Korean J. Math. **25** (2017), No. 2, pp. 147–162
https://doi.org/10.11568/kjm.2017.25.2.147

REGULARIZED PENALTY METHOD FOR NON-STATIONARY SET VALUED EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. In this research works, we consider the general regularized penalty method for non-stationary set valued equilibrium problem in a Banach space. We define weak coercivity conditions and show that the weak and strong convergence problems of the regularized penalty method.

1. Introduction

Regularized penalty methods give basic techniques for solutions of various type of nonlinear problems *via* substitution of an initial problem with a sequence of auxiliary problems with suitable assumptions. It is well-known from the optimization theory that they are mutually dual in the sense that the penalty method applied to the primal problem is equivalent to the regularization method for its dual and vice versa; *see, e.g.,* [[21], Chap. X, §3]. In principle, they can be applied within a unique algorithmic scheme for nonlinear constrained optimization, hierarchical optimizations, cooperative games theories, Hadamard manifolds, variational inequalities (VIs), and equilibrium problems (EPs);

Received November 30, 2016. Revised February 11, 2017. Accepted February 28, 2017.

2010 Mathematics Subject Classification: 49J40, 47H09, 47J20.

Key words and phrases: Non-stationary set valued equilibrium problems, Set valued mappings, Non-monotone bi-functions, General regularized penalty method, Coercivity conditions, Strong convergence, Hausdorff metric, Banach spaces.

see, e.g., [1, 17, 18, 20, 22, 24, 27, 29, 30] and references therein. However, such a combined method usually requires restrictive concordance rules for penalty and regularization parameters. Besides, many existing penalty and regularization methods applied to VIs and EPs are based on (generalized) monotonicity conditions for convergence; *see, e.g.,* [1,8,9,19,23,26,30]. Rather recently, the convergence of regularization methods was established under weak conditions that are sufficient for existence of solutions of EPs instead of (generalized) monotonicity assumptions in a finite-dimensional space setting; *see* [12, 14] and references therein.
In this research works, we deal to the convergence analysis of the general regularized penalty method for SVEPs in reflexive Banach space without any monotonicity conditions and concordance rules for penalty and regularization parameters. To this end, we first obtain an existence result for SVEPs, which can be viewed as a modification of those in [1, 9, 25, 26, 28, 30]. Then, we define coercivity conditions that provide weak and strong convergence analysis of the regularized penalty method.

2. Preliminaries

We first recall some auxiliary properties. Let X be a nonempty subset of a Banach space E. Recall that a function $\phi : X \longrightarrow \mathbb{R}$ is said to be

(a) quasiconvex on a convex set $K \subseteq X$, iff
$$\phi(\alpha x + (1-\alpha)y) \leq \max\{\phi(x), \phi(y)\}, \forall x, y \in K \text{ and } \forall \alpha \in [0, 1];$$

(b) explicitly quasiconvex on a convex set $K \subseteq X$, iff it is quasiconvex and it holds that
$$\phi(\alpha x+(1-\alpha)y) < \max\{\phi(x), \phi(y)\}, \forall x, y \in K \text{ with } \phi(x) \neq \phi(y) \text{ and } \forall \alpha \in]0, 1[;$$

(c) convex on a convex set $K \subseteq X$, iff
$$\phi(\alpha x+(1-\alpha)y) \leq \alpha\phi(x)+(1-\alpha)\phi(y), \forall x, y \in K \text{ and } \forall \alpha \in [0, 1];$$

(d) strictly convex on a convex set $K \subseteq X$, iff
$$\phi(\alpha x + (1-\alpha)y) < \alpha\phi(x) + (1-\alpha)\phi(y), \forall x, y \in K \text{ with } x \neq y \text{ and } \forall \alpha \in]0, 1[;$$

(e) uniformly convex on a convex set $K \subseteq X$, iff there exists a continuous and increasing function $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ with $\theta(0) = 0$ such that

$$\phi(\alpha x + (1-\alpha)y) \leq \alpha\phi(x) + (1-\alpha)\phi(y) \\ -\alpha(1-\alpha)\theta(\|x-y\|), \forall x, y \in K \text{ and } \forall \alpha \in [0,1];$$

(f) strongly convex with constant $\varkappa > 0$ on a convex set $K \subseteq X$, iff

$$\phi(\alpha x + (1-\alpha)y) \leq \alpha\phi(x) + (1-\alpha)\phi(y) \\ -0.5\varkappa\alpha(1-\alpha)\|x-y\|^2, \forall x, y \in K \text{ and } \forall \alpha \in [0,1].$$

Besides, we say that a function $\phi : X \longrightarrow \mathbb{R}$ is

(g) coercive iff $\phi(x) \longrightarrow +\infty$ as $\|x\| \longrightarrow \infty$;

(h) weakly coercive with respect to a set $K \subseteq E$ iff there exists a number γ such that the set $K_\gamma = \{x \in K \mid \phi(x) \leq \gamma\}$ is nonempty and bounded.

Clearly, we have

$$(f) \Rightarrow (e) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a), \text{ and } (e) \Rightarrow (g) \Rightarrow (h),$$

but converse is not true in general.

Let $CB(X)$ be the family of all nonempty closed bounded subsets of X and the Hausdorff metric defined by

$$\mathfrak{H}(A, B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x,y) \, , \, \sup_{y\in B}\inf_{x\in A} d(x,y)\}, \ \forall A, B \in CB(X).$$

A function $\phi : X \longrightarrow \mathbb{R}$ is said to be upper (lower) semicontinuous at a point $z \in X$, iff, for each sequence $\{x^k\} \longrightarrow z, x^k \in X$, it holds that

$$\limsup_{k\longrightarrow\infty} \phi(x^k) \leq \phi(z) \ (\liminf_{k\longrightarrow\infty} \phi(x^k) \geq \phi(z)).$$

Similarly, a function $\phi : X \longrightarrow \mathbb{R}$ is said to be weakly upper (lower) semicontinuous at a point $z \in X$, iff, for each sequence $\{x^k\} \xrightarrow{w} z, \ x^k \in X$, it holds that

$$\limsup_{k\longrightarrow\infty} \phi(x^k) \leq \phi(z) \ (\liminf_{k\longrightarrow\infty} \phi(x^k) \geq \phi(z)).$$

Here and below $\{x^k\} \longrightarrow z$ ($\{x^k\} \xrightarrow{w} z$) denotes the strong (weak) convergence of $\{x^k\}$ to z. We say that any of the above properties holds on a set $K \subseteq X$, iff it holds at any point of K. Recall that a set X is said to be

(i) weakly sequentially closed, iff for each sequence $\{x^k\} \xrightarrow{w} z,\ x^k \in X$, it holds that $z \in X$;
(ii) weakly sequentially compact, iff each sequence $\{x^k\} \subset X$ contains a subsequence converging weakly to a point of X.

Next, we give the existence result for EPs on compact sets known as the Ky Fan inequality; *see* [6].

PROPOSITION 2.1. *If X is a nonempty, convex, and compact subset of a real topological vector space, $F : X \times X \longrightarrow \mathbb{R}$ is an equilibrium bi-function (i.e., $F(x,x) = 0$ for each $x \in X$), $F(\cdot, y)$ is upper semicontinuous for each $y \in X$, and $F(x, \cdot)$ is quasiconvex for each $x \in X$; then there exists a point $x^* \in X$ such that*

$$F(x^*, y) \geq 0,\ \forall y \in X.$$

For deriving similar results on not necessarily compact sets, one needs certain coercivity conditions.

3. Existence of Solutions

Let W be a nonempty set. Let $\phi : W \times W \longrightarrow \mathbb{R}$ be a set valued equilibrium bi-function and let $G : W \longrightarrow \Pi(W)$ be a set valued mapping where $\Pi(W)$ denotes the family of all nonempty subsets of a set W. Then, we can define the following set valued equilibrium problems (SVEPs): find a point $x^* \in W$ such that there exists $g^* \in G(x^*)$ and

$$\Phi(g^*, y) \geq 0,\ \forall y \in W. \tag{3.1}$$

This formulation of SVEPs gives a suitable common format for investigation of various nonlinear problems. In particular, it contains optimization, fixed point, variational inequality, saddle point, and noncooperative game equilibrium problems; *see, e.g.,* [2, 5, 10, 11, 19, 25] and references therein.
A set valued mapping $G : X \longrightarrow \Pi(X)$ is said to be

(A1) upper semicontinuous on X if for each point $z \in X$ and for each open set U such that $G(z) \subset U$ there is neighbourhood W of z such that $G(x) \subseteq U$ whenever $x \in X \cap W$;
(A2) a K(Kakutani)-mapping on X if it is upper semicontinuous on X and has nonempty convex and compact values.

In this section, we will consider SVEPs (3.1) under the following basic assumptions.
(B1) W is a nonempty, convex, and closed subset of a reflexive Banach space E, $\Phi : W \times W \longrightarrow \mathbb{R}$ is an equilibrium bi-function, $\phi(\cdot, y)$ is weakly upper semicontinuous for each $y \in W$, and $\Phi(x, \cdot)$ is explicitly quasiconvex for each $x \in W$ and convex function $G : X \longrightarrow \Pi(W)$ is a K-mapping.
By a simple specialization of Proposition 2.1, we obtain an existence result for SVEPs (3.1) on bounded sets.

Proposition 3.1. *If* (A1)-(A2), (B1) *holds and* W *is bounded, then SVEPs* (3.1) *has a solution.*

Now we turn to the unbounded case. Then, we should utilize a suitable coercivity condition; *see, e.g.,* [4,5,12,14,15] and references therein. We now modify one of the weakest conditions from [12, 14, 15]. For a function $\mu : E \longrightarrow \mathbb{R}$ and a number r, we define the level sets

$$B_r = \{x \in E \mid \mu(x) \le r\}$$

and

$$L_r = \{x \in E \mid \mu(x) < r\}.$$

(C1) There exists a lower semicontinuous and convex function $\mu : E \longrightarrow \mathbb{R}$, which is weakly coercive with respect to the set W, and a number r, such that, for any point $x \in W \backslash B_r$, there exists $g \in G(x)$ with

$$\inf \Phi(g, \mu, x) \ge 0, \text{ for } x \in W(\mu, r),$$

then there is a point $z \in W$ with

$$\begin{aligned} \min\{\Phi(g, z), \mu(z) - \mu(x)\} < 0, \forall g \in G(x) \ \text{ and} \\ \max\{\Phi(g, z), \mu(z) - \mu(x)\} \le 0, \forall g \in G(x). \end{aligned} \tag{3.2}$$

We recall that, for any lower semicontinuous and convex function $\mu : E \longrightarrow \mathbb{R}$, its weak coercivity with respect to the set W is equivalent to boundedness of any set $W_\rho = B_\rho \bigcap W$ for each ρ. Set

$$r(m) = \inf_{x \in W} \mu(x).$$

Lemma 3.2. *If* (B1) *and* (C1) *hold, then there exists* $\bar{x} \in W$ *such that*

$$\mu(\bar{x}) = r(m).$$

Proof. The function μ is weakly lower semicontinuous; hence the set W_ρ is weakly sequentially compact. Moreover, there exists ρ' such that it is nonempty for all $\rho \geq \rho'$. Due to the Weierstrass theorem, there exists $\bar{x} \in W_\rho$ such that

$$r(m) = \inf_{x \in W_\rho} \mu(x) = \mu(\bar{x}).$$

□

We show that condition (C1) is well defined.

LEMMA 3.3. *If* (A1)-(A2), (B1) *and* (C1) *hold, then W_r is nonempty.*

Proof. From Lemma 3.2, the set $W_{r(m)}$ is nonempty, convex, closed, and bounded. Applying Proposition 3.1 with $W = W_{r(m)}$, we see that there exists $\bar{x} \in W_{r(m)}$ and $\bar{g} \in G(\bar{x})$ such that

$$\phi(\bar{g}, y) \geq 0, \forall y \in W_{r(m)}.$$

If $W_r = \phi$, then $r < r(m)$ and $\bar{x} \notin W_r$. Using now (3.2) with $x = \bar{x}$ and noting that $\mu(\bar{x}) \leq \mu(x)$ for every $x \in W$ by definition, we obtain $\mu(\bar{x}) \leq \mu(z)$ in (C1), hence $\mu(\bar{x}) = \mu(z)$ and $\Phi(\bar{g}, z) < 0,$ for $g \in G(\bar{x})$, which is a contradiction. □

Next basic property of solutions of SVEPs on reduced sets was given in [12], Proposition 2.4 and [16], Proposition 3.2, but we give its proof here for the convenience of the reader.

PROPOSITION 3.4. *Suppose that $\mu : E \longrightarrow \mathbb{R}$ is a lower semicontinuous and convex function,* (A1)-(A2), (B1) *hold, and, for some ρ, there exist $x^\rho \in W_\rho$ and $g^\rho \in G(x^\rho)$ such that*

$$\Phi(g^\rho, y) \geq 0, \forall y \in W_\rho, \tag{3.3}$$

and $w \in L_\rho \bigcap W$ such that

$$\Phi(g^\rho, w) \leq 0, \ \forall g^\rho \in G(x^\rho).$$

Then, x^ρ is a solution of SVEPs (3.1).

Proof. Set $\phi(x) = \Phi(g^\rho, x)$, for $g^\rho \in G(x^\rho)$ then $\phi(x^\rho) = \phi(w) = 0$; moreover, $w \in W_\rho$, hence x^ρ and w are minimizers for the function ϕ over W_ρ. Suppose that there exists a point $x' \in W \backslash B_\rho$ such that $\phi(x') < \phi(w)$, and set $x(\alpha) = \alpha x' + (1 - \alpha)w$. Clearly, $x(\alpha) \in W$ for each $\alpha \in]0, 1[$. By convexity of μ, we have

$$\mu[x(\alpha)] \leq \alpha\mu(x') + (1 - \alpha)\mu(w) = \mu(w) + \alpha[\mu(x') - \mu(w)] \leq \rho$$

for $\alpha > 0$ small enough. Then, $x(\alpha) \in W_\rho$ for $\alpha > 0$ small enough, but, due to the explicit quasiconvexity of $\Phi(x, \cdot)$, we have

$$\phi[x(\alpha)] < \max\{\phi(x'), \phi(w)\} = \phi(w),$$

which is a contradiction. Therefore,

$$\phi(x^\rho) = \phi(w) \le \phi(y), \forall y \in W,$$

i.e., x^ρ solves SVEPs (3.1). □

First we introduce the following approximation assumptions:

(i) there exists a sequence of nonempty convex closed set $\{D_\rho\}$ which is Mosco convergent to the set D;

(ii) there exists a sequence of ρ-mappings $G_\rho : D_\rho \longrightarrow \Pi(D_\rho), \rho = 1, 2, \cdots$ such that the relations $\{y^\rho\} \longrightarrow \bar{y}$, $y^\rho \in D_\rho$ and $g^\rho \in G_\rho(y^\rho)$ imply $\{g^\rho\}$ is bounded and $\{g^\rho\} \longrightarrow \bar{g}$ yields $\bar{y} \in G(\bar{y})$.

We are now ready to establish the general existence result.

Theorem 3.5. *If* (A1)-(A2), (B1) *and* (C1) *are fulfilled, then SVEPs* (3.1) *has a solution.*

Proof. Since (C1) holds, we can take any $\rho > r$, then the set W_ρ is nonempty, convex, closed, and bounded. From Proposition 3.1, we obtain that there exists a solution x^ρ and $g^\rho \in G(x^\rho)$ of problem (3.3). If $x^\rho \in L_\rho$, we set $w = x^\rho$. Otherwise, we have $\mu(x^\rho) = \rho$ and $x^\rho \notin B_r$. From (C1) with $x = x^\rho$ and $g = g^\rho$, it now follows that there exists a point $z \in W$ such that $\mu(z) < \mu(x^\rho) = \rho$ and $\Phi(g^\rho, z) = 0$ where for $g^\rho \in G(x^\rho)$, due to (3.2). Hence, we can set $w = z$. The result now follows from Proposition 3.4. □

This existence result appears to be suitable for application to penalty methods.

4. Weak Convergence

In this section, we will consider SVEPs (3.1) under the following basic assumptions.

(B2) W is a nonempty set of the form

$$W = D \bigcap V, \tag{4.1}$$

where D and V are convex and closed sets in a reflexive Banach space E and $G : D \longrightarrow \Pi(D)$ is a set valued mapping, $\Phi(\cdot, y)$ is weakly upper

semicontinuous for each $y \in D, \Phi(x,\cdot)$ is convex for each $x \in D$, and $\Phi : D \times D \longrightarrow \mathbb{R}$ is an equilibrium bi-function.
In this partition, D stands for a "simply" constrained set, whereas V usually include "functional" constraints. For this reason, we will utilize a general penalty function $P : E \longrightarrow \mathbb{R}$ just for the set V, *i.e.,*

$$P(x) = 0, \text{ if } x \in V \text{ and } P(x) > 0, \text{ if } x \notin V,$$

which is supposed to be convex and lower semicontinuous on D. Then

$$\Phi_\tau(g, y) = \Phi(g, y) + \tau[P(y) - P(x)], \text{ for } g \in G(x),$$

determines the penalized bi-function on the set D.

REMARK 4.1. *In* [[13], Theorem 1], *a convergence of the usual penalty method for EPs was established in a finite-dimensional space setting under the following coercivity condition.*

There exists a convex function $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$, which is weakly coercive with respect to the set D and lower semicontinuous on D, and a number r such that, for any point $x \in D \backslash B_r$, there is a point $\tilde{y} \in L_r \bigcap D$ such that $P(\tilde{y}) \leq P(x)$ and

$$\Phi(x, \tilde{y}) < 0.$$

Taking into account the result of Theorem 3.5, we conclude that all the assertions of Theorem 1 of [13] remain true if we replace the above condition with the following weaker one.
There exists a convex function $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$, which is weakly coercive with respect to the set D and lower semicontinuous on D, and a number r such that, for any number $\tau > 0$ and for any point $x \in D \backslash B_r$, there is a point $\tilde{y} \in D$ such that $\mu(\tilde{y}) \leq \mu(x)$ and

$$\Phi(g, \tilde{y}) + \tau[P(\tilde{y}) - P(x)] < 0, \text{ for } g \in G(x)$$

where $G : D \longrightarrow \Pi(D)$ is a set valued mapping.
The proof is obtained along the same lines as in [13].
However, we intend to apply the regularized penalty method. Together with (B2) we consider the following coercivity condition.
(C2) There exists a lower semicontinuous and convex function $\mu : E \longrightarrow \mathbb{R}$, which is weakly coercive with respect to the set D, $G : D \longrightarrow \Pi(D)$ a set valued mapping, an equilibrium bi-function $\Psi : D \times D \longrightarrow \mathbb{R}$ such that $\Psi(\cdot, y)$ is weakly upper semicontinuous for each $y \in D, \Psi(x,\cdot)$ is convex for each $x \in D$ and a number r such that, for any number

$\tau > 0$ and for any point $x \in D \backslash B_r$, there is a point $z \in D$ such that $\mu(z) \le \mu(x)$ and

$$\begin{aligned} \min\{\Phi_\tau(g,z), \Psi(g,z)\} < 0, \forall g \in G(x) \ \text{ and} \\ \max\{\Phi_\tau(g,z), \Psi(g,z)\} \le 0, \forall g \in G(x). \end{aligned} \tag{4.2}$$

We note that (C2) is a modification of condition (**S2**) in [15]. If we choose

$$\Psi(x,y) = \mu(y) - \mu(x), \forall x, y \in D,$$

then (C2) modifies slightly condition (C1) for the penalized SVEPs. Furthermore, we can take

$$\Psi(x,y) \ge \mu(y) - \mu(x), \forall x, y \in D,$$

then (4.2) implies $\mu(z) \le \mu(x)$.
Given numbers $\tau > 0$ and $\varepsilon > 0$, now we consider the perturbed problem of finding a point $x(\tau,\varepsilon) \in D$ and $g(\tau,\varepsilon) \in G(x(\tau,\varepsilon))$ such that

$$\Phi(g(\tau,\varepsilon), y) + \tau[P(y) - P(x(\tau,\varepsilon))] + \varepsilon\Psi(g(\tau,\varepsilon), y) \ge 0, \forall y \in D. \tag{4.3}$$

For brevity, set

$$\Phi_{\tau,\varepsilon}(g,y) = \Phi(x,y) + \tau[P(y) - P(x)] + \varepsilon\Psi(g,y), \forall g \in G(x).$$

Our aim is to prove that the trajectory $\{x(\tau,\varepsilon)\}$ approximates a solution of SVEPs (3.1),(4.1) as $\tau \longrightarrow +\infty$ and $\varepsilon \longrightarrow 0$. Again, for brevity, we set $x^k = x(\tau_k, \varepsilon_k)$. This means that the point x^k is an arbitrary solution of SVEPs (4.3) with $\tau = \tau_k$ and $\varepsilon = \varepsilon_k$. The rules for choice of the sequences $\{\tau_k\}$ and $\{\varepsilon_k\}$ appear very simple.

Theorem 4.2. *Suppose that* (A1)-(A2), (B2) *and* (C2) *are fulfilled, the sequences* $\{\tau_k\}$ *and* $\{\varepsilon_k\}$ *satisfy*

$$\{\tau_k\} \nearrow +\infty, \ \{\varepsilon_k\} \searrow 0. \tag{4.4}$$

Then

(i) *SVEPs* (3.1), (4.1) *has a solution;*
(ii) *SVEPs* (4.3) *has a solution for each pair* $\tau > 0$ *and* $\varepsilon > 0$ *and all these solutions belong to* $B_r \bigcap D$;
(iii) *Each sequence* $\{x^k\}$ *of solutions of SVEPs* (4.3) *has weak limit points, and all these weak limit points belong to* $B_r \bigcap W$ *and are solutions of SVEPs* (3.1), (4.1).

Proof. We first show that, for any $\tau > 0$ and $\varepsilon > 0$, (C1) is true with $\Phi = \Phi_{\tau,\varepsilon}$ and $W = D$. Take any $x \in D \backslash B_r$, then there is $z \in D, \mu(z) \leq \mu(x)$ such that (4.2) holds and we have

$$\Phi_{\tau,\varepsilon}(g,z) = \Phi(g,z) + \tau[P(z) - P(x)] + \varepsilon\Psi(g,z) < 0.$$

Since $\Phi_{\tau,\varepsilon}(\cdot, y)$ is weakly upper semicontinuous for each $y \in D$, and $\Phi_{\tau,\varepsilon}(x,\cdot)$ is convex for each $x \in D$, SVEPs (4.3) has a solution for any $\tau > 0$ and $\varepsilon > 0$ due to Theorem 3.5 with $\Phi = \Phi_{\tau,\varepsilon}$ and $W = D$. It also follows that $x(\tau,\varepsilon) \in B_r \bigcap D$. Hence, assertion (ii) is true. By (ii), the sequence $\{x^k\}$ exists and is bounded. Therefore, it has weak limit points. Since $B_r \bigcap D$ is convex and closed, and all these weak limit points must belong to $Br \bigcap D$. Let x' be an arbitrary weak limit point of $\{x^k\}$, *i.e.,* $\{x^{k_s}\} \xrightarrow{w} x'$ and also $\{g^{k_s}\} \xrightarrow{w} g'$. Then, by assumption,

$$0 \leq P(x^{k_s}) \leq \tau_{k_s}^{-1}\Phi(g^{k_s}, y) + P(y) + \varepsilon_{k_s}\tau_{k_s}^{-1}\Psi(g^{k_s}, y), \forall y \in D, g^{k_s} \in G(x^{k_s}).$$

Taking $y \in W$ and using (4.4), we obtain

$$0 \leq P(x') \leq \liminf_{s\longrightarrow\infty} P(x^{k_s}) \leq \limsup_{s\longrightarrow\infty}[\tau_{k_s}^{-1}\Phi(g^{k_s}, y) + \varepsilon_{k_s}\tau_{k_s}^{-1}\Psi(g^{k_s}, y)]$$

$$\leq \limsup_{s\longrightarrow\infty}[\tau_{k_s}^{-1}\Phi(g^{k_s}, y)] + \limsup_{s\longrightarrow\infty}[\varepsilon_{k_s}\tau_{k_s}^{-1}\Psi(g^{k_s}, y)] \leq 0, \forall g^{k_s} \in G(x^{k_s})$$

i.e., $x' \in V$ and $x' \in W$. This means that all the weak limit points of $\{x^k\}$ belong to $B_r \bigcap W$. It follows that

$$0 \leq \tau_{k_s} P(x^{k_s}) \leq \Phi(g^{k_s}, x') + \tau_{k_s} P(x') + \varepsilon_{k_s}\Psi(g^{k_s}, x')$$

$$= \Phi(g^{k_s}, x') + \varepsilon_{k_s}\Psi(g^{k_s}, x'), \forall g^{k_s} \in G(x^{k_s})$$

hence

$$0 \leq \liminf_{s\longrightarrow\infty}[\tau_{k_s} P(x^{k_s})] \leq \limsup_{s\longrightarrow\infty}[\tau_{k_s} P(x^{k_s})]$$

$$\leq \limsup_{s\longrightarrow\infty} \Phi(g^{k_s}, x') + \limsup_{s\longrightarrow\infty}[\varepsilon_{k_s}\Psi(g^{k_s}, x')] \leq \Phi(x', x') = 0.$$

Therefore,

$$\lim_{s\longrightarrow\infty}[\tau_{k_s} P(x^{k_s})] = 0.$$

However, for each $y \in W$, for some $g^{k_s} \in G(x^{k_s})$, we have

$$\Phi(g^{k_s}, y) - \tau_{k_s} P(x^{k_s}) + \varepsilon_{k_s}\Psi(g^{k_s}, y)$$

$$= \Phi(g^{k_s}, y) + \tau_{k_s}[P(y) - P(x^{k_s})] + \varepsilon_{k_s}\Psi(g^{k_s}, y) \geq 0, \ \forall g^{k_s} \in G(x^{k_s}).$$

By (A2) we suppose that $\{g^{k_s}\} \longrightarrow g'$ without loss of generality, then $g' \in G(x')$. It follows that

$$\Phi(g', y) \geq \limsup_{s \longrightarrow +\infty} \Phi(g^{k_s}, y) \geq \liminf_{s \longrightarrow +\infty} [\tau_{k_s} P(x^{k_s})] + \liminf_{s \longrightarrow +\infty} [-\varepsilon_{k_s} \Psi(g^{k_s}, y)]$$

$$\geq \lim_{s \longrightarrow +\infty} [\tau_{k_s} P(x^{k_s})] - \limsup_{s \longrightarrow +\infty} [\varepsilon_{k_s} \Psi(g^{k_s}, y)] \geq 0, \ \forall g' \in G(x'), g^{k_s} \in G(x^{k_s}).$$

Therefore, x' and g' solve SVEPs (3.1), (4.1) and assertion (iii) is true. Since x' exists, assertion (i) is also true. The proof is complete. □

It should be noted that SVEPs (3.1), (4.1) can have also solutions out of B_r under the conditions of Theorem 4.2; moreover, the solution set may be unbounded in general.

5. Strong Convergence

Denote by W^* the solution set of SVEPs (3.1), (4.1) and set

$$r(n) = \inf_{x \in W^*} \mu(x)$$

and

$$W_n^* = \{\bar{x} \in W^* \mid \mu(\bar{x}) = r(n)\}.$$

Note that the assumptions of Theorem 4.2 imply (C1) for SVEPs (4.3) and the nonemptiness of $B_r \bigcap W$; *see* also Lemma 3.3. Now we discuss conditions for the nonemptiness of W_n^*. First of all we note that assumptions (B2) and (C2) imply the nonemptiness of W^*; moreover, W^* is weakly closed since $\Phi(\cdot, y)$ is weakly upper semicontinuous. From (B2) and (C2) we also obtain that $B_r \bigcap W$ is weakly sequentially compact with $r \geq r(n)$ and so is $B_r \bigcap W^*$, which is also nonempty due to Theorem 4.2 (iii). Since μ is weakly lower semicontinuous, it attains its minimal value at $B_r \bigcap W^*$, hence $W_n^* \neq \emptyset$. This assertion remains true if (B2) holds, $W^* \neq \emptyset$, and μ is convex, lower semicontinuous, and weakly coercive with respect to the set W. The choice of r in (C2) is rather arbitrary. By fixing this parameter at $r(n)$, we can attain the strong convergence.

Theorem 5.1. *Suppose that* (A1)-(A2),(B2) *and* (C2) *with* $r = r(n)$ *are fulfilled, the sequences* $\{\tau_k\}$ *and* $\{\varepsilon_k\}$ *satisfy* (4.4). *Then*

(i) *SVEPs* (4.3) *has a solution for each pair* $\tau > 0$ *and* $\varepsilon > 0$ *and all these solutions belong to* $B_{r(n)} \bigcap D$;

(ii) *Each sequence* $\{x^k\}$ *and* $\{g^k\}$ *of solutions of SVEPs* (4.3) *has weak limit points and all these weak limit points belong to* W_n^**, besides,*

$$\lim_{k\longrightarrow+\infty}\mu(x^k)=r(n); \tag{5.1}$$

(iii) *If, additionally,* W^* *is convex and* μ *is strictly convex, the sequence* $\{x^k\}$ *converges weakly to a point* z_n^* *such that* $W_n^*=\{z_n^*\}$.

Proof. Assertion (i) follows from Theorem 4.2 (ii). Besides, from Theorem 4.2 (iii) we obtain that all the weak limit points of $\{x^k\}$ belong to W_n^*. It now follows from properties of μ that

$$r(n)\le\liminf_{k\longrightarrow\infty}\mu(x^k)\le\limsup_{k\longrightarrow\infty}\mu(x^k)\le r(n),$$

i.e., (5.1) holds true. Next, in case (iii), W_n^* must be a singleton $\{z_n^*\}$ that implies weak convergence of $\{x^k\}\longrightarrow z_n^*$. Similarly, $\{g^k\}\longrightarrow g_n^*$ with $g_n^*\in G(z_n^*)$. □

Assertions (ii) and (iii) of Theorem 5.1 remain true when there exists a number $\tau'>0$ such that (C2) holds if $0<\tau<\tau'$ and (C2) with $r=r(n)$ holds if $\tau\ge\tau'$. In order to derive strong convergence, we impose additional conditions on μ and rewrite the modified condition (C2) for convenience of the reader.
(C2′) There exist set valued mapping $G:D\longrightarrow\Pi(D)$ and an equilibrium bi-function $\Psi:D\times D\longrightarrow\mathbb{R}$ such that $\Psi(\cdot,y)$ is weakly upper semicontinuous for each $y\in D,\Psi(x,\cdot)$ is convex for each $x\in D$ and a lower semicontinuous and uniformly convex function $\mu:E\longrightarrow\mathbb{R}$, such that, for any number $\tau>0$ and for any point $x\in D\backslash B_{r(n)}$, there is a point $z\in D$ such that $\mu(z)\le\mu(x)$ and (4.2) holds true.

Remark 5.2. *The condition* (C2′) *is weaker essentially than the usual (generalized) monotonicity assumptions used for providing strong convergence of existing regularization and regularized penalty methods; see, e.g., [1,3,7,8]. To see this, let us consider a simplified version of* (C2′) *applied for the pure regularization method. Then, we suppose that* $V=E$ *and* $P(x)\equiv 0$*. Moreover, we set* $\Psi(x,y)=\mu(y)-\mu(x)$ *for simplicity. This gives the following coercivity condition.*

There exists a lower semicontinuous and uniformly convex function $\mu:E\longrightarrow\mathbb{R}$ such that, for any point $x\in W\backslash B_{r(n)}$, there is a point $z\in D$ such that (3.3) holds.
At the same time, the weakest known sufficient condition was $W^*=$

$W^e \neq \emptyset$, where W^e is the solution set of the minimax dual SVEPs: find $y^* \in W$ such that

$$\Phi(g, y^*) \leq 0, \ \forall x \in W, g \in G(x).$$

It holds if $W^* \neq \emptyset$ and Φ is pseudomonotone. Clearly, they imply the above coercivity condition. In the proof we utilize partially the technique from [[30], Chap. IV, 4, Lemma 2], which was applied to optimization problems. For a point x and a set X, we define the distance value

$$d(x, X) = \inf_{y \in X} \|x - y\|.$$

THEOREM 5.3. *Suppose that* (A1)-(A2), (B2) *and* (C2′) *are fulfilled, the sequences* $\{\tau_k\}$ *and* $\{\varepsilon_k\}$ *satisfy* (4.4). *Then*

(i) *SVEPs* (4.3) *has a solution for each pair* $\tau > 0$ *and* $\varepsilon > 0$ *and all these solutions belong to* $B_{r(n)} \bigcap D$;

(ii) *Each sequence* $\{x^k\}$ *of solutions of SVEPs* (4.3) *has strong limit points and all these limit points belong to* W_n^*, *besides,* (5.1) *holds and*

$$\lim_{k \longrightarrow +\infty} d(x^k, W_n^*) = 0; \tag{5.2}$$

(iii) *If, additionally,* W^* *is convex, the sequence* $\{x^k\}$ *converges strongly to a point* z_n^* *such that* $W_n^* = \{z_n^*\}$;

(iv) *If, additionally,* W^* *is convex, the sequence* $\{g^k\}$ *converges strongly to a point* g_n^* *such that* $W_n^* = \{g_n^*\}$ *and* $g_n^* \in G(z_n^*)$.

Proof. Assertion (i) again follows from Theorem 4.2 (ii). Besides, from Theorem 5.1 (ii) we obtain that the sequence $\{x^k\}$ has weak limit points and all the weak limit points of $\{x^k\}$ belong to W_n^* and that (5.2) holds. Take an arbitrary subsequence $\{x^{k_s}\}$, without any loss of generality suppose that $\{x^{k_s}\} \xrightarrow{w} \bar{z} \in W_n^*$. By the uniform convexity of μ we have

$$\frac{1}{4}\theta(\|x^{k_s} - \bar{z}\|) \leq \frac{1}{2}\mu(x^{k_s}) + \frac{1}{2}\mu(\bar{z}) - \mu((x^{k_s} + \bar{z})/2).$$

Setting $w^{k_s} = (x^{k_s} + \bar{z})/2$, we note that $\{w^{k_s}\} \xrightarrow{w} \bar{z}$. Hence,

$$\liminf_{s \longrightarrow \infty} \mu(w^{k_s}) \geq \mu(\bar{z}) = r(n),$$

and

$$0 \leq \frac{1}{4} \liminf_{s \longrightarrow \infty} \theta(\|x^{k_s} - \bar{z}\|) \leq \frac{1}{4} \limsup_{s \longrightarrow \infty} \theta(\|x^{k_s} - \bar{z}\|)$$

$$\leq \frac{1}{2} \lim_{s \longrightarrow \infty} [\mu(x^{k_s}) + r(n)] - r(n) = 0.$$

This yields

$$\lim_{s \longrightarrow \infty} \theta(\|x^{k_s} - \bar{z}\|) = 0$$

and

$$\lim_{s \longrightarrow \infty} \|x^{k_s} - \bar{z}\| = 0.$$

Therefore, the subsequence $\{x^{k_s}\}$ converges strongly to $\bar{z} \in W_n^*$. So, the sequence $\{x^k\}$ has strong limit points and all these limit points belong to W_n^*. Next,

$$0 \leq \liminf_{k \longrightarrow \infty} d(x^k, W_n^*) \leq \limsup_{k \longrightarrow \infty} d(x^k, W_n^*) = \lim_{s \longrightarrow \infty} d(x^{k_s}, W_n^*)$$

for some subsequence $\{x^{k_s}\}$. However, without any loss of generality we can suppose that $\{x^{k_s}\} \longrightarrow \bar{z} \in W_n^*$, then

$$\lim_{s \longrightarrow \infty} d(x^{k_s}, W_n^*) \leq \lim_{s \longrightarrow \infty} \|x^{k_s} - \bar{z}\| = 0,$$

Again, since the sequence $\{g^{k_s}\}$ belong to W_n^* and

$$0 \leq \liminf_{k \longrightarrow \infty} d(g^k, W_n^*) \leq \limsup_{k \longrightarrow \infty} d(g^k, W_n^*) = \lim_{s \longrightarrow \infty} d(g^{k_s}, W_n^*)$$

for some subsequence $\{g^{k_s}\}$. However, without any loss of generality, we can suppose that $\{g^{k_s}\} \longrightarrow \bar{g}$ and $\bar{g} \in G(\bar{z}) \in W_n^*$, then

$$\begin{aligned} \lim_{s \longrightarrow \infty} d(g, \bar{g}) &\leq \lim_{s \longrightarrow \infty} \{\|g - g^{k_s}\| + d(g^{k_s}, G(\bar{z}))\} \\ &\leq \lim_{s \longrightarrow \infty} \{\|g - g^{k_s}\| + \mathfrak{H}(G(x^{k_s}), G(\bar{z}))\} \\ &\leq \lim_{s \longrightarrow \infty} \{\|g - g^{k_s}\| + \xi \|x^{k_s} - \bar{z}\|\} \longrightarrow 0, \end{aligned}$$

where $g \in G(z)$, $\mathfrak{H}(\cdot, \cdot)$ is a Hausdorff metric and $\xi > 0$ is a constant, *i.e.,* (5.2) holds and part (ii) is true. Next, in case (iii), W_n^* must be a singleton $\{z_n^*\}$ as in Theorem 5.1. This implies the strong convergence of $\{x^k\} \longrightarrow z_n^*$ and $\{g^{k_s}\} \longrightarrow g_n^*$ where $g_n^* \in G(z_n^*)$. □

It should be noted that assertions (ii) and (iii) of Theorem 5.3 remain true when there exists a number $\tau' > 0$ such that (C2) holds if $0 < \tau < \tau'$ and (C2$'$) holds if $\tau \geq \tau'$.

6. Conclusions

We considered a general regularized penalty method for set valued equilibrium problems in reflexive Banach space without monotonicity assumptions and concordance rule for penalty and regularization parameters. We suggest new coercivity conditions that provide weak and strong convergence properties of the methods.

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