BOUNDEDNESS IN NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS VIA $t^\infty$-SIMILARITY

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Abstract. This paper shows that the solutions to nonlinear perturbed differential system

$$y' = f(t, x) + \int_{t_0}^{t} g(s, y(s)) ds + h(t, y(t) , Ty(t))$$

have bounded properties. To show the bounded properties, we impose conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s)) ds, h(t, y(t), Ty(t)),$ and on the fundamental matrix of the unperturbed system $y' = f(t, y)$ using the notion of $h$-stability.

1. Introduction and preliminaries

We consider the nonautonomous differential system

$$x' = f(t, x), \quad x(t_0) = x_0,$$  \hspace{1cm} (1.1)

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean $n$-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed differential systems of (1.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s)) ds + h(t, y(t) , Ty(t)) , \quad y(t_0) = y_0,$$  \hspace{1cm} (1.2)

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where \( g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n) \), \( h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n) \) , \( g(t, 0) = 0 \), \( h(t, 0, 0) = 0 \), and \( T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n) \) is a continuous operator.

The symbol \( |\cdot| \) will be used to denote any convenient vector norm in \( \mathbb{R}^n \). For an \( n \times n \) matrix \( A \), define the norm \( |A| \) of \( A \) by

\[
|A| = \sup_{|x| \leq 1} |Ax|.
\]

Let \( x(t, t_0, x_0) \) denote the unique solution of (1.1) with \( x(t_0, t_0, x_0) = x_0 \), existing on \([t_0, \infty)\). Then we can consider the associated variational systems around the zero solution of (1.1) and around \( x(t) \), respectively,

\[
v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0
\]  

and

\[
z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0.
\]  

The fundamental matrix \( \Phi(t, t_0, x_0) \) of (1.4) is given by

\[
\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),
\]

and \( \Phi(t, t_0, 0) \) is the fundamental matrix of (1.3).

We recall some notions of \( h \)-stability [16].

**Definition 1.1.** The system (1.1) (the zero solution \( x = 0 \) of (1.1)) is called an \( h \)-system if there exist a constant \( c \geq 1 \), and a positive continuous function \( h \) on \( \mathbb{R}^+ \) such that

\[
|x(t)| \leq c |x_0| h(t) h(t_0)^{-1}
\]

for \( t \geq t_0 \geq 0 \) and \( |x_0| \) small enough (here \( h(t)^{-1} = \frac{1}{h(t)} \)).

**Definition 1.2.** The system (1.1) (the zero solution \( x = 0 \) of (1.1)) is called \( (hS)h \)-stable if there exists \( \delta > 0 \) such that (1.1) is an \( h \)-system for \( |x_0| \leq \delta \) and \( h \) is bounded.

Pinto [15, 16] introduced the notion of \( h \)-stability (hS) with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called \( h \)-systems. Choi, Ryu [5] and Choi et al. [6] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [8,9,10,11] and Choi and Goo [3,4] investigated boundedness of solutions for nonlinear perturbed systems.
In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of $t_\infty$-similarity.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class $C^1$ with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of $t_\infty$-similarity in $\mathcal{M}$ was introduced by Conti [7].

**Definition 1.3.** A matrix $A(t) \in \mathcal{M}$ is $t_\infty$-similar to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \quad (1.5)$$

for some $S(t) \in \mathcal{N}$.

The notion of $t_\infty$-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [7, 12].

For the proof we prepare some related properties.

**Lemma 1.4.** [16] The linear system

$$x' = A(t)x, \quad x(t_0) = x_0, \quad (1.6)$$

where $A(t)$ is an $n \times n$ continuous matrix, is an $h$-system (respectively $h$-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function $h$ defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c h(t) h(t_0)^{-1} \quad (1.7)$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \quad y(t_0) = y_0, \quad (1.8)$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point $(t_0, y_0)$ in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].
Lemma 1.5. [2] Let \( x \) and \( y \) be a solution of (1.1) and (1.8), respectively. If \( y_0 \in \mathbb{R}^n \), then for all \( t \geq t_0 \) such that \( x(t, t_0, y_0) \in \mathbb{R}^n \),
\[
y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) g(s, y(s)) ds.
\]

Theorem 1.6. [5] If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.

Theorem 1.7. [6] Suppose that \( f_x(t, 0) \) is \( t_{\infty} \)-similar to \( f_x(t, x(t_0, x_0)) \) for \( t \geq t_0 \geq 0 \) and \( |x_0| \leq \delta \) for some constant \( \delta > 0 \). If the solution \( v = 0 \) of (1.3) is hS, then the solution \( z = 0 \) of (1.4) is hS.

Lemma 1.8. (Bihari − type inequality) Let \( u, \lambda \in C(\mathbb{R}^+) \), \( w \in C((0, \infty)) \) and \( w(u) \) be nondecreasing in \( u \). Suppose that, for some \( c > 0 \),
\[
u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \quad t \geq t_0 \geq 0.
\]
Then
\[
u(t) \leq W^{-1}[W(c) + \int_{t_0}^{t} \lambda(s) ds],
\]
where \( t_0 \leq t < b_1 \), \( W(u) = \int_{u_0}^{u} \frac{ds}{w(s)} \), \( W^{-1}(u) \) is the inverse of \( W(u) \), and
\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \text{dom} W^{-1} \right\}.
\]

Lemma 1.9. [3] Let \( u, \lambda_1, \lambda_2, \lambda_3 \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+) \), \( w \in C((0, \infty)) \) and \( w(u) \) be nondecreasing in \( u \), \( u \leq w(u) \). Suppose that for some \( c > 0 \),
\[
u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds
\]
\[
+ \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) w(u(\tau)) d\tau ds
\]
\[
+ \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) w(u(\tau)) d\tau ds, \quad 0 \leq t_0 \leq t.
\]
Then
\[
  u(t) \leq W^{-1}\left[ W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau)d\tau \\
  + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau)d\tau)ds \right],
\]
where \( t_0 \leq t < b_1 \), \( W, W^{-1} \) are the same functions as in Lemma 1.8, and

\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau)d\tau \\
  + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau)d\tau)ds \in \text{dom}W^{-1} \right\}.
\]

**Lemma 1.10.** [3] Let \( u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C([0, \infty)), w \in C((0, \infty)) \), and \( w(u) \) be nondecreasing in \( u \), \( u \leq w(u) \). Suppose that for some \( c > 0 \) and \( 0 \leq t_0 \leq t \),

\[
u(t) \leq c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)u(\tau)d\tau \\
  + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(\tau)w(u(\tau))d\tau d\tau ds + \int_{t_0}^t \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)w(u(\tau))d\tau d\tau ds.
\]

Then
\[
u(t) \leq W^{-1}\left[ W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(\tau)d\tau) \\
  + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau)ds \right],
\]
where \( t_0 \leq t < b_1 \), \( W, W^{-1} \) are the same functions as in Lemma 1.8, and

\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(\tau)d\tau) \\
  + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau)ds \in \text{dom}W^{-1} \right\}.
\]

For the proof we need the following corollary from Lemma 1.10.

**Corollary 1.11.** Let \( u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C([0, \infty)), w \in C((0, \infty)) \), and \( w(u) \) be nondecreasing in \( u \), \( u \leq w(u) \). Suppose that for some \( c > 0 \)
and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) u(\tau)) d\tau d\sigma + \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) w(u(\tau)) d\sigma d\tau.$$

Then

$$u(t) \leq W^{-1} \left[ W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r) dr) d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) d\sigma ds \right],$$

where $t_0 \leq t < b_1$, $W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r) dr) d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) d\sigma ds \in \text{dom} W^{-1} \right\}.$$

**Lemma 1.12.** [8] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+), w \in C((0, \infty))$, and $w(u)$ be nondecreasing in $u, u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) u(\tau)) d\sigma d\tau + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) w(u(\tau)) ds + \int_{t_0}^{t} \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau) w(u(\tau)) d\sigma ds.$$  

Then

$$u(t) \leq W^{-1} \left[ W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_7(\tau) d\sigma ds \right],$$

where $t_0 \leq t < b_1$, $W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_7(\tau) d\sigma ds \in \text{dom} W^{-1} \right\}.$$

We prepare the following corollary from Lemma 1.12 that is used in proving the theorem.
Corollary 1.13. Let \( u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+) \), \( w \in C((0, \infty)) \), and \( w(u) \) be nondecreasing in \( u \), \( u \leq w(u) \). Suppose that for some \( c > 0 \) and \( 0 \leq t_0 \leq t \),

\[
\begin{align*}
 u(t) &\leq c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) u(\tau) d\tau ds \\
 &\quad + \int_{t_0}^{t} \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau) w(u(\tau)) d\tau ds.
\end{align*}
\]

Then

\[
\begin{align*}
 u(t) &\leq W^{-1} \left[ W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau \\
 &\quad + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau) d\tau) ds \right],
\end{align*}
\]

where \( t_0 \leq t < b_1 \), \( W, W^{-1} \) are the same functions as in Lemma 1.8, and

\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau \\
 &\quad + \lambda_4(s) \int_{t_0}^{s} \lambda_5(\tau) d\tau) ds \in \text{dom}W^{-1} \right\}.
\]

2. Main Results

In this section, we investigate boundedness for solutions of nonlinear perturbed differential systems via \( t_\infty \)-similarity.

To obtain the bounded result, the following assumptions are needed:

(H1) \( f_x(t,0) \) is \( t_\infty \)-similar to \( f_x(t,x(t,t_0,x_0)) \) for \( t \geq t_0 \geq 0 \) and \( |x_0| \leq \delta \) for some constant \( \delta > 0 \).

(H2) The solution \( x = 0 \) of (1.1) is hS with the increasing function \( h \).

(H3) \( w(u) \) is nondecreasing in \( u \) such that \( u \leq w(u) \) and \( \frac{1}{v} w(u) \leq w\left(\frac{u}{v}\right) \) for some \( v > 0 \).

Theorem 2.1. Let \( a, b, c, k, w \in C(\mathbb{R}^+) \). Suppose that (H1), (H2), (H3), and \( g \) in (1.2) satisfies

\[
|g(t,y(t))| \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)w(|y(s)|) ds \tag{2.1}
\]

and

\[
|h(t,y(t),Ty(t))| \leq \int_{t_0}^{t} c(s)|y(s)| ds, \tag{2.2}
\]

\[
|g(t,y(t))| \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)w(|y(s)|) ds \tag{2.1}
\]

and

\[
|h(t,y(t),Ty(t))| \leq \int_{t_0}^{t} c(s)|y(s)| ds, \tag{2.2}
\]
where \( a, b, c, k \in L^1(\mathbb{R}^+) \). Then, any solution \( y(t) = y(t, t_0, y_0) \) of (1.2) is bounded on \([t_0, \infty)\) and it satisfies

\[
|y(t)| \leq h(t)W^{-1}\left[ W(c) + c_2 \int_{t_0}^{t} \int_{s}^{t} a(\tau) + c(\tau) + b(\tau) \int_{s}^{\tau} k(r)dr d\tau ds \right],
\]

where \( W, W^{-1} \) are the same functions as in Lemma 1.8, and

\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} \int_{s}^{t} a(\tau) + c(\tau) + b(\tau) \int_{s}^{\tau} k(r)dr d\tau ds \in \text{dom} W^{-1} \right\}.
\]

**Proof.** Let \( x(t) = x(t, t_0, y_0) \) and \( y(t) = y(t, t_0, y_0) \) be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution \( x = 0 \) of (1.1) is hS, the solution \( v = 0 \) of (1.3) is hS. Using (H1), by Theorem 1.7, the solution \( z = 0 \) of (1.4) is hS. Using the nonlinear variation of constants formula Lemma 1.5, Lemma 1.4, together with (2.1), and (2.2), we have

\[
|y(t)| \leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left( \int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds
\]

\[
\leq c_1|y_0|h(t_0)h^{-1} + \int_{t_0}^{t} c_2 h(t)h^{-1}\left( \int_{t_0}^{s} (a(\tau)w(|y(\tau)|)) + b(\tau) \int_{t_0}^{s} k(r)w(|y(\tau)|)dr d\tau \right) ds.
\]

Applying (H2) and (H3), we obtain

\[
|y(t)| \leq c_1|y_0|h(t_0)h^{-1} + \int_{t_0}^{t} c_2 h(t)\left( \int_{t_0}^{s} c(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau \right.
\]

\[
+ \left. \int_{t_0}^{s} (a(\tau)w\left( \frac{|y(\tau)|}{h(\tau)} \right)) + b(\tau) \int_{t_0}^{s} k(r)w\left( \frac{|y(\tau)|}{h(\tau)} \right)dr d\tau \right) ds.
\]

Set \( u(t) = |y(t)||h(t)^{-1} \). Then, by Corollary 1.11, we have

\[
|y(t)| \leq h(t)W^{-1}\left[ W(c) + c_2 \int_{t_0}^{t} \int_{s}^{t} a(\tau) + c(\tau) + b(\tau) \int_{s}^{\tau} k(r)dr d\tau ds \right],
\]

where \( c = c_1|y_0|h(t_0)^{-1} \). From the above estimation, we obtain the desired result. Thus, the theorem is proved. \(\square\)
Remark 2.2. Letting $c(s) = 0$ in Theorem 2.1, we obtain the similar result as that of Theorem 3.6 in [9].

Theorem 2.3. Let $a, b, c, k, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and $g$ in (1.2) satisfies
\[
\int_{t_0}^t |g(s, y(s))| \, ds \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^t k(s)|y(s)| \, ds, \quad t \geq t_0 \geq 0,
\]
and
\[
|h(t, y(t), Ty(t))| \leq \int_{t_0}^t c(s)w(|y(s)|) \, ds,
\]
where $a, b, c, k \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies
\[
|y(t)| \leq h(t)W^{-1}\left[W(c) + c_2 \int_{t_0}^t (a(s) + \int_{s}^{t} c(\tau) \, d\tau) + b(s) \int_{t_0}^{s} k(\tau) \, d\tau \right],
\]
where $t_0 \leq t < b_1$ and $W, W^{-1}$ are the same functions as in Lemma 1.8 and
\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (a(s) + \int_{s}^{t} c(\tau) \, d\tau) + b(s) \int_{t_0}^{\tau} k(\tau) \, d\tau \in \text{dom} W^{-1} \right\}.
\]

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Applying Lemma 1.4, Lemma 1.5, together with (2.3) and (2.4) we have
\[
|y(t)| \leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| (\int_{t_0}^s |g(\tau, y(\tau))| \, d\tau + |h(s, y(s), Ty(s))|) \, ds
\]
\[
\leq c_1 |y_0|h(t_0)^{-1} + \int_{t_0}^t c_2 h(t)h(s)^{-1} \left(a(s)w(|y(s)|) + b(s) \int_{t_0}^{s} k(\tau)|y(\tau)| \, d\tau + \int_{t_0}^{\tau} c(\tau)w(|y(\tau)|) \, d\tau \right) ds.
\]
It follows from (H2) and (H3), we obtain
\[
|y(t)| \leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) a(s) w\left(\frac{|y(s)|}{h(s)}\right) ds \\
+ \int_{t_0}^{t} c_2 h(t) \left( \int_{t_0}^{s} c(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau + b(s) \int_{t_0}^{s} k(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau \right) ds.
\]
Set \( u(t) = |y(t)||h(t)|^{-1} \). Now an application of Corollary 1.13 yields
\[
|y(t)| \leq h(t) W^{-1}\left[W(c) + c_2 \int_{t_0}^{t} (a(s) + \int_{t_0}^{s} c(\tau) d\tau + b(s) \int_{t_0}^{s} k(\tau) d\tau) ds,\right]
\]
where \( c = c_1 |y_0| h(t_0)^{-1} \). Thus, any solution \( y(t) = y(t, t_0, y_0) \) of (1.2) is bounded on \([t_0, \infty)\), and so the proof is complete.

**Remark 2.4.** Letting \( c(s) = 0 \) in Theorem 2.3, we obtain the same result as that of Theorem 3.4 in [10].

**Theorem 2.5.** Let \( a, b, c, k, q, w \in C(\mathbb{R}^+) \). Suppose that (H1), (H2), (H3), and \( g \) in (1.2) satisfies
\[
|g(t, y(t))| \leq a(t) w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)|y(s)| ds \quad (2.5)
\]
and
\[
|h(t, y(t), Ty(t))| \leq c(t) w(|y(t)|) + |Ty(t)|, |Ty(t)| \leq \int_{t_0}^{t} q(s)|y(s)| ds, \quad (2.6)
\]
where \( t \geq t_0 \geq 0 \) and \( a, b, c, k, q \in L^1(\mathbb{R}^+) \). Then, any solution \( y(t) = y(t, t_0, y_0) \) of (1.2) is bounded on \([t_0, \infty)\) and it satisfies
\[
|y(t)| \leq h(t) W^{-1}\left[W(c) + c_2 \int_{t_0}^{t} (c(s) + \int_{t_0}^{s} a(\tau) + b(s) \int_{t_0}^{s} k(\tau) d\tau) d\tau \\
+ c(s) \int_{t_0}^{s} q(\tau) d\tau] ds,\right]
\]
where \( W, W^{-1} \) are the same functions as in Lemma 1.8, and
\[
b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} (c(s) + \int_{t_0}^{s} a(\tau) + b(s) \int_{t_0}^{s} k(\tau) d\tau \\
+ c(s) \int_{t_0}^{s} q(\tau) d\tau] ds \in \text{dom}W^{-1} \right\},
\]
Proof. Let \( x(t) = x(t, t_0, y_0) \) and \( y(t) = y(t, t_0, y_0) \) be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution \( z = 0 \) of (1.4) is \( hS \). From Lemma 1.4, Lemma 1.5, together with (2.5), and (2.6), we have

\[
|y(t)| \leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\left[\int_{s}^{\tau} |g(\sigma, y(\sigma))|d\sigma + |h(s, y(s), Ty(s))|\right]ds
\]

\[
\leq c_1|y_0|h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)h(s)^{-1}\left(\int_{s}^{\tau} (a(\sigma)w(|y(\sigma)|)) + b(\tau)\int_{t_0}^{\tau} k(\rho)|y(\rho)|d\rho + c(s)\int_{t_0}^{\tau} q(\tau)|y(\tau)|d\tau\right)ds
\]

Using the assumptions (H2) and (H3), we obtain

\[
|y(t)| \leq c_1|y_0|h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)c(s)\left(\frac{|y(s)|}{h(s)}\right) + \int_{t_0}^{s} (a(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right) + b(\tau)\int_{t_0}^{\tau} k(\rho)\frac{|y(\rho)|}{h(\rho)}d\rho + c(s)\int_{t_0}^{\tau} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\right)d\tau.
\]

Set \( u(t) = |y(t)||h(t)|^{-1} \). Then, by Lemma 1.12, we have

\[
|y(t)| \leq h(t)W^{-1}\left[W(c) + c_2\int_{t_0}^{t} c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau)\int_{t_0}^{\tau} k(\rho)d\rho)\tau
\]

\[
+c(s)\int_{t_0}^{\tau} q(\tau)d\tau\right],
\]

where \( c = c_1|y_0|h(t_0)^{-1} \). The above estimation yields the desired result since the function \( h \) is bounded. Hence the proof is complete. \( \square \)

Remark 2.6. Letting \( a(t) = b(t) \) and \( c(t) = 0 \) in Theorem 2.5, we obtain the similar result as that of Theorem 3.7 in [10].

Theorem 2.7. Let \( a, b, k, q, w \in C(\mathbb{R}^+) \). Suppose that (H1), (H2), (H3), and \( g \) in (1.2) satisfies

\[
\int_{t_0}^{t} |g(s, y(s))|ds \leq a(t)(|y(t)| + \int_{t_0}^{t} k(s)w(|y(s)|)ds)
\]

and

\[
|h(t, y(t), Ty(t))| \leq b(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \leq \int_{t_0}^{t} q(s)w(|y(s)|)ds,
\]

\[
(2.7)
\]

(2.8)
where $t \geq t_0 \geq 0$ and $a, b, k, q \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1}\left[ W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s)) \int_{t_0}^{s} k(\tau)d\tau + b(s) \int_{t_0}^{s} q(\tau)d\tau ds \right],$$

where $t_0 \leq t < b_1$ and $W, W^{-1}$ are the same functions as in Lemma 1.8 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s)) \int_{t_0}^{s} k(\tau)d\tau + b(s) \int_{t_0}^{s} q(\tau)d\tau ds \in \text{dom}W^{-1} \right\}.$$

**Proof.** Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. By Lemma 1.4, Lemma 1.5, together with (2.7), and (2.8), we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| (\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))|) ds$$

$$\leq c_1 |y_0|h(t) h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) h(s)^{-1} \left( a(s)|y(s)| + b(s)w(|y(s)|) \right.$$ 

$$+ a(s) \int_{t_0}^{s} k(\tau)w(|y(\tau)|) d\tau + b(s) \int_{t_0}^{s} q(\tau)w(|y(\tau)|) d\tau \big) ds.$$

By the assumptions (H2) and (H3), we obtain

$$|y(t)| \leq c_1 |y_0|h(t) h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) \left( a(s) \frac{|y(s)|}{h(s)} + b(s)w\left( \frac{|y(s)|}{h(s)} \right) \right.$$ 

$$+ b(s) \int_{t_0}^{s} q(\tau)w\left( \frac{|y(\tau)|}{h(\tau)} \right) d\tau + a(s) \int_{t_0}^{s} k(\tau)w\left( \frac{|y(\tau)|}{h(\tau)} \right) d\tau \big) ds.$$

Set $u(t) = |y(t)||h(t)^{-1}$. Now an application of Lemma 1.9 yields

$$|y(t)| \leq h(t)W^{-1}\left[ W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s)) \int_{t_0}^{s} k(\tau)d\tau + b(s) \int_{t_0}^{s} q(\tau)d\tau ds \right].$$
Thus any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. This completes the proof.

Remark 2.8. Letting $b(t) = 0$ in Theorem 2.7, we obtain the similar result as that of Theorem 3.3 in [9].

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