\,\|\xi\|^2+\mu|<\varepsilon} \frac{|\mathcal{F}_D(f)(\xi)|^2}{|\mu+\|\xi\|^2|^{2n}} \omega_k(\xi) d\xi \\
&\geq \frac{1}{\varepsilon^{2n}} \int_{|\,\|\xi\|^2+\mu|<\varepsilon} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi,
\end{aligned}$$

that yields

$$\begin{aligned}
\liminf_{n\to\infty} \Big\| \frac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n} \Big\|^{1/n}_{L^2_k(\mathbb{R}^d)} &\geq \liminf_{n\to\infty} \frac{1}{\varepsilon} \Big\{ \int_{|\mu+\|\xi\|^2|<\varepsilon} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi \Big\}^{1/2n} \\
&= \frac{1}{\varepsilon}.
\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we obtain

$$\liminf_{n\to\infty} \Big\| \frac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n} \Big\|^{1/n}_{L^2_k(\mathbb{R}^d)} = \infty.$$

Second case : $\sigma_\mu > 0$. We have

$$\begin{aligned}
\Big\| \frac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n} \Big\|^2_{L^2_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \frac{|\mathcal{F}_D(f)(\xi)|^2}{|\mu+\|\xi\|^2|^{2n}} \omega_k(\xi) d\xi \\
&= \int_{\sigma_\mu<|\mu+\|\xi\|^2|<\infty} \frac{|\mathcal{F}_D(f)(\xi)|^2}{|\mu+\|\xi\|^2|^{2n}} \omega_k(\xi) d\xi \\
&\leq \frac{1}{\sigma_\mu^{2n}} \int_{\sigma_\mu<|\mu+\|\xi\|^2|<\infty} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi \\
&\leq \frac{1}{\sigma_\mu^{2n}} \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi = \frac{\|\mathcal{F}_D(f)\|^2_{L^2_k(\mathbb{R}^d)}}{\sigma_\mu^{2n}}.
\end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \Big\| \frac{\mathcal{F}_D(f)(\xi)}{(\mu+\|\xi\|^2)^n} \Big\|^{1/n}_{L^2_k(\mathbb{R}^d)} \leq \limsup_{n\to\infty} \frac{1}{\sigma_\mu} \|\mathcal{F}_D(f)\|^{1/n}_{L^2_k(\mathbb{R}^d)} = \frac{1}{\sigma_\mu}.$$

On the other hand, from the definition of σ_μ, for any $\varepsilon > 0$

$$\int_{\sigma_\mu < | \, \|\xi\|^2 + \mu| < \sigma_\mu + \varepsilon} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi > 0.$$

Therefore,

$$\begin{aligned} \Big\| \frac{\mathcal{F}_D(f)(\xi)}{(\|\xi\|^2 + \mu)^n} \Big\|^2_{L^2_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \frac{|\mathcal{F}_D(\xi)|^2 \omega_k(\xi) d\xi}{| \, \|\xi\|^2 + \mu|^{2n}} \\ &\geq \int_{\sigma_\mu < | \, \|\xi\|^2 + \mu| < \sigma_\mu + \varepsilon} \frac{|\mathcal{F}_D(f)(\xi)|^2}{| \, \|\xi\|^2 + \mu|^{2n}} \omega_k(\xi) d\xi \\ &\geq \frac{1}{(\sigma_\mu + \varepsilon)^{2n}} \int_{\sigma_\mu < | \, \|\xi\|^2 + \mu| < \sigma_\mu + \varepsilon} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi \end{aligned}$$

that yields

$$\begin{aligned} &\liminf_{n \to \infty} \Big\| \frac{\mathcal{F}_D(f)(\xi)}{(\mu + \|\xi\|^2)^n} \Big\|^{1/n}_{L^2_k(\mathbb{R}^d)} \\ &\quad \geq \liminf_{n \to \infty} \frac{1}{\sigma_\mu + \varepsilon} \left\{ \int_{\sigma_\mu < |\mu + \|\xi\|^2| < \sigma_\mu + \varepsilon} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi \right\}^{1/2n} \\ &\quad = \frac{1}{\sigma_\mu + \varepsilon}. \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, the inequality (4.1) follows. Lemma 4.1 is thus proved. □

The following theorem describes the image of a function $g \in L^2_k(\mathbb{R}^d)$ that vanishes in a neighborhood of a point ξ_0 under the Dunkl transform.

Theorem 4.2. i) *For $f \in L^2_k(\mathbb{R}^d)$ let $g = \mathcal{F}_D(f)$, then g vanishes in a neighborhood of ξ_0 if and only if $R^n_{-\|\xi_0\|^2} f \in L^2_k(\mathbb{R}^d)$ for all $n = 0, 1, 2, ..,$ and*

$$\lim_{n \to \infty} \|R^n_{-\|\xi_0\|^2} f\|^{1/n}_{L^2_k(\mathbb{R}^d)} = \frac{1}{\sigma_{-\|\xi_0\|^2}} < \infty. \tag{4.2}$$

ii) *For $f \in L^2_k(\mathbb{R}^d)$ let $g = \mathcal{F}_D(f)$, then g vanishes in a neighborhood of ξ_0 if and only if $f \in L^2_k(\mathbb{R}^d)$ and*

$$\lim_{n \to \infty} \|R^n_\mu f\|^{1/n}_{L^2_k(\mathbb{R}^d)} < \frac{1}{\tilde{\sigma}_\mu}\ , \tag{4.3}$$

for some non-real μ with $Re\mu = -\|\xi_0\|^2$.

iii*) For* $f \in L^2_k(\mathbb{R}^d)$ *let* $g = \mathcal{F}_D(f)$*, then* g *vanishes in a neighborhood of* 0 *if and only if* $f \in L^2_k(\mathbb{R}^d)$ *and relation (4.3) holds for some non real* μ *with* $Re\mu > 0$*.*

Proof. i) Necessity : Let $g = \mathcal{F}_D(f) \in L^2_k(\mathbb{R}^d)$ vanish a.e. in a neighborhood of ξ_0. Then $\sigma_{-\|\xi_0\|^2} > 0$ and $\dfrac{\mathcal{F}_D(f)(\xi)}{(\|\xi\|^2 - \|\xi_0\|^2)^n}$ belongs to $L^2_k(\mathbb{R}^d)$ for any $n = 0, 1, \dots$.
Applying the representation for the resolvent function in $L^2_k(\mathbb{R}^d)$ n times we obtain

$$R^n_{-\|\xi_0\|^2} f = \mathcal{F}_D^{-1}\Big(\frac{\mathcal{F}_D(f)(\xi)}{(\|\xi\|^2 - \|\xi_0\|^2)^n}\Big). \tag{4.4}$$

So the Dunkl-Plancherel theorem yields

$$R^n_{-\|\xi_0\|^2} f \in L^2_k(\mathbb{R}^d) \quad \text{for any} \quad n = 0, 1, \dots$$

and

$$\|R^n_{-\|\xi_0\|^2} f\|^2_{L^2_k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \frac{|\mathcal{F}_D(f)(\xi)|^2}{(\|\xi_0\|^2 - \|\xi\|^2)^{2n}} \omega_k(\xi) d\xi. \tag{4.5}$$

Thus

$$\lim_{n\to\infty} \|R^n_{-\|\xi_0\|^2} f\|^{1/n}_{L^2_k(\mathbb{R}^d)} = \lim_{n\to\infty} \Big\|\frac{\mathcal{F}_D(f)(\xi)}{(\|\xi_0\|^2 - \|\xi\|^2)^n}\Big\|^{1/n}_{L^2_k(\mathbb{R}^d)}. \tag{4.6}$$

Therefore, from Lemma 4.1 one can see that

$$\lim_{n\to\infty} \|R^n_{-\|\xi_0\|^2} f\|^{1/n}_{L^2_k(\mathbb{R}^d)} = \frac{1}{\sigma_{-\|\xi_0\|^2}} < \infty.$$

Sufficiency: Let $R^n_{-\|\xi_0\|^2} f$ be in $L^2_k(\mathbb{R}^d)$ for any $n = 0, 1, \dots$, and formula (4.2) holds. Then each $R^n_{-\|\xi_0\|^2} f$ is the Dunkl inverse transformation of some g_n in $L^2_k(\mathbb{R}^d)$.
Since

$$(-\triangle_k - \|\xi_0\|^2)^n R^n_{-\|\xi_0\|^2} f = f,$$

we have

$$\mathcal{F}_D(f)(\xi) = (\|\xi\|^2 - \|\xi_0\|^2)^n \, g_n(\xi).$$

Hence $\dfrac{\mathcal{F}_D(f)(\xi)}{(\|\xi_0\|^2-\|\xi\|^2)^n}$ belongs to $L_k^2(\mathbb{R}^d)$, and formula (4.5) holds. Therefore

$$\begin{aligned}\frac{1}{\sigma_{-\|\xi_0\|^2}} &= \lim_{n\to\infty}\Big\|\frac{\mathcal{F}_D(f)(\xi)}{(\|\xi_0\|^2-\|\xi\|^2)^n}\Big\|^{1/n}_{L_k^2(\mathbb{R}^d)}\\ &= \lim_{n\to\infty}\|R^n_{-\|\xi_0\|^2}f\|^{1/n}_{L_k^2(\mathbb{R}^d)}<\infty.\end{aligned}$$

Thus $\sigma_{-\|\xi_0\|^2}>0$, and $\mathcal{F}_D(f)$ vanishes a.e. in a neighborhood of ξ_0.

ii) Let $g=\mathcal{F}_D(f)$ be in $L_k^2(\mathbb{R}^d)$ vanish a.e. in a neighborhood of ξ_0, and $\mathrm{Re}\mu=-\|\xi_0\|^2$. Then $\sigma_\mu>\tilde{\sigma}_\mu$ and $\dfrac{\mathcal{F}_D(f)(\xi)}{(\|\xi_0\|^2-\|\xi\|^2)^n}$ belongs to $L_k^2(\mathbb{R}^d)$ for any $n=0,1,....$ Hence, $R^n_{-\|\xi_0\|^2}f$ belongs to $L_k^2(\mathbb{R}^d)$ for any $n=0,1,...$ and formula (4.3) holds. This, together with Lemma 4.1, gives formula (4.3). Conversely, let (4.3) hold. Then Lemma 4.1 and formulas (4.3), (4.6) yield that $\sigma_\mu>\widetilde{\sigma}_\mu$. Thus $\mathcal{F}_D(f)$ vanishes in some neighborhood of ξ_0 with $Re\mu=-\|\xi_0\|^2$.

iii) Let $g=\mathcal{F}_D(f)$ be in $L_k^2(\mathbb{R}^d)$ vanish a.e. in a neighborhood of 0. Then for $Re\mu>0$, $\sigma_\mu>\tilde{\sigma}_\mu$ and $\dfrac{\mathcal{F}_D(f)(\xi)}{\mu+\|\xi\|^{2n}}\in L_k^2(\mathbb{R}^d)$ for any $n=0,1,....$ Hence, $R^n_\mu f$ belongs to $L_k^2(\mathbb{R}^d)$ for any $n=0,1,...$ and formula (4.6) holds. This, together with Lemma 4.1, gives the result. Conversely, let (4.3) hold. Then Lemma 4.1 and formulas (4.3), (4.6) yield that $\sigma_\mu>\widetilde{\sigma}_\mu$. Thus $\mathcal{F}_D(f)$ vanishes in some neighborhood of 0. $\square$

5. Characterization for the support of the Dunkl transform on $L_k^p(\mathbb{R})$ via the Dunkl potentials

In this section, we extend the definition of the Dunkl potentials on the space of tempered distributions as follows:

Definition 5.1. Let $f\in\mathcal{S}'(\mathbb{R}^d)$. The tempered generalized function R_0f is termed the Dunkl potential of f if $-\triangle_k(R_0f)=f$, that is

$$\langle R_0f,\triangle_k\varphi\rangle=-\langle f,\varphi\rangle,\quad \text{for all }\varphi\in\mathcal{S}(\mathbb{R}^d).$$

Theorem 5.1. *We assume that $d=1$ and $W=\mathbb{Z}_2$. Let $1\le p\le\infty$. If $R_0^nf\in L_k^p(\mathbb{R})$ for all $n\in\mathbb{N}_0$, then*

$$\lim_{n\to\infty}||R_0^nf||^{\frac{1}{n}}_{L_k^p(\mathbb{R})}=\frac{1}{\sigma_0^2},\tag{5.1}$$

where

$$\sigma_0 = \inf\Big\{|\xi| \,:\, \xi \in supp\mathcal{F}_D(f)\Big\}.$$

To prove this theorem we need the following lemmas.

LEMMA 5.2. *If* $\sigma_0 > 0$, *then*

$$supp\,\mathcal{F}_D\Big(R_0^n f\Big) = supp\,\mathcal{F}_D(f), \quad n = 1, ... \tag{5.2}$$

Proof. As

$$(-\triangle_k)^n (R_0^n f) = f$$

we deduce that

$$\mathcal{F}_D(f) = \xi^{2n}\mathcal{F}_D\Big(R_0^n f\Big).$$

Therefore,

$$supp\,\mathcal{F}_D(f) \subset supp\,\mathcal{F}_D\Big(R_0^n f\Big) \subset \mathcal{F}_D(f) \cup \Big\{0\Big\}.$$

So, to obtain (5.2), it is enough to show that $0 \notin supp\,\mathcal{F}_D\Big(R_0^n f\Big)$.
We choose numbers $a, b : 0 < a < b < \sigma_0$ and a function $h \in D(\mathbb{R})$ such that
$supp\,h \subset (-b, b)$ and $h(x) \equiv 1$ in $(-a, a)$. Then

$$supp\Big(\,h\mathcal{F}_D(R_0^n f)\Big) \subset \Big\{0\Big\}.$$

Suppose that $supp\Big(\,h\mathcal{F}_D(R_0^n f)\Big) = \Big\{0\Big\}$, then there is a number $N(n) \in \mathbb{N}$ such that

$$h\mathcal{F}_D\Big(R_0^n f\Big) = \sum_{j=0}^{N(n)} C_j(N(n))\triangle_k^j \delta.$$

Hence,

$$\mathcal{F}_D^{-1}(h) *_D R_0^n f = \sum_{j=0}^{N(n)} C_j(N(n))(-\xi^2)^j.$$

As $R_0^n f \in L_k^p(\mathbb{R})$ and $\mathcal{F}_D^{-1}(h) \in L_k^q(\mathbb{R})$, we get $\mathcal{F}_D^{-1}(h) *_D R_0^n f \in L_k^\infty(\mathbb{R})$.
Therefore

$$\mathcal{F}_D^{-1}(h) *_D R_0^n f = C_0(N(n)),\ n \in \mathbb{N}.$$

Note that

$$\begin{aligned} C_0(N(n)) &= \mathcal{F}_D^{-1}(h) *_D R_0^n f(x) \\ &= \mathcal{F}_D^{-1}(h) *_D \Big(-\triangle_k\Big) R_0^{n+1} f(x) \\ &= \Big(-\triangle_k\Big)\Big(\mathcal{F}_D^{-1}(h) *_D R_0^{n+1} f(x)\Big) \\ &= \Big(-\triangle_k\Big)(C_0(N(n+1))) = 0. \end{aligned}$$

Thus we deduce that $C_0(N(n)) = 0$. So $h\mathcal{F}_D\Big(R_0^n f\Big) = 0$.
Assume now the contrary that

$$\Big\{0\Big\} \subset supp\, \mathcal{F}_D\Big(R_0^n f\Big).$$

Then there is a function $\chi \in D(\mathbb{R})$, with $supp\,\chi \subset (-a, a)$ and such that

$$\langle \mathcal{F}_D\Big(R_0^n f\Big), \chi\rangle \neq 0.$$

So, as $h(x) = 1$ for $|x| < a$, we get

$$0 \neq \langle \mathcal{F}_D\Big(R_0^n f\Big), \chi\rangle = \langle \mathcal{F}_D\Big(R_0^n f\Big), h\chi\rangle = \langle h\mathcal{F}_D\Big(R_0^n f\Big), \chi\rangle = 0,$$

which is impossible. Thus we have proved (5.2). □

LEMMA 5.3. *If $\sigma_0 > 0$, then*

(5.3) $$\limsup_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} \leq \frac{1}{\sigma_0^2}.$$

Proof. From (5.2) we have

(5.4) $$supp\, \mathcal{F}_D\Big(R_0^n f\Big) \subset \mathbb{R}\backslash(-\sigma_0, \sigma_0).$$

For any $\varepsilon > 0$, $\varepsilon < \frac{\sigma_0}{2}$ we choose a function $h \in \mathcal{E}(\mathbb{R})$ satisfying

$$h(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \geq \sigma_0 - \varepsilon \\ 0 & \text{if} \quad |\xi| < \sigma_0 - 2\varepsilon. \end{cases}$$

Let χ be an arbitrary element in $\mathcal{S}(\mathbb{R})$. Then it follows from (5.4) that

$$\begin{aligned} \langle R_0^n f, \chi\rangle &= \langle \mathcal{F}_D\Big(R_0^n f\Big), \mathcal{F}_D^{-1}(\chi)\rangle \\ &= \langle \mathcal{F}_D\Big(R_0^n f\Big), h\mathcal{F}_D^{-1}(\chi)\rangle \\ &= \langle R_0^n f, \mathcal{F}_D\Big(h\mathcal{F}_D^{-1}(\chi)\Big)\rangle. \end{aligned}$$

Therefore,

$$\langle R_0^n f, \chi\rangle = \langle R_0^n f, \varphi\rangle, \tag{5.5}$$

where

$$\varphi = \mathcal{F}_D\Big(h\mathcal{F}_D^{-1}(\chi)\Big).$$

We put

$$\varphi_n = \mathcal{F}_D\Big(\frac{h(\xi)}{\xi^{2n}}\mathcal{F}_D^{-1}(\chi)\Big).$$

Then $\varphi_n \in \mathcal{S}(\mathbb{R})$ and

$$\begin{array}{rcl} |\langle f, \varphi_n\rangle| & = & |\langle(-\triangle_k)^n R_0^n f, \varphi_n\rangle| \\ & = & |\langle R_0^n f, (-\triangle_k)^n \varphi_n\rangle| \\ & = & |\langle R_0^n f, \varphi\rangle|. \end{array} \tag{5.6}$$

Combining (5.5) and (5.6), we get

$$|\langle R_0^n f, \chi\rangle| = |\langle f, \varphi_n\rangle| = |\langle f, \chi *_D \mathcal{F}_D(\frac{h(\xi)}{\xi^{2n}})\rangle|. \tag{5.7}$$

Therefore, we have

$$\begin{array}{rcl} ||R_0^n f||_{L_k^p(\mathbb{R})} & = & \sup\limits_{\Big\{\chi\in\mathcal{S}(\mathbb{R}):\quad ||\chi||_{L_k^q(\mathbb{R})}\leq 1\Big\}} \Big|\langle f, \chi *_D \mathcal{F}_D(\frac{h(\xi)}{\xi^{2n}})\rangle\Big| \\ & \leq & \sup\limits_{\Big\{\chi\in\mathcal{S}(\mathbb{R}):\quad ||\chi||_{L_k^q(\mathbb{R})}\leq 1\Big\}} ||f||_{L_k^p(\mathbb{R})} ||\chi *_D \mathcal{F}_D(\frac{h(\xi)}{\xi^{2n}})||_{L_k^q(\mathbb{R})} \\ & \leq & C||f||_{L_k^p(\mathbb{R})} ||\mathcal{F}_D(\frac{h(\xi)}{\xi^{2n}})||_{L_k^1(\mathbb{R})}. \end{array}$$

Hence

$$\limsup_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} \leq \limsup_{n\to\infty} ||\mathcal{F}_D(\frac{h(\xi)}{\xi^{2n}})||_{L_k^1(\mathbb{R})}^{\frac{1}{n}}. \tag{5.8}$$

By a simple calculation we prove that

$$\limsup_{n\to\infty} ||\mathcal{F}_D(\frac{h(\xi)}{\xi^{2n}})||_{L_k^1(\mathbb{R})}^{\frac{1}{n}} \leq \frac{1}{(\sigma_0 - 2\varepsilon)^2}. \tag{5.9}$$

Combining (5.8) and (5.9), we get

$$\limsup_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} \leq \frac{1}{(\sigma_0 - 2\varepsilon)^2}$$

and then (5.3) by letting $\varepsilon \to 0$. $\square$

LEMMA 5.4. *If* $\sigma_0 > 0$, *then*

$$\liminf_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} \geq \frac{1}{\sigma_0^2}. \tag{5.10}$$

Proof. From the definition of σ_0, there exists a function $\chi \in D(\mathbb{R})$ such that

$$supp\chi \subset \Big\{\xi : \sigma_0 - \varepsilon < |\xi| < \sigma_0 + \varepsilon\Big\} \quad \text{and} \quad \langle \mathcal{F}_D(f), \chi\rangle \neq 0.$$

Therefore,

$$\begin{aligned} 0 \neq |\langle f, \chi\rangle| &= |\langle (-\triangle_k)^n R_0^n f, \chi\rangle| \\ &= |\langle R_0^n f, (-\triangle_k)^n \chi\rangle| \\ &\leq ||R_0^n f||_{L_k^p(\mathbb{R})} ||(-\triangle_k)^n \chi||_{L_k^q(\mathbb{R})}. \end{aligned} \tag{5.11}$$

So

$$\liminf_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} \geq \frac{1}{\limsup\limits_{n\to\infty} ||(-\triangle_k)^n \chi||_{L_k^q(\mathbb{R})}}. \tag{5.12}$$

We proceed as above to prove

$$\limsup_{n\to\infty} ||(-\triangle_k)^n \chi||_{L_k^q(\mathbb{R})}^{\frac{1}{n}} \leq (\sigma_0 + \varepsilon)^2.$$

So by (5.12) we obtain

$$\liminf_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} \geq \frac{1}{(\sigma + \varepsilon)^2}, \quad \varepsilon > 0,$$

and then (5.10). □

Proof. of Theorem 5.1.

We divide our proof into two cases.

Case 1. $\sigma_0 = 0$. We have $\xi_0 \in supp\mathcal{F}_D(f)$. Hence, for any $\varepsilon > 0$ there is a function $\chi \in D(\mathbb{R})$ such that $supp\,\chi \subset (-\varepsilon, \varepsilon)$ and $\langle \mathcal{F}_D(f), \chi\rangle \neq 0$. Arguing as above we obtain

$$\liminf_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} \geq \frac{1}{\limsup\limits_{n\to\infty} ||(-\triangle_k)^n (\chi)||_{L_k^q(\mathbb{R})}^{\frac{1}{n}}} \geq \frac{1}{\varepsilon^2}.$$

Therefore

$$\liminf_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} = \infty.$$

So we always have

$$\lim_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R})}^{\frac{1}{n}} = \frac{1}{\sigma_0^2}.$$

Case 2. $\sigma_0 > 0$. Combining (5.3) and (5.10), we arrive to (5.1).

□

We proceed as above theorem, we characterize the support of the Dunkl transform on $L_k^p(\mathbb{R}^d)$ via the Dunkl potentials by the following result.

THEOREM 5.5. *Let $1 \leq p \leq \infty$ and $R_0^n f \in L_k^p(\mathbb{R}^d)$ for all $n \in \mathbb{N}_0$. If $0 \notin supp\, \mathcal{F}_D\Big(R_0^n f\Big)$, then*

$$\lim_{n\to\infty} ||R_0^n f||_{L_k^p(\mathbb{R}^d)}^{\frac{1}{n}} = \frac{1}{\sigma_0^2}, \tag{5.13}$$

where

$$\sigma_0 = \inf\Big\{||\xi|| \;:\; \xi \in supp\mathcal{F}_D(f)\Big\}.$$

6. Real Paley-Wiener theorems for the Dunkl transform on $\mathcal{S}'(\mathbb{R}^d)$

We start this section by stating the following result.

THEOREM 6.1. *Let P be a non-constant polynomial with complex coefficients on $\mathbb{R}^d$. Let $u \in \mathcal{E}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$, and suppose the set*

$$V_r := \Big\{\xi \in \mathbb{R}^d : |P(\xi)| \leq r\Big\}$$

is compact for a constant $r \geq 0$. Then the support of $\mathcal{F}_D(u)$ is contained in V_r, if and only if, for each $R > r$, there exist N_R and a positive constant $C(R)$ such that

$$|P^n(-iT)(u)(x)| \leq C(R)R^n(1 + ||x||)^{N_R}, \tag{6.1}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$.

Proof. Assume that the support of $\mathcal{F}_D(u)$ is contained in the compact V_r. Let $R > r$ and let $\varepsilon \in (0, R - r)$. We choose $\chi \in D(\mathbb{R}^d)$ such that $\chi \equiv 1$ on an open neighborhood of support of $\mathcal{F}_D(u)$, and $\chi \equiv 0$ outside

$V_{R-\frac{\varepsilon}{3}}$. As $\mathcal{F}_D(u)$ is of order N, there exists a positive constant C such that for all $x \in \mathbb{R}^d$

$$\begin{array}{rcl} |P^n(-iT)(u)(x)| & = & \Big|(\mathcal{F}_D)^{-1}\Big(P^n(\xi)\mathcal{F}_D(u)\Big)(x)\Big| \\ & = & \Big|(\mathcal{F}_D)^{-1}\Big(\chi(\xi)P^n(\xi)\mathcal{F}_D(u)\Big)(x)\Big| \\ & = & |\langle \chi(\xi)P^n(\xi)\mathcal{F}_D(u)(\xi), K(i\xi,x)\rangle| \\ & = & |\langle \mathcal{F}_D(u)(\xi), \chi(\xi)P^n(\xi)K(i\xi,x)\rangle| \\ & \leq & C\sup_{\xi\in\mathbb{R}^d} \sum_{0\leq|l|\leq N} \Big|D^l\Big(\chi(\xi)P^n(\xi)K(i\xi,x)\Big)\Big|. \end{array}$$

Thus from the Leibniz formula (2.9) we obtain that

$$\begin{aligned} \forall\, n \in \mathbb{N}_0, \quad |P^n(-iT)(u)(x)| &\leq C_1(R)n^N(R-\frac{\varepsilon}{3})^n(1+||x||)^N \\ &\leq C_2(R)R^n(1+||x||)^N. \end{aligned}$$

Conversely we assume that we have (6.1).
Suppose $\xi_0 \in \mathbb{R}^d$ is fixed and such that $|P(\xi_0)| \geq R+\varepsilon$, for some $\varepsilon > 0$. Choose and fix $\chi \in D(\mathbb{R}^d)$ such that $\mathrm{supp}\chi \subset \Big\{\xi \in \mathbb{R}^d : |P(\xi)| \geq R+\frac{\varepsilon}{3}\Big\}$, and put $\chi_n = P^{-n}(\xi)\chi$. We have

$$\begin{array}{rcl} \langle \mathcal{F}_D(u), \chi\rangle & = & \langle \mathcal{F}_D(u), P^n(\xi)\chi_n\rangle = \langle P^n(\xi)\mathcal{F}_D(u), \chi_n\rangle \\ & = & \langle \mathcal{F}_D(P^n(-iT)u), \chi_n\rangle \\ & = & \langle\Big((1+||x||)^{-N}P^n(-iT)u\Big), (1+||x||)^N(\mathcal{F}_D)^{-1}(\chi_n)\rangle. \end{array}$$

Hence, from the Hölder inequality we obtain

$$\begin{aligned} &|\langle \mathcal{F}_D(u), \chi\rangle| \\ &\leq C||(1+||x||)^{-N+d+1}P^n(-iT)u||_{L^\infty_k(\mathbb{R}^d)}||(1+||x||)^N(\mathcal{F}_D)^{-1}(\chi_n)||_{L^2_k(\mathbb{R}^d)}. \end{aligned}$$

We proceed as in Theorem 4 [16], to prove that

$$||(1+||x||)^N(\mathcal{F}_D)^{-1}(\chi_n)||_{L^2_k(\mathbb{R}^d)} \;\leq\; Cn^M(|P(\xi_0)|+\frac{\varepsilon}{3})^{-n} \leq Cn^M(R+\frac{\varepsilon}{3})^{-n}.$$

Thus

$$|\langle \mathcal{F}_D(u), \chi\rangle| \leq C(R)n^{M+N}\Big(\frac{R}{R+\frac{\varepsilon}{3}}\Big)^n.$$

Thus we deduce $\langle \mathcal{F}_D(u), \chi\rangle = 0$, which implies that $\xi_0 \notin$ *supp* $\mathcal{F}_D(u)$. Thus the support of $\mathcal{F}_D(u)$ is contained in the compact V_r. $\square$

Corollary 6.2. *Let P be a non-constant polynomial with complex coefficients on $\mathbb{R}^d$. Let $u \in \mathcal{E}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ such that supp$\mathcal{F}_D(u)$ is compact. Then*

$$\sup_{y \in supp\mathcal{F}_D(f)} |P(y)| = \mathcal{R}_u,$$

where $\mathcal{R}_u$ is defined as the infimum of all $R \geq 0$ for which there exist N and $C(N,R) \geq 0$, such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$

$$|P^n(-iT)(u)(x)| \leq C(R,N) n^N R^n (1 + ||x||)^N.$$

Notations. We denote by

$$B_r := \Big\{\xi \in \mathbb{R}^d : |P(\xi)| < r\Big\}, \quad S_r := \Big\{\xi \in \mathbb{R}^d : |P(\xi)| = r\Big\}.$$

Theorem 6.3. *Let $u = u_0 \in \mathcal{E}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$, and consider the infinite series $\{u_{-n}\}_{n\in\mathbb{N}}$ of generalized tempered distributions defined as $u_{-n+1} = P(-iT)u_n$, for a polynomial P and for all $n \in \mathbb{N}$. Let $r > 0$. Assume, for all $R \in (0,r)$ there exist constants $N \in \mathbb{N}_0$ and $C > 0$, such that*

$$\forall\, x \in \mathbb{R}^d, \quad |u_{-n}(x)| \leq C R^{-n} (1 + ||x||)^N, \tag{6.2}$$

for all $n \in \mathbb{N}$. Then supp$\mathcal{F}_D(u) \cap B_r = \emptyset$. On the other hand, if supp$\mathcal{F}_D(u) \cap B_r = \emptyset$ and supp$\mathcal{F}_D(u)$ is compact, then (6.2) holds, for all $R \in (0,r)$.

Proof. Assume that supp$\mathcal{F}_D(u) \cap B_r = \emptyset$ and supp$\mathcal{F}_D(u)$ is compact. Let $R \in (0,r)$ and let $\varepsilon \in (0, r-R)$. Choose $\chi \in D(\mathbb{R}^d)$ such that $\chi \equiv 1$ on an open neighborhood of the support of $\mathcal{F}_D(u)$, and $\chi \equiv 0$ outside $V_{R+\frac{\varepsilon}{3}}$. As $u = P^n(-iT)u_{-n}$, we have

$$\begin{aligned}
|u_{-n}(x)| = \;&= \Big|\mathcal{F}_D^{-1}\Big(P^{-n}(\xi)\mathcal{F}_D(u)\Big)(x)\Big| \\
&= \Big|\mathcal{F}_D^{-1}\Big(\chi(\xi)P^{-n}(\xi)\mathcal{F}_D(u)\Big)(x)\Big| \\
&= |\langle \chi(\xi)P^{-n}(\xi)\mathcal{F}_D(u)(\xi), K(i\xi,x)\rangle| \\
&= |\langle \mathcal{F}_D(u)(\xi), \chi(\xi)P^{-n}(\xi)K(i\xi,x)\rangle| \\
&\leq C \sup\nolimits_{\xi\in\mathbb{R}^d} \sum_{0\leq|j|\leq N} \Big|D^j\Big(\chi(\xi)P^{-n}(\xi)K(i\xi,x)\Big)\Big|.
\end{aligned}$$

Thus from Leibniz formula (2.9) we obtain

$$\forall\, n \in \mathbb{N}_0, \quad |u_{-n}(x)| \leq C_1(R) n^N (R+\frac{\varepsilon}{3})^{-n}(1+||x||)^N \leq C_2(R) R^{-n} (1+||x||)^N.$$

Assume that we have (6.2). For a fixed $R \in (0, r)$, let $\varepsilon > 0$. Choose and fix $\chi \in D(\mathbb{R}^d)$ such that $\mathrm{supp}\chi \subset \left\{\xi \in \mathbb{R}^d : |P(\xi)| \geq R - \frac{\varepsilon}{3}\right\}$, and put $\chi_n = P^n(\xi)\chi$. We have

$$\begin{aligned} \langle \mathcal{F}_D(u), \chi\rangle &= \langle \mathcal{F}_D(u), P^{-n}(\xi)\chi_n\rangle = \langle P^{-n}(\xi)\mathcal{F}_D(u), \chi_n\rangle \\ &= \langle \mathcal{F}_D(u_{-n}), \chi_n\rangle \\ &= \langle \Big((1+||x||)^{-N}u_{-n}\Big), (1+||x||)^N \mathcal{F}_D^{-1}(\chi_n)\rangle. \end{aligned}$$

Hence, from the Hölder inequality we obtain

$$|\langle \mathcal{F}_D(u), \chi\rangle| \leq ||(1+||x||)^{-N}u_{-n}||_{L_k^\infty(\mathbb{R}^d)}||(1+||x||)^N \mathcal{F}_D^{-1}(\chi_n)||_{L_k^1(\mathbb{R}^d)}.$$

We proceed as in Theorem 4 [16] to prove

$$||(1+||x||)^N \mathcal{F}_D^{-1}(\chi_n)||_{L_k^1(\mathbb{R}^d)} \leq Cn^M(|P(\xi_0)| + \frac{\varepsilon}{3})^{-n} \leq Cn^M(R + \frac{\varepsilon}{3})^{-n}.$$

Thus

$$|\langle \mathcal{F}_D(u), \chi\rangle| \leq C(R)n^{M+N}\Big(\frac{R}{R+\frac{\varepsilon}{3}}\Big)^n.$$

Thus we deduce $\langle \mathcal{F}_D(u), \chi\rangle = 0$, which implies that $\mathrm{supp}\mathcal{F}_D(u) \cap B_r = \emptyset$. □

Putting Theorem 6.1 and Theorem 6.3 together, we get the following.

Corollary 6.4. *Let $u = u_0 \in \mathcal{E}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$, and consider the infinite series $\{u_n\}_{n\in\mathbb{Z}}$ of generalized tempered distributions defined as $u_{n+1} = P(-iT)u_n$, for a polynomial P and for all $n \in \mathbb{Z}$. Let $R > 0$. Then supp$\mathcal{F}_D(u)$ is contained in S_R, if and only if for all $\varepsilon > 0$, there exist constants $N \in \mathbb{N}_0$ and $C > 0$, such that*

$$\forall\, x \in \mathbb{R}, \quad |u_n(x)| \leq CR^n(1+\varepsilon)^{|n|}(1+||x||)^N \tag{6.3}$$

for all $n \in \mathbb{Z}$.

Remark 6.1. Theorem 6.3 and Corollary 6.4 are the analogue of the new real Paley-Wiener theorems for the Fourier transform, proved by Andersen (see [2]).

7. Roe's Theorem associated with the Dunkl operators

In [19] Roe proved that if a doubly-infinite sequence $(f_j)_{j\in\mathbb{Z}}$ of functions on $\mathbb{R}$ satisfies

$\frac{df_j}{dx} = f_{j+1}$ and $|f_j(x)| \leq M$ for all $j = 0, \pm 1, \pm 2, ...$ and $x \in \mathbb{R}$, then $f_0(x) = a\sin(x+b)$ where a and b are real constants. This result was extended to $\mathbb{R}^d$ by Strichartz [21] where $\frac{d}{dx}$ is substituted by the Laplacian on $\mathbb{R}^d$ as follows.

Theorem. (Strichartz). Let $(f_j)_{j\in\mathbb{Z}}$ be a doubly infinite sequence of measurable functions on $\mathbb{R}^d$ such that for all $j \in \mathbb{Z}$, (i) $||f_j||_{L^\infty(\mathbb{R}^d)} \leq C$ for some constant $C > 0$ and (ii) for some $a > 0, \triangle f_j = af_{j+1}$. Then $\triangle f_0 = -af_0$.

The purpose of this section is to generalize this theorem in the context of Dunkl setting. We now state our main result.

THEOREM 7.1. *Suppose* $P(\xi) = \sum_n a_n \xi^n$ *is real-valued and let* $\{f_j\}_{-\infty}^{\infty}$ *be a sequence of complex-valued functions on* $\mathbb{R}$ *such that*

$$f_{j+1} = P(-iT)f_j.$$

(i) Let $a \geq 0$, $R > 0$, *and assume that* $\{f_j\}_{-\infty}^{\infty}$ *satisfies*

$$|f_j(x)| \leq M_j R^j (1 + ||x||)^a, \tag{7.1}$$

where $(M_j)_{j\in\mathbb{Z}}$ *satisfies the sublinear growth condition*

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0. \tag{7.2}$$

Then $f = f_+ + f_-$ *where* $P(-iT)f_+ = Rf_+$ *and* $P(-iT)f_- = -Rf_-$. *If* R *(or* $-R$*) is not in the range of* P *then* $f_+ = 0$ *(or* $f_- = 0$*).*

(ii) If we replace (7.2) by

$$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^{|j|}} = 0, \tag{7.3}$$

for all $j > 0$, *then the span of* $(f_j)_j$ *is finite dimensional. Moreover,* $f_0 = f_+ + f_-$, *where, for some integer* N , $(P(-iT) - R)^N f_+ = 0$ *and* $(P(-iT) + R)^N f_- = 0$. *Thus* f_+ *(or* f_- *) is a generalized eigenfunction of* $P(-iT)$ *with eigenvalue* R *(or* $-R$*).*

We break the proof up into three steps. In the first step we consider the Dunkl transform $\mathcal{F}_D(f_0)$ of f_0, which exists as a distribution.

LEMMA 7.2. *Let* $a \geq 0$. *If* $(f_j)_{j\in\mathbb{Z}}$ *is a sequence of functions on* $\mathbb{R}^d$ *satisfying*

$$P(-iT)f_j = f_{j+1}, \tag{7.4}$$

(7.5) $$\forall\, x \in \mathbb{R}^d, \quad |f_j(x)| \le M_j R^j (1 + ||x||)^a,$$

and

(7.6) $$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^{|j|}} = 0,$$

for all $\varepsilon > 0$*, then*

$$\text{support}(\mathcal{F}_D(f_0)) \subset S_R := \Big\{\xi :\ |P(\xi)| = R\Big\}.$$

Proof. First we show that $\mathcal{F}_D(f_0)$ is supported in $\Big\{\xi :\ |P(\xi)| \le R\Big\}$. To do this we need to show that $\langle \mathcal{F}_D(f_0), \phi\rangle = 0$ if $\phi \in D(\mathbb{R}^d)$ and $\text{support}(\phi) \cap \Big\{\xi :\ |P(\xi)| \le R\Big\} = \emptyset$. Since support$(\phi)$ is compact, there is some $r < R$ so that $\frac{1}{|P(\xi)|} \le r$, for all $\xi \in \text{support}(\phi)$. Then

$$\begin{aligned} \langle \mathcal{F}_D(f_0), \phi\rangle &= \langle P^j \mathcal{F}_D(f_0), \tfrac{\phi}{P^j}\rangle \\ &= \langle (\mathcal{F}_D)\Big(P(-iT)^j f_0\Big), \tfrac{\phi}{P^j}\rangle \\ &= \langle P(-iT)^j f_0, (\mathcal{F}_D)^{-1}(\tfrac{\phi}{P^j})\rangle. \end{aligned}$$

Choose an integer m with $2m \ge 2a + 2\gamma + d + 1$. A calculation, using the hypothesis of Lemma 7.2 and Cauchy-Schwartz inequality, implies

$$\begin{aligned} |\langle \mathcal{F}_D(f_0), \phi\rangle| &\le \int_{\mathbb{R}^d} |P(-iT)^j f_0(x)||(\mathcal{F}_D)^{-1}(\frac{\phi}{P^j})(x)|\omega_k(x)dx \\ &\le CM_j \sup_{x\in\mathbb{R}^d} |(1 + ||x||^2)^{m+1} (\mathcal{F}_D)^{-1}(\frac{\phi}{P^j})(x)]|. \end{aligned}$$

Using the continuity of $(\mathcal{F}_D)^{-1}$ and the fact that ϕ is supported in $\Big\{\xi :\ |P(\xi)| \ge R + \varepsilon\Big\}$ for some fixed $\varepsilon > 0$, it is not hard to prove that the right-hand side of this goes to zero as $j \to \infty$ and so $\langle \mathcal{F}_D(f_0), \phi\rangle = 0$. To complete the proof we need to show that $\mathcal{F}_D(f_0)$ is also supported in $\Big\{\xi :\ |P(\xi)| \ge R\Big\}$, which means that $\langle \mathcal{F}_D(f_0), \phi\rangle = 0$ if ϕ is supported in $\Big\{\xi :\ |P(\xi)| \le R\Big\}$. Here we use (7.4) to obtain

$$\langle \mathcal{F}_D(f_0), \phi\rangle = \langle \mathcal{F}_D(f_{-j}), P^j\phi\rangle$$

and the argument proceeds as before. □

The next step in the proof we assume firstly that $-R$ is not a value of $P(\xi)$, and we show that $L_k f_0 = R f_0$.

Lemma 7.3. *There exists an integer N such that*

$$(P(\xi) - R)^{N+1}\mathcal{F}_D(f_0) = 0. \tag{7.7}$$

Proof. Using Lemma 7.2 and proceeding as in [13], we prove the result. □

Proof. of Theorem 7.1

We want to prove (i). Indeed, inverting the Dunkl transform in (7.7) yields that

$$(P(-iT) - R)^{N+1} f_0 = 0. \tag{7.8}$$

This equation implies

$$\begin{aligned}\text{span}\Big\{f_0, f_1, f_2, ...\Big\} &= \text{span}\Big\{f_0, P(-iT)f_0, P(-iT)^2 f_0, ...\Big\}\\ &= \text{span}\Big\{f_0, P(-iT)f_0, ..., P^N(-iT)f_0\Big\}.\end{aligned}$$

We shall now show that we can take $N = 0$ in (7.8). If not then $(P(-iT) - R)f_0 \neq 0$. Let p be the largest positive integer so that $(P(-iT) - R)^p f_0 \neq 0$. Clearly $p \leq N$. Thus

$$f := (P(-iT) - R)^{p-1} f_0 \in \text{span}\Big\{f_0, f_1, ..., f_N\Big\}$$

will satisfy

$$(P(-iT) - R)^2 f = 0 \quad \text{and} \quad (P(-iT) - R)f \neq 0. \tag{7.9}$$

Write

$$f = a_0 f_0 + ... + a_N f_N,$$

for constants $a_0, ..., a_N$. Then

$$P^j(-iT)f = a_0 f_j + ... + a_N f_{N+j}.$$

If

$$C_j = |a_0|R^0 M_j + ... + |a_N|R^N M_{j+N},$$

then this and (7.1) imply

$$|P^j(-iT)f(x)| \leq C_j R^j (1 + ||x||)^a. \tag{7.10}$$

By (7.2) these satisfy the sublinear growth condition

$$\lim_{j\to\infty} \frac{C_j}{j} = 0. \tag{7.11}$$

An induction using (7.9) implies for $j \geq 2$ that

$$P^j(-iT)f = R^{j-1} jP(-iT)f - R^j(j-1)f = R^{j-1}j(P(-iT)-R)f + R^j f.$$

Thus

$$|(P(-iT)-R)f(x)| \leq \frac{1}{jR^{j-1}}|P^j(-iT)f(x)| + \frac{R|f(x)|}{j}$$
$$\leq \frac{C_jR}{j}(1+||x||)^a + \frac{R|f(x)|}{j}.$$

Letting $j \to \infty$ and using (7.11) implies $(P(-iT)-R)f = 0$. But this contradicts (7.9). Consequently, $N = 0$ in (7.8). This completes the proof in the case that $-R$ is not in the range of P.

In the case that R is not in the range of P we apply the same argument to $-P(-iT)$ to conclude $P(-iT)f_0 = -Rf_0$.
In the general case, let $\mathfrak{L} = P^2(-iT)$. Then $\mathcal{F}_D(\mathfrak{L}f)(\xi) = P^2(\xi)\mathcal{F}_D(f)(\xi)$. $\mathfrak{L}f_{2p} = f_{2(p+1)}$ and $P^2(\xi) \neq -R$. Thus we can (as before) conclude, for the sequence $(f_{2p})_{p\in\mathbb{Z}}$ that

$$\mathfrak{L}f_0 = P^2(-iT)f_0 = R^2 f_0.$$

Set

$$f_+ = \frac{1}{2}(f_0 + \frac{1}{R}P(-iT)f_0) \text{ and } f_- = \frac{1}{2}(f_0 - \frac{1}{R}P(-iT)f_0).$$

Then

$$f = f_+ + f_-,\ P(-iT)f_+ = Rf_+, \text{ and } P(-iT)f_- = -Rf_-.$$

This completes the proof of (i).

Now we want to prove (ii). Indeed the proof will be based on the following result from linear algebra. (cf. [5], Chapter 10.)

Lemma 7.4. *Let X be a finite dimensional complex vector space, and let $T : X \to X$ be a linear map with eigenvalues $\lambda_1, ..., \lambda_p$. Then $X = X_1 \oplus ... \oplus X_p$, where $X_j = ker((T-\lambda_j)^N)$ and $dim X = N$.*

We first prove (ii) under the assumption that $P(\xi) \neq -R$. Using the growth condition (7.3) and Lemma 7.4, we may still conclude that

$$\text{support}(\mathcal{F}_D(f_0)) \subset S_R := \Big\{\xi : P(\xi) = R\Big\}.$$

But then, as before, we can conclude that (7.8) holds. But this is enough to complete the proof in this case. A similar argument shows that if $P(\xi) \neq R$, then $(P(-iT)+R)^N f_0 = 0$.
In the general case we again let $\mathfrak{L} = P^2(-iT)$ and $P_0 = P^2$. Then $P_0(\xi) \neq -R$ and the span of $(f_{2j})_j$ is finite dimensional. The map $P(-iT)$ takes the span of $(f_{2j})_j$ onto the span of $(f_{2j+1})_j$. Thus X is

finite dimensional. Any $f \in X$ will have support(f) inside the set defined by $P(\xi) = \pm R$. From this it is not hard to show that the only possible eigenvalues of $P(-iT)$ restricted to X are R and $-R$. The result now follows from the last lemma. $\square$

REMARK 7.1. (i) If we take $P(y) = -||y||^2$, then $L_k = \triangle_k$ and Theorem 7.1 gives $\triangle_k f_0 = -Rf_0$. This characterizes eigenfunctions f of generalized Laplace operator $\triangle_k$ with polynomial growth in terms of the size of the powers $\triangle_k^j f$, $-\infty < j < \infty$.

(ii) We note that the results presented in this section, inspired by [13], generalizes and improves the version presented in [15]. This version was established, for $R = 1$.

In the rest of this section we state another version of the Roe's theorem associated to the Dunkl operator on the real line. This version is proved in the context of the Dunkl-type operator, which is more general that the Dunkl operator in real line. (See [17]).

For the sake of simplification, we denote the Dunkl operator on the real line by Λ. This operator is defined by

$$\Lambda f(x) = f'(x) + \frac{2k}{x}(f(x) - f(-x)).$$

THEOREM 7.5. *Suppose* $P(\xi) = \sum_n a_n \xi^n$ *is a non-constant polynomial with complex coefficients. Let* $\{f_j\}_{-\infty}^{\infty}$ *be a sequence of complex-valued functions on* $\mathbb{R}$ *such that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(\Lambda) f_j.$$

1) Let $a \geq 0$ *and let* $R > 0$. *Assume that for all* $\varepsilon > 0$, *there exist constants* $N \in \mathbb{N}_0$ *and* $C > 0$, *so that*

$$\forall\, x \in \mathbb{R}, \quad |f_n(x)| \leq CR^n (1+\varepsilon)^{|n|} (1+|x|)^N \tag{7.12}$$

is satisfied for all $n \in \mathbb{Z}$. *Then*

$$f_0 = \sum_{\lambda \in S_R} \sum_{j=0}^{N} c(\lambda, j) \frac{d^j}{d\xi^j}_{|\xi=\lambda} K(i\xi, .), \tag{7.13}$$

for constants $c(\lambda, j) \in \mathbb{C}$ *and* $N \in \mathbb{N}$.

2) Let $a \geq 0$ *and let* $R > 0$ *and assume that* $\{f_j\}_{-\infty}^{\infty}$ *satisfies*

$$|f_j(x)| \leq M_j R^j (1+|x|)^a, \tag{7.14}$$

where $(M_j)_{j\in\mathbb{Z}}$ *satisfies the subpotential growth condition*

$$\lim_{j\to\infty} \frac{M_{|j|}}{j^m} = 0, \tag{7.15}$$

for some $m \geq 0$.

We have

(i) If $P'(\lambda_p) \neq 0$, *for all* $\lambda_p \in S_R$, *then* $N < m$ *in (7.13).*

In particular, if $m = 1$, *then*

$$f_0 = \sum_{\lambda_p\in S_R} f_{\lambda_p}, \quad \text{where } f_{\lambda_p} = c(\lambda_p)K(i\lambda_p, .)$$

.

(ii) If S_R *consists of one point* λ_0 *and* $m = 1$ *in (7.15), then* $P(\Lambda)f_0 = P(\lambda_0)f_0$.

Remark 7.2. The previous theorem is the analogue for Theorems 1 and 6 of [2].

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