SUBNORMALITY OF THE WEIGHTED CESÀRO OPERATOR $C_h \in \ell^2(h)$

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Abstract. The subnormality of some classes of operators is a very interesting property. In this paper, we prove that the weighted Cesàro operator $C_h \in \ell^2(h)$ is subnormal and we described completely the set of the extended eigenvalues for the weighted Cesàro operator, some other important results are also given.

1. Introduction and preliminaries

In this paper we discuss the Cesàro operator in weighted $\ell^2$ spaces. For a sequence $h = (h(n))_{n \in \mathbb{N}}$ of positive numbers, called weights and a sequence $a = (a(n))_{n \in \mathbb{N}}$ of complex numbers the discrete weighted Cesàro operator $C_h$ is defined by

$$(C_ha)(n) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k)a(k), \text{ with } H(n) = \sum_{k=0}^{n} h(k)$$

(1)

Let $1 < p < \infty$ and

$$\ell^p(h) = \left\{ a = (a(n))_{n \in \mathbb{N}_0} : a(n) \in \mathbb{C}, \|a\|_{p,h}^p = \sum_{n=0}^{\infty} h(n)|a(n)|^p < \infty \right\}$$

(2)

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It is well known that the Cesàro operator in $\ell^p(h)$ is bounded by $\|C_h\| \leq \frac{p}{p-1}$, see [9]. An easy computation shows that the dual operator $C^*_h$ of $C_h$ in $\ell^q(h), \frac{1}{p} + \frac{1}{q} = 1$, is

$$\begin{align*}
(C^*_h a)(n) &= \sum_{k=n}^{\infty} \frac{h(k) a(k)}{H(k)}.
\end{align*}
$$

In the Hilbert space $\ell^2(h)$ the inner product is defined by

$$\begin{align*}
\langle a, b \rangle_h &= \sum_{n=0}^{\infty} h(n) a(n) \overline{b(n)}, & a, b \in \ell^2(h).
\end{align*}
$$

**Theorem 1.1.** [8] Let $T \in B(H)$. The following conditions on $T$ are equivalent.

1. $T$ is subnormal.
2. There exists a unitary operator $U \in B(H \oplus H)$ such that for $n = 0, 1, \ldots$: $T^*n = P_H U^n T^n$, where $P_H$ is the orthogonal projection of $H \oplus H$ onto $H \oplus 0$.
3. For $n = 0, 1, \ldots$:

$$T^*n = \left[ \int_{\partial D} e^{int} dQ(t) \right] T^n
$$

where $Q$ is a positive operator measure (denoted by POM) defined on the boundary of the unit disc, $\partial D$.
4. There exists a sequence of operators $K_n \in B(H)$ satisfying $T^*n = K_n T^n$ for $n = 0, 1, \ldots$, Moreover if we define

$$L_n = \begin{cases}
K_n & \text{if } n \geq 0 \\
K^*_n & \text{if } n < 0.
\end{cases}
$$

then for any finite set $\{x_0, x_1, \ldots, x_n\}$ contained in $H$,

$$\sum_{j,k \geq 0}^{n} \langle L_{j-k} x_j, x_k \rangle \geq 0.
$$

5. There exists a sequence of operators $K_n \in B(H)$ satisfying $T^*n = K_n T^n$ for $n = 0, 1, \ldots$, Moreover if we define
Subnormality of the weighted Cesàro operator $C_h \in l^2(h)$

$$L_n = \begin{cases} K_n & \text{if } n \geq 0 \\ K_n^* & \text{if } n < 0. \end{cases}$$

then for each $x \in H$ and each $n = 0, 1, \ldots$, the matrix $[(L_{j-k}x, x)]_{j,k \geq 0}^n$ is positive definite.

**Lemma 1.2.** [6] Let $T$ be a bounded linear operator on a complex Banach space $E$ and let us suppose that there is an analytic mapping $f : \text{int}\sigma_p(T) \rightarrow E$ with $f(z) \in \ker(T - z)/\{0\}$ for all $z \in \text{int}\sigma_p(T)$ and such that $f(z) : z \in \text{int}\sigma_p(T)$ is a total subset of $E$. Then $T$ has rich point spectrum.

**Theorem 1.3.** [6] Let $T$ be a bounded linear operator with rich point spectrum and such that $\sigma_p(T) = D(r, r)$ for some $r > 0$. If $\lambda$ is an extended eigenvalue for $T$ then $\lambda$ is real and $0 < \lambda \leq 1$.

2. Main Results

2.1. Subnormality of $C_h$. As $C_h$ and $C_h^*$ are operators in a sequence space, they have matrix representations with respect to the basis $(e_j)_{j \in \mathbb{N}}$ of $l^2(h)$ (in the following also denoted by $C_h$ and $C_h^*$, respectively). From (1) and (3) we can infer that

$$C_h = \begin{pmatrix} h(0) & 0 & 0 & \ldots \\ \overline{h(0)} & h(1) & 0 & \ldots \\ \overline{h(1)} & \overline{h(1)} & h(2) & \ldots \\ \overline{h(2)} & \overline{h(2)} & \overline{h(2)} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (6)$$

$$C_h^* = \begin{pmatrix} h(0) & h(1) & h(2) & \ldots \\ 0 & \overline{h(1)} & \overline{h(1)} & \ldots \\ 0 & 0 & \overline{h(2)} & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (7)$$
Direct computation yields the matrix representations of $C_h^{-1}$ and $C_h^{*-1}$ with respect to $(e_j)_{j \in \mathbb{N}}$:

$$C_h^{-1} = \begin{pmatrix}
\frac{H(0)}{h(0)} & 0 & 0 & \ldots \\
\frac{-H(1)}{h(1)} & \frac{H(1)}{h(1)} & 0 & \ldots \\
0 & \frac{-H(2)}{h(2)} & \frac{H(2)}{h(2)} & \ldots \\
0 & 0 & \frac{-H(3)}{h(3)} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \quad (8)$$

and

$$C_h^{*-1} = \begin{pmatrix}
\frac{H(0)}{h(0)} & \frac{H(0)}{h(1)} & 0 & \ldots \\
0 & \frac{-H(1)}{h(1)} & \frac{H(1)}{h(1)} & \ldots \\
0 & 0 & \frac{-H(2)}{h(2)} & \ldots \\
0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \quad (9)$$

**Theorem 2.1.** The weighted Cesàro operator $C_h \in \ell^2(h)$ is subnormal

**Proof.** Let $C_h^* = K_1C_h$, where $K_1 = C_h^*C_h^{-1}$.

Matrix $K_1$ knowing as follows

$$K_1 = \begin{pmatrix}
\frac{h(1)}{H(1)} & \frac{h(2)}{H(1)} & \frac{h(3)}{H(1)} & \ldots & 1 \\
\frac{-H(0)}{h(1)} & \frac{h(2)}{h(1)} & \frac{h(3)}{h(1)} & \ldots & 1 \\
0 & \frac{-H(1)}{h(2)} & \frac{h(3)}{h(2)} & \ldots & 1 \\
0 & 0 & \frac{-H(2)}{h(3)} & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & 1
\end{pmatrix} \quad (10)$$

And for each natural number $n$:

$$K_n = C_h^{*-n}C_h^{-n} = C_h^{*-n-1}K_1C_h^{1-n}. \quad (11)$$

In [4] the positivity of the matrix $T$ acting on $\ell^2$ was proved by considering the determinants of its finite sections. In order to include the case when the matrix $T$ is positive semidefinite, we give a more detailed proof for the positivity of the operator $T$ here.
The bilinear form $\langle \cdot, K_1 \cdot \rangle_h$ is defined for all sequences 

$$a, b \in \ell^2(h).$$

Using the vector representations for $a$ and $b$, the matrix representation for $K_1$ and the inner product as defined in (4), we obtain

$$\langle a, K_1 b \rangle_h = \left( \begin{array}{c} a(0) \\ a(1) \\ \vdots \end{array} \right)^t \left( \begin{array}{ccc} h(0) & 0 & \\ 0 & h(1) & \\ & \ddots & \end{array} \right) \times \left( \begin{array}{ccc} h(1) & h(2) & \cdots \\ -H(0) & H(2) & \cdots \\ H(1) & H(2) & \cdots \\ & \ddots & \end{array} \right) \left( \begin{array}{c} b(0) \\ b(1) \\ \vdots \end{array} \right),$$

with

$$K_{1h} = \left( \begin{array}{ccc} \frac{h(0)h(1)}{H(1)} & \frac{h(0)h(2)}{H(2)} & \frac{h(0)h(3)}{H(3)} & \cdots \\ \frac{-h(1)H(0)}{H(1)} & \frac{h(1)b(2)}{H(2)} & \frac{h(1)h(3)}{H(3)} & \cdots \\ 0 & \frac{-h(2)H(1)}{H(2)} & \frac{h(2)b(3)}{H(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right).$$

In particular, for all $n \in \mathbb{N}$

$$\langle e_i, K_1 e_i \rangle_h = \frac{h(i-1)h(i)}{H(i)} \geq 0, \quad i \geq 1.$$  

If $i = n$ then

$$\langle e_n, K_1 e_n \rangle_h = h(n) \geq 0.$$  

As well as for all $a \in \ell^2(h)$,  

$$\langle a, K_1 a \rangle_h \geq 0.$$  

And so $K_1$ is positive.

We have also $C_h$ is positive so $C_h^{-1}$ is positive. And this for any natural $n$, $C_h^{-n}$ is positive.

The same applies to $C_h^*$ so $C_h^* n$ is positive.

And since, for each natural $n : \quad K_n = C_h^{*n-1}K_1C_h^{1-n}$ is positive.

By Theorem (1.1), the operator $C_h$ is subnormal. □
2.2. Extended eigenvalues for The weighted Cesàro operator $C_h \in \ell^p$. We shall prove in this section that the set of the extended eigenvalues for the weighted Cesàro operator is the interval $[1, \infty)$ when $p = 2$ and that it is contained in the interval $[1, \infty)$ when $1 < p < \infty$. Let us recall that the weighted Cesàro operator $C_h$ is defined on the complex Banach space $\ell^p$ by the sequence of arithmetic means (1).

**Theorem 2.2.**

1. The point spectrum of $C_h$ is empty.
2. If $|1 - \lambda| < 1$ then $\lambda$ is a simple proper value of $C_h^*$.
3. The spectrum of $C_h$ is the closed disc $\{\lambda : |1 - \lambda| \leq 1\}$.
4. The point spectrum of $C_h^*$ is the open disc $\{\lambda : |1 - \lambda| < 1\}$.

**Proof.**

1. Observe first if $C_h f = g$ then $f(0) = g(0)$ and if $n \geq 1$ then

$$C_h f(n) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k) f(k) = g(n)$$

And therefore

$$\frac{h(0)f(0)}{H(n)} + \frac{h(1)f(1)}{H(n)} + \ldots + \frac{h(n)f(n)}{H(n)} = g(n)$$

$$\frac{h(0)f(0)}{H(n-1)} + \frac{h(1)f(1)}{H(n-1)} + \ldots + \frac{h(n-1)f(n-1)}{H(n-1)} = g(n-1)$$

And it

$$h(n) f(n) = H(n) g(n) - H(n-1) g(n-1)$$

Which

$$f(n) = \frac{H(n) g(n)}{h(n)} - \frac{H(n-1) g(n-1)}{h(n)}$$

consequently, if $C_h f = \lambda f$ then

$$f(n) = \lambda \left( \frac{H(n) f(n)}{h(n)} - \frac{H(n-1) f(n-1)}{h(n)} \right)$$

Or

$$\left( \lambda \frac{H(n)}{h(n)} - 1 \right) f(n) = \lambda \frac{H(n-1) f(n-1)}{h(n)}$$

Whenever $n \geq 1$, if $m$ is the smallest integer for which $f(m) \neq 0$, then $\lambda = \frac{h(m)}{H(m)}$ so that $0 < \lambda \leq 1$. It follows that if $n \geq 1$ then
\[ |f(n)| = \left| \frac{\lambda H(n-1)f(n-1)}{\lambda h(n) - 1} \right| \]
\[ = \left| \frac{\lambda H(n-1)f(n-1)}{\lambda H(n) - h(n)} \right| \geq f(n-1) \]

Which, for a non zero \( f \in \ell^2 \); is impossible.

2. Observe first that \((C^*_h f)(n) = \sum_{k=n}^{\infty} \frac{h(k)f(k)}{H(k)}\).

If \(C^*_h f = g\) then
\[
\frac{h(n)f(n)}{H(n)} + \frac{h(n+1)f(n+1)}{H(n+1)} + \ldots = g(n)
\]
\[
\frac{h(n+1)f(n+1)}{H(n+1)} + \frac{h(n+2)f(n+2)}{H(n+2)} + \ldots = g(n+1)
\]
\[\implies f(n) = \frac{H(n)g(n)}{h(n)} - \frac{H(n)g(n+1)}{h(n)}, \quad n = 0, 1, 2, \ldots\]

consequently; if \(C^*_h f = \lambda f\) then
\[f(n) = \lambda \left( \frac{H(n)f(n)}{h(n)} - \frac{H(n)f(n+1)}{h(n)} \right)\]

And it
\[\lambda \frac{H(n)f(n+1)}{h(n)} = \left( \frac{\lambda H(n)}{h(n)} - 1 \right) f(n)\]

It follows that 0 is not a proper value of \(C^*_h\); if \(\lambda = 0\) then \(f(n) = 0, n = 0, 1, 2, \ldots\) and it follows also that
\[f(n+1) = \left( 1 - \frac{h(n)}{\lambda H(n)} \right) f(n).\]

This implies that if \(n \geq 1\) then
\[f(n) = \prod_{j=1}^{n} \left( 1 - \frac{h(j)}{\lambda H(j)} \right) f(0).\]

And we can conclude, even before we know which values of \(\lambda\) can be proper values of \(C^*_h\) that all the proper values are simple.
3. Since \( \|1 - C_h\| \leq 1 \), the spectrum of \( 1 - C_h \) is included in the closed disc \( \{ \lambda : |\lambda| \leq 1 \} \), and consequently, the spectrum of \( C_h \) is included in the closed disc \( \{ \lambda : |1 - \lambda| \leq 1 \} \).

4. The preceding paragraph implies that the spectrum of \( 1 - C_h^* \) includes the open disc \( \{ \lambda : |\lambda| < 1 \} \), and hence that the same is true of the spectrum of \( C_h \). This, in turn implies that the spectrum of \( C_h \) includes the open disc \( \{ \lambda : |1 - \lambda| < 1 \} \), and the proof of (4) is complete.

Theorem 2.3. The adjoint of the weighted Cesaro operator \( C_h^* \in B(\ell_q) \) has rich point spectrum.

Proof. Notice that \( \sigma_p(C_h^*) = D(q/2, q/2) \) is open and connected. It is easy to see that the mapping \( f : \sigma_p(C_h^*) \rightarrow \ell_q \) defined by

\[
    f_0(z) = 1, \quad f_n(z) = \prod_{j=1}^{n} \left( 1 - \frac{h(j)}{zH(j)} \right) \quad \text{for } n \geq 1
\]

is analytic, and \( f(z) \in \ker(C_h^* - z)/0 \). It is a standard fact that the family of eigenvectors \( \{ f(z) : z \in D(q/2, q/2) \} \) is total in \( \ell_q \). Then \( C_h^* \in B(\ell_q) \) has rich point spectrum.

Lemma 2.4. If \( \lambda \) is an extended eigenvalue for \( C_h \) on \( \ell_p \) then \( \lambda \) is real and \( \lambda \geq 1 \).

Proof. First of all, we have \( \lambda \neq 0 \) because \( C_h \) is injective. Also, notice that \( \lambda \) is an extended eigenvalue for \( C_h \) if and only if \( 1/\lambda \) is an extended eigenvalue for \( C_h^* \), and therefore it is enough to show that if \( \lambda \) is an extended eigenvalue for \( C_h^* \) then \( \lambda \) is real and \( 0 < \lambda \leq 1 \).

Our next goal is to show in the Hilbertian case \( p = 2 \) that if \( \lambda \) is real and \( \lambda \geq 1 \) then \( \lambda \) is an extended eigenvalue for \( C_h \). In section 2.1 we showed that \( C_h \) is subnormal using the following construction. Let \( \mu \) be a positive finite measure defined on the Borel subsets of the complex plane with compact support and let \( H^2(\mu) \) be the closure of the polynomials on the Hilbert space \( \ell^2(\mu) \). Consider the shift operator \( M_z \) defined on the Hilbert space \( H^2(\mu) \) by the expression \( (M_z f)(z) = z f(z) \) there is a is a positive finite measure defined on the Borel subsets of the complex plane and supported on \( \overline{D} \), and there is a unitary operator \( U : \ell^2 \rightarrow H^2(\mu) \) such that \( I - C_h = U^* M_z U \), or in other words
\[ C_h = U^*(I - M_z)U \]

Then, the extended eigenvalues of \( C_h \) are the extended eigenvalues of \( I - M_z \) and the corresponding extended eigenoperators of \( C_h \) are in one to one correspondence with the extended eigenoperators of \( I - M_z \) under conjugation with \( U \), that is, if a non-zero operator \( X \) satisfies \( (I - M_z)X = \lambda X(I - M_z) \) then the operator \( Y = U^*XU \) satisfies \( C_h Y = \lambda Y C_h \).

**Theorem 2.5.** If \( \lambda_k = \frac{H(k)}{h(k)} \geq 1 \) then \( \lambda_k \) is an extended eigenvalue for \( I - M_z \) and a corresponding extended eigenoperator is the composition operator \( X_k \) defined by the expression

\[
(X_k f)(z) = f \left( \frac{H(k) - h(k)}{H(k)} + \frac{zh(k)}{H(k)} \right). \tag{13}
\]

Where \( H(k), h(k) \) defined by (1) and \( k \in \mathbb{N} \)

**Proof.** Let \( f_n = X_k z^n = \left( \frac{H(k) - h(k)}{H(k)} + \frac{zh(k)}{H(k)} \right)^n \).

We have \( f_{n+1} = \left( \frac{H(k) - h(k)}{H(k)} + \frac{zh(k)}{H(k)} \right) f_n \) so that

\[
\frac{H(k)}{h(k)} f_{n+1} = \left( \frac{H(k)}{h(k)} - 1 + z \right) f_n
\]

\[
= \left( \frac{H(k)}{h(k)} - 1 + M_z \right) f_n
\]

\[
= \frac{H(k)}{h(k)} f_n - (I - M_z) f_n
\]

and it follows that

\[
(I - M_z) f_n = \frac{H(k)}{h(k)} (f_n - f_{n+1})
\]

So that

\[
(I - M_z) X_k z^n = \frac{H(k)}{h(k)} (f_n - f_{n+1})
\]

\[
= \frac{H(k)}{h(k)} (X_k z^n - X_k M_z z^n)
\]

\[
= \frac{H(k)}{h(k)} X_k (I - M_z) z^n.
\]
and since the family of monomials \( \{z^n : n \in \mathbb{N}\} \) is a total set in \( H^2(\mu) \), and \( \lambda_k = \frac{H(k)}{h(k)} \) it follows that \( (I - M_z)X_k = \lambda_k X_k (I - M_z) \).

\[ \square \]

**Corollary 2.6.** If \( \lambda_k = \frac{H(k)}{h(k)} \geq 1 \) then \( \lambda_k \) is an extended eigenvalue for the weighted Cesàro operator \( C_h \in \ell^2 \).

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